Canadian Mathematics Competition

An activity of the Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario

2009 Galois Contest

Wednesday, April 8, 2009

Solutions

1. (a) Since the total number of students in the class is $8 + 7 + 3 + 2 = 20$, then the fraction of students having blonde hair is $\frac{8}{20}$.
Thus, $\frac{8}{20} \times 100\% = 40\%$ of the students in the class have blonde hair.

(b) Since there are 3 students with red hair and 2 students with black hair, then the number of students that have red or black hair is 5.
Thus, the fraction of students with red or black hair is $\frac{5}{20}$.
The percentage of students in the class with red or black hair is $\frac{5}{20} \times 100\% = 25\%$.

(c) First note that if some number of students in the class with blonde hair dye their hair black, the total number of students in the class remains unchanged at 20.
For 20% of the class to have black hair, 20% of 20 or 4 students need to have black hair.
Since there are presently 2 students with black hair, 2 students with blonde hair would need to dye their hair black.

(d) If x students with red hair join the class, the number of students with red hair will be $3 + x$, and the total number of students in the class will be $20 + x$.
Thus, the new fraction of students having red hair will be $\frac{3+x}{20+x}$.
Since 32% is equivalent to $\frac{32}{100} = \frac{8}{25}$, we need to find the value of x satisfying $\frac{3+x}{20+x} = \frac{8}{25}$.
Solving, we get $25(3 + x) = 8(20 + x)$ or $75 + 25x = 160 + 8x$ or $17x = 85$ or $x = 5$.
Thus, 5 students with red hair would have to join the class so the percentage of students in the class with red hair is equal to 32%.

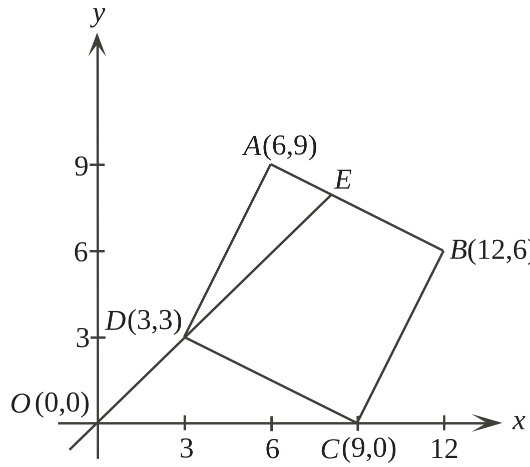
2. (a) *Solution 1*
Since $ABCD$ is a square, the path travelling from A to B is the same as the path travelling from D to C. To travel from A to B, we go 6 units right and 3 units down.
Thus, C has coordinates $(3 + 6, 3 - 3) = (9, 0)$, so $t = 9$.

(b) *Solution 2*
The slope of CD is $\frac{3-0}{3-t}$ and the slope of AB is $\frac{6-9}{12-6} = \frac{-3}{6} = -\frac{1}{2}$.
Since $ABCD$ is a square, CD is parallel to AB.
Since parallel line segments have equal slopes, we get

$$\begin{aligned} \frac{3}{3-t} &= -\frac{1}{2} \\ (3)(2) &= (-1)(3-t) \\ 6 &= -3+t \\ t &= 9 \end{aligned}$$

Therefore, the x-coordinate of vertex C is 9.

(c)



First, we find the equation of the line through O and D.
The slope of the line through $O(0,0)$ and $D(3,3)$ is $\dfrac{3-0}{3-0} = 1$.
Since this line has y-intercept 0, the equation of the line is $y = x$.
Next, we find the equation of the line through A and B.
As in part (a), the slope of the line through A and B is $-\frac{1}{2}$.
Therefore, the line has equation $y = -\frac{1}{2}x + b$ for some b.
Since $B(12,6)$ lies on this line, then $6 = -\frac{1}{2}(12) + b$ so $6 = -6 + b$ or $b = 12$.
Thus, the equation of the line is $y = -\frac{1}{2}x + 12$.
These lines, $y = x$ and $y = -\frac{1}{2}x + 12$, intersect when $x = -\frac{1}{2}x + 12$ or $\frac{3}{2}x = 12$ or $x = 8$.
Therefore, point E has coordinates (8,8).

