Canadian
Mathematics
Competition

An activity of the Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario

2009 Euclid Contest

Tuesday, April 7, 2009

Solutions

1. (a) We rewrite $6x + 3y = 21$ as $3y = -6x + 21$ or $y = -2x + 7$.
Thus, the slope is -2.

(b) *Solution 1*

Since the slope of the line segment is 3, then $\frac{c-0}{5-1} = 3$, and so $\frac{c}{4} = 3$ or $c = 12$.

Solution 2

Since the slope of the line segment is 3, then for every unit that we move to the right, we move 3 units up.
Since $(5, c)$ is 4 units to the right of $(1, 0)$, then it is $3(4) = 12$ units up from $(1, 0)$, so $c = 0 + 12 = 12$.

(c) *Solution 1*

The given line segment joins $(0, 4)$ to $(8, -4)$, so has slope $\frac{4-(-4)}{0-8} = \frac{8}{-8} = -1$.
Since the y-intercept of the line segment is 4, then the equation of the line passing through A and B is $y = -x + 4$.
Since the point (k, k) lies on the line, then $k = -k + 4$ or $2k = 4$ and so $k = 2$.

Solution 2

We label the point (k, k) as K.
Since K lies on the line segment AB, then the slope of AK equals the slope of AB.
Line segment AB joins $(0, 4)$ to $(8, -4)$, so has slope $\frac{4-(-4)}{0-8} = \frac{8}{-8} = -1$.
Line segment AK joins $(0, 4)$ to (k, k), so has slope $\frac{k-4}{k-0}$.
Therefore, $\frac{k-4}{k} = -1$ or $k - 4 = -k$ or $2k = 4$ and so $k = 2$.

2. (a) *Solution 1*

If a quadratic equation has the form $ax^2 + bx + c = 0$, then the sum of its roots is $-\frac{b}{a}$.
Here, the sum of the roots must be $-\left(\frac{(-6)}{1}\right) = 6$.

Solution 2

Since $x^2 - 6x - 7 = 0$, then $(x - 7)(x + 1) = 0$.
Thus, the roots are $x = 7$ and $x = -1$.
The sum of these roots is $7 + (-1) = 6$.

(b) *Solution 1*

If a quadratic equation has the form $ax^2 + bx + c = 0$, then the product of its roots is $\frac{c}{a}$.
Here, the product of the roots must be $\frac{-20}{5} = -4$.

Solution 2

Since $5x^2 - 20 = 0$, then $x^2 - 4 = 0$ or $(x - 2)(x + 2) = 0$.
Thus, the roots are $x = 2$ and $x = -2$.
The product of these roots is $2(-2) = -4$.

(c) *Solution 1*

If a cubic equation has the form $a^3 + bx^2 + cx + d = 0$, then the sum of its roots is $-\frac{b}{a}$.
Here, the sum of the three roots is $-\left(\frac{-6}{1}\right) = 6$.
The average of three numbers is their sum divided by 3, so the average of the three roots is $\frac{6}{3} = 2$.

Solution 2
Since $x^3 - 6x^2 + 5x = 0$, then $x(x^2 - 6x + 5) = 0$ or $x(x-5)(x-1) = 0$.
The three roots of this equation are $x = 0$, $x = 1$ and $x = 5$.
The average of these numbers is $\frac{1}{3}(0+1+5) = \frac{1}{3}(6) = 2$.

3. (a) Since $AB = AD = BD$, then $\triangle BDA$ is equilateral.
Thus, $\angle ABD = \angle ADB = \angle DAB = 60^\circ$.
Also, $\angle DAE = 180^\circ - \angle ADE - \angle AED = 180^\circ - 60^\circ - 90^\circ = 30^\circ$.
Since CAE is a straight line, then $\angle CAD = 180^\circ - \angle DAE = 180^\circ - 30^\circ = 150^\circ$.
Now $AC = AD$ so $\triangle CAD$ is isosceles, which gives $\angle CDA = \angle DCA$.
Since the sum of the angles in $\triangle CAD$ is 180° and $\angle CDA = \angle DCA$, then

$$\angle CDA = \tfrac{1}{2}(180^\circ - \angle CAD) = \tfrac{1}{2}(180^\circ - 150^\circ) = 15^\circ$$

Thus, $\angle CDB = \angle CDA + \angle ADB = 15^\circ + 60^\circ = 75^\circ$.

(b) *Solution 1*
Since $ABCD$ is a rectangle, then $AB = CD = 40$ and $AD = BC = 30$.
By the Pythagorean Theorem, $BD^2 = AD^2 + AB^2$ and since $BD > 0$, then

$$BD = \sqrt{30^2 + 40^2} = \sqrt{900 + 1600} = \sqrt{2500} = 50$$

We calculate the area of $\triangle ADB$ is two different ways.
First, using AB as base and AD as height, we obtain an area of $\frac{1}{2}(40)(30) = 600$.
Next, using DB as base and AF as height, we obtain an area of $\frac{1}{2}(50)x = 25x$.
We must have $25x = 600$ and so $x = \frac{600}{25} = 24$.