(d) The required lengths are as follows,

$$ED = \sqrt{(8-3)^2 + (8-3)^2} = \sqrt{50} = 5\sqrt{2}$$

and

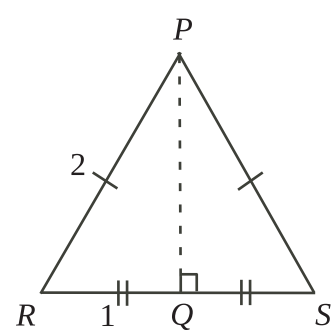
$$EB = \sqrt{(8-12)^2 + (8-6)^2} = \sqrt{20} = 2\sqrt{5}$$

and

$$CD = CB = \sqrt{(12-9)^2 + (6-0)^2} = \sqrt{45} = 3\sqrt{5}.$$

Thus, the perimeter of quadrilateral $EBCD$ is $5\sqrt{2} + 2\sqrt{5} + 2 \times 3\sqrt{5} = 5\sqrt{2} + 8\sqrt{5}$.

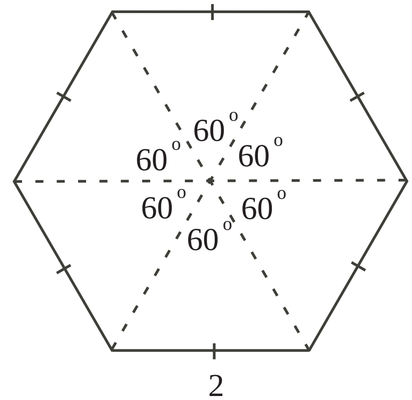
3. (a) Equilateral triangle PRS has side lengths equal to 2.
Since $PR = PS$, the perpendicular from P meets RS at its midpoint Q as shown. Thus, $RQ = QS = 1$ and $\triangle PRQ$ is a right triangle.
Using the Pythagorean Theorem,



$$\begin{aligned} PR^2 &= RQ^2 + QP^2 \\ 2^2 &= 1^2 + QP^2 \\ 4 &= 1 + QP^2 \\ 3 &= QP^2 \\ QP &= \sqrt{3} \quad (\text{since } QP > 0) \end{aligned}$$

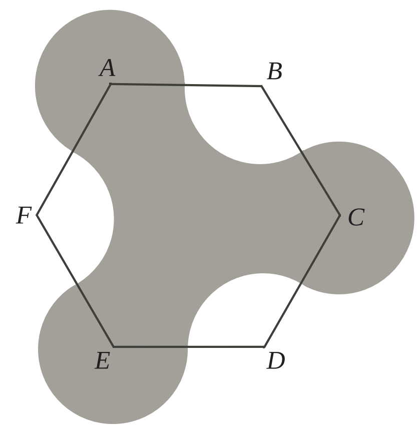
The area of an equilateral triangle with side length 2 is, $\frac{1}{2}(RS)(QP) = \frac{1}{2}(2)(\sqrt{3}) = \sqrt{3}$.

(b) Using 6 equilateral triangles with side length 2, we can create the regular hexagon as shown.



Let us justify that these 6 equilateral triangles will meet in a common point at the hexagon centre without any overlap or gaps between the triangles.
The angle at each vertex of an equilateral triangle is 60°. When 6 of these vertices meet at a common point, the sum of the angles is $6\times 60^\circ = 360^\circ$, a complete rotation as required. Also, note that the sides of the hexagon formed each have length 2 and each interior angle of the hexagon measures $60^\circ + 60^\circ = 120^\circ$. Thus, we can be assured that exactly 6 equilateral triangles with side length 2 can produce a regular hexagon with side length 2.
Then, the area of the regular hexagon is 6 times the area of the equilateral triangle from part (a) or
$A = 6 \times \sqrt{3} = 6\sqrt{3}$.

(c) The construction of the hexagon in part (b) gives each interior angle measuring 120°.