Solution 2
Since $ABCD$ is a rectangle, then $AB = CD = 40$ and $AD = BC = 30$.
By the Pythagorean Theorem, $BD^2 = AD^2 + AB^2$ and since $BD > 0$, then

$$BD = \sqrt{30^2 + 40^2} = \sqrt{900 + 1600} = \sqrt{2500} = 50$$

Since $\triangle DAB$ is right-angled at A, then $\sin(\angle ADB) = \dfrac{AB}{BD} = \dfrac{40}{50} = \dfrac{4}{5}$.
But $\triangle ADF$ is right-angled at F and $\angle ADF = \angle ADB$.
Therefore, $\sin(\angle ADF) = \dfrac{AF}{AD} = \dfrac{x}{30}$.
Thus, $\dfrac{x}{30} = \dfrac{4}{5}$ and so $x = \frac{4}{5}(30) = 24$.

Solution 3
Since $ABCD$ is a rectangle, then $AB = CD = 40$ and $AD = BC = 30$.
By the Pythagorean Theorem, $BD^2 = AD^2 + AB^2$ and since $BD > 0$, then

$$BD = \sqrt{30^2 + 40^2} = \sqrt{900 + 1600} = \sqrt{2500} = 50$$

Note that $\triangle BFA$ is similar to $\triangle BAD$, since each is right-angled and they share a common angle at B.
Thus, $\dfrac{AF}{AB} = \dfrac{AD}{BD}$ and so $\dfrac{x}{30} = \dfrac{40}{50}$ which gives $x = \dfrac{30(40)}{50} = 24$.

4. (a) *Solution 1*

The sum of the terms in an arithmetic sequence is equal to the average of the first and last terms times the number of terms.

If n is the number of terms in the sequence, then $\frac{1}{2}(1+19)n = 70$ or $10n = 70$ and so $n = 7$.

Solution 2

Let n be the number of terms in the sequence and d the common difference.

Since the first term is 1 and the nth term equals 19, then $1 + (n-1)d = 19$ and so $(n-1)d = 18$.

Since the sum of the terms in the sequence is 70, then $\frac{1}{2}n(1 + 1 + (n-1)d) = 70$.

Thus, $\frac{1}{2}n(2+18) = 70$ or $10n = 70$ and so $n = 7$.

(b) *Solution 1*

Since the given equation is true for all values of x, then it is true for any particular value of x that we try.

If $x = -3$, the equation becomes $a(-3 + b(0)) = 2(3)$ or $-3a = 6$ and so $a = -2$.

If $x = 0$, the equation becomes $-2(0 + b(3)) = 2(6)$ or $-6b = 12$ and so $b = -2$.

Therefore, $a = -2$ and $b = -2$.

Solution 2

We expand both sides of the equation:

$$\begin{aligned} a(x + b(x+3)) &= 2(x+6) \\ a(x + bx + 3b) &= 2x + 12 \\ ax + abx + 3ab &= 2x + 12 \\ (a+ab)x + 3ab &= 2x + 12 \end{aligned}$$

Since this equation is true for all values of x, then the coefficients on the left side and right side must be equal, so $a + ab = 2$ and $3ab = 12$.

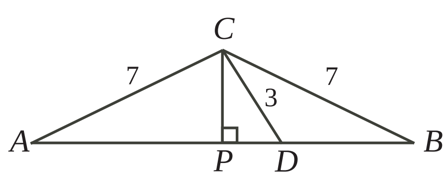
From the second equation, $ab = 4$ so the first equation becomes $a + 4 = 2$ or $a = -2$.

Since $ab = 4$, then $-2b = 4$ and so $b = -2$.

Thus, $a = b = -2$.

5. (a) *Solution 1*

Drop a perpendicular from C to P on AD.



Since $\triangle ACB$ is isosceles, then $AP = PB$.

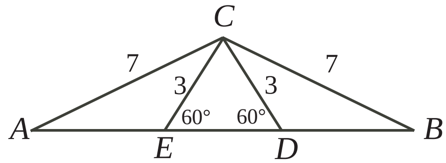
Since $\triangle CDP$ is a 30°-60°-90° triangle, then $PD = \frac{1}{2}(CD) = \frac{3}{2}$.

Thus, $AP = AD - PD = 8 - \frac{3}{2} = \frac{13}{2}$.

This tells us that $DB = PB - PD = AP - PD = \frac{13}{2} - \frac{3}{2} = 5$.

Solution 2

Since $\triangle ACB$ is symmetric about the vertical line through C, we can reflect CD in this vertical line, finding point E on AD with $CE = 3$ and $\angle CED = 60°$.



Then $\triangle CDE$ has two 60° angles, so must have a third, and so is equilateral.
Therefore, $ED = CD = CE = 3$ and so $DB = AE = AD - ED = 8 - 3 = 5$.