Since the interior angle at each of B, D and F is 120°, then the unshaded sector inside the hexagon at each of these points is $\frac{120^\circ}{360^\circ} = \frac{1}{3}$ of a full circle.
Thus, the area of each of these sectors is $\frac{1}{3} \times \pi(1)^2 = \frac{1}{3}\pi$.
Therefore, the total area of the 3 unshaded sectors inside the hexagon, is $3 \times \frac{1}{3}\pi = \pi$.
Since each interior angle of the hexagon measures 120°, the measure of the reflex angle at each of A, C and E is $360^\circ - 120^\circ = 240^\circ$.
Thus, the shaded sector outside the hexagon at each of these points is $\frac{240^\circ}{360^\circ} = \frac{2}{3}$ of a full circle.
Thus, the area of each of these sectors is $\frac{2}{3} \times \pi(1)^2 = \frac{2}{3}\pi$.
Therefore, the total area of the 3 shaded sectors outside the hexagon, is $3 \times \frac{2}{3}\pi = 2\pi$.
The shaded area consists of the entire hexagon, minus the three unshaded sectors at B, D and F, plus the three shaded sectors at A, C and E, or $6\sqrt{3}-\pi+2\pi = 6\sqrt{3}+\pi$.

4. (a) The largest positive integer N that can be written in this form is obtained by maximizing the values of the integers a, b, c, d, and e. Thus, $a = 1$, $b = 2$, $c = 3$, $d = 4$, and $e = 5$, which gives $N = 1(1!)+2(2!)+3(3!)+4(4!)+5(5!) = 1+2(2)+3(6)+4(24)+5(120) = 719$.

(b) For any two positive integers n and m, it is always possible to write a division statement of the form,

$$n = qm + r,$$

where the *quotient* q and *remainder* r are non-negative integers and $0 \le r < m$.
The following table shows some examples of this.

n	m	q	r	$n = qm + r$
20	6	3	2	$20 = 3(6) + 2$
12	13	0	12	$12 = 0(13) + 12$
9	7	1	2	$9 = 1(7) + 2$
36	9	4	0	$36 = 4(9) + 0$

Notice that in each of the 4 examples, the inequality $0 \leq r < m$ has been satisfied.
We can always satisfy this inequality by beginning with n and then subtracting multiples of m from it until we get a number in the range 0 to $m-1$. We let r be this number, or $r = n - qm$, so that $n = qm + r$.
Further, this process is repeatable. For example, beginning with $n = 653$ and $m = 5! = 120$, we get $653 = 5(120) + 53$. We can now repeat the process using remainder $r = 53$ as our next n, and $4! = 24$ as our next m. This process is shown in the table below with each new remainder becoming our next n and m taking the successive values of 5!, 4!, 3!, 2!, and 1!.

n	m	q	r	$n = qm + r$
653	120	5	53	$653 = 5(120) + 53$
53	24	2	5	$53 = 2(24) + 5$
5	6	0	5	$5 = 0(6) + 5$
5	4	1	1	$5 = 2(2) + 1$
1	1	1	0	$1 = 1(1) + 0$

From the 5th column of the table above,

$$\begin{aligned}
653 &= 5(120) + 53 \\
&= 5(120) + 2(24) + 5 \\
&= 5(120) + 2(24) + 0(6) + 5 \\
&= 5(120) + 2(24) + 0(6) + 2(2) + 1 \\
&= 5(120) + 2(24) + 0(6) + 2(2) + 1(1) + 0 \\
&= 5(5!) + 2(4!) + 0(3!) + 2(2!) + 1(1!)
\end{aligned}$$

Thus, $n = 653$ is written in the required form with $a = 1$, $b = 2$, $c = 0$, $d = 2$, and $e = 5$.

(c) Following the process used in (b) above, we obtain the more general result shown here.

n	m	q	r	$n = qm + r$	restriction on r
n	120	e	r_1	$n = e(120) + r_1$	$0 \leq r_1 < 120$
r_1	24	d	r_2	$r_1 = d(24) + r_2$	$0 \leq r_2 < 24$
r_2	6	c	r_3	$r_2 = c(6) + r_3$	$0 \leq r_3 < 6$
r_3	2	b	r_4	$r_3 = b(2) + r_4$	$0 \leq r_4 < 2$
r_4	1	a	r_5	$r_4 = a(1) + r_5$	$0 \leq r_5 < 1$

From the 5th column of this table,

$$\begin{aligned}
n &= e(120) + r_1 \\
&= e(120) + d(24) + r_2 \\
&= e(120) + d(24) + c(6) + r_3 \\
&= e(120) + d(24) + c(6) + b(2) + r_4 \\
&= e(120) + d(24) + c(6) + b(2) + a(1) + r_5 \\
&= e(5!) + d(4!) + c(3!) + b(2!) + a(1!) \text{ (since } r_5 = 0)
\end{aligned}$$

We must justify that the integers a, b, c, d, and e satisfy their required inequality.
From part (b), each of these quotients is a non-negative integer. Therefore, it remains to show that $a \leq 1$, $b \leq 2$, $c \leq 3$, $d \leq 4$, and $e \leq 5$.
From part (a), $N = 719$, therefore $0 \leq n < 720$.
From the table above, we have $n = e(120)+r_1$. Therefore $e(120)+r_1 < 720$ or $e(120) < 720$ (since $r_1 \geq 0$), and so $e < 6$. Thus, $e \leq 5$, as required.
Also from the table above, $r_1 < 120$, so $d(24) + r_2 < 120$ or $d(24) < 120$ (since $r_2 \geq 0$), and therefore $d < 5$. Thus, $d \leq 4$, as required.
Also, $r_2 < 24$, so $c(6) + r_3 < 24$ or $c(6) < 24$ (since $r_3 \geq 0$), and therefore $c < 4$.
Thus, $c \leq 3$, as required.
Continuing, $r_3 < 6$, so $b(2) + r_4 < 6$ or $b(2) < 6$ (since $r_4 \geq 0$), and therefore $b < 3$.
Thus, $b \leq 2$, as required.
Finally, $r_4 < 2$, so $a(1) + r_5 < 2$ or $a(1) < 2$ (since $r_5 = 0$), and therefore $a < 2$.
Thus, $a \leq 1$, as required.
Therefore, all integers n, with $0 \leq n \leq N$, can be written in the required form.

(d) Since $c = 0$, we are required to find the sum of all integers n of the form $n = a + 2b + 24d + 120e$, with the stated restrictions on the integers a, b, d, and e.
Since $n = a + 2b + 24d + 120e = (a + 2b) + 24(d + 5e)$, let $n_1 = a + 2b$ and $n_2 = d + 5e$ so that $n = n_1 + 24n_2$. First, consider all possible values of n_1.
Since $0 \leq a \leq 1$ and $0 \leq b \leq 2$ and $n_1 = a + 2b$, we have that n_1 can equal any of the numbers in the set $\{0, 1, 2, 3, 4, 5\}$. Each of these comes from exactly one pair (a, b).
Next, find all possible values for $n_2 = d + 5e$. Since $0 \leq d \leq 4$ and $0 \leq e \leq 5$, we have that $d + 5e$ can equal any of the numbers in the set $\{0, 1, 2, 3, 4, 5, 6, 7, \ldots, 29\}$.
Each of these comes from exactly one pair (d, e).
Therefore, $24n_2$ can equal any of the numbers in the set $\{24 \times 0, 24 \times 1, 24 \times 2, \ldots, 24 \times 29\} = \{0, 24, 48, \ldots, 696\}$, the multiples of 24 from 0 to 696.
Adding each of these possible values of $24n_2$ in turn to each of the 6 possible values of n_1, we get the set of all possible $n = n_1 + 24n_2$:

$$\{0, 1, 2, 3, 4, 5, 24, 25, 26, 27, 28, 29, 48, 49, 50, 51, 52, 53, \ldots, 696, 697, 698, 699, 700, 701\}$$

Because each of the 6 possible values of n_1 comes from exactly one pair (a, b) and each of the 30 possible values of n_2 comes from exactly one pair (d, e), then each of these integers above occurs exactly once as a, b, d, and e move through their possible values.
It remains to find the sum of these possible values for n:

$$\begin{aligned}
& 0+1+2+3+4+5+24+25+26+27+28+29+48+49+\cdots+699+700+701 \\
= \; & 0+1+2+3+4+5+(24+0)+(24+1)+(24+2)+(24+3)+(24+4)+(24+5) \\
& +(48+0)+(48+1)+\cdots+(696+3)+(696+4)+(696+5) \\
= \; & (0+1+2+3+4+5)+24\times 6+(0+1+2+3+4+5)+48\times 6 \\
& +(0+1+2+3+4+5)+\cdots+696\times 6+(0+1+2+3+4+5) \\
= \; & 30(0+1+2+3+4+5)+24\times 6+48\times 6+\cdots+696\times 6 \\
= \; & 30(15)+24(6)[1+2+3+\cdots+29] \\
= \; & 30(15)+24(6)\left[\frac{29\times 30}{2}\right] \\
= \; & 63090
\end{aligned}$$