Solution 3
Since $\angle CDB = 180^\circ - \angle CDA = 180^\circ - 60^\circ = 120^\circ$, then using the cosine law in $\triangle CDB$, we obtain

$$\begin{aligned} CB^2 &= CD^2 + DB^2 - 2(CD)(DB)\cos(\angle CDB) \\ 7^2 &= 3^2 + DB^2 - 2(3)(DB)\cos(120^\circ) \\ 49 &= 9 + DB^2 - 6(DB)\left(-\tfrac{1}{2}\right) \\ 0 &= DB^2 + 3DB - 40 \\ 0 &= (DB-5)(DB+8) \end{aligned}$$

Since $DB > 0$, then $DB = 5$.

(b) *Solution 1*
Since $\triangle ABC$ is right-angled at C, then $\sin B = \cos A$.
Therefore, $2\cos A = 3\tan A = \dfrac{3\sin A}{\cos A}$ or $2\cos^2 A = 3\sin A$.
Using the fact that $\cos^2 A = 1 - \sin^2 A$, this becomes $2 - 2\sin^2 A = 3\sin A$ or $2\sin^2 A + 3\sin A - 2 = 0$ or $(2\sin A - 1)(\sin A + 2) = 0$.
Since $\sin A$ is between -1 and 1, then $\sin A = \frac{1}{2}$.
Since A is an acute angle, then $A = 30^\circ$.

Solution 2
Since $\triangle ABC$ is right-angled at C, then $\sin B = \dfrac{b}{c}$ and $\tan A = \dfrac{a}{b}$.
Thus, the given equation is $\dfrac{2b}{c} = \dfrac{3a}{b}$ or $2b^2 = 3ac$.
Using the Pythagorean Theorem, $b^2 = c^2 - a^2$ and so we obtain $2c^2 - 2a^2 = 3ac$ or $2c^2 - 3ac - 2a^2 = 0$.
Factoring, we obtain $(c - 2a)(2c + a) = 0$.
Since a and c must both be positive, then $c = 2a$.
Since $\triangle ABC$ is right-angled, the relation $c = 2a$ means that $\triangle ABC$ is a 30°-60°-90° triangle, with $A = 30^\circ$.

6. (a) The number of integers between 100 and 999 inclusive is $999 - 100 + 1 = 900$.
An integer n in this range has three digits, say a, b and c, with the hundreds digit equal to a.
Note that $0 \leq b \leq 9$ and $0 \leq c \leq 9$ and $1 \leq a \leq 9$.
To have $a + b + c = 24$, then the possible triples for a, b, c in some order are 9,9,6; 9,8,7; 8,8,8. (There cannot be three 9's. If there are two 9's, the the other digit equals 6. If there is one 9, the second and third digits add to 15 but are both less than 9, so must equal 8 and 7. If there are zero 9's, the maximum for each digit is 8, and so each digt must be 8 in order for the sum of all three to equal 24.)
If the digits are 9, 9 and 6, there are 3 arrangements: 996, 969, 699.

If the digits are 9, 8 and 7, there are 6 arrangements: 987, 978, 897, 879, 798, 789.
If the digits are 8, 8 and 8, there is only 1 arrangement: 888.
Therefore, there are $3+6+1=10$ integers n in the range 100 to 999 with the sum of the digits of n equal to 24.
The required probability equals the number of possible values of n with the sum of digits equal to 24 divided by the total number of integers in the range, or $\frac{10}{900}=\frac{1}{90}$.

(b) Since Alice drives at 60 km/h, then she drives 1 km every minute.
Since Alice drove from G to F in 45 minutes, then the distance from G to F is 45 km.
Let the distance from E to G be d km and let Bob's speed be B km/h.
Since Bob drove from G to E in 20 minutes (or $\frac{1}{3}$ of an hour), then $\dfrac{d}{B}=\dfrac{1}{3}$. Thus, $d=\frac{1}{3}B$.
The time that it took Bob to drive from F to G was $\frac{45}{B}$ hours.
The time that it took Alice to drive from E to G was $\frac{d}{60}$ hours.
Since the time that it took each of Alice and Bob to reach G was the same, then $\dfrac{d}{60}=\dfrac{45}{B}$ and so $Bd=45(60)=2700$.
Thus, $B\left(\frac{1}{3}B\right)=2700$ so $B^2=8100$ or $B=90$ since $B>0$.
Therefore, Bob's speed was 90 km/h.

7. (a) Completing the square on the original parabola, we obtain

$$y=x^2-2x+4=x^2-2x+1-1+4=(x-1)^2+3$$

Therefore, the vertex of the original parabola is $(1,3)$.
Since the new parabola is a translation of the original parabola and has x-intercepts 3 and 5, then its equation is $y=1(x-3)(x-5)=x^2-8x+15$.
Completing the square here, we obtain

$$y=x^2-8x+15=x^2-8x+16-16+15=(x-4)^2-1$$

Therefore, the vertex of the new parabola is $(4,-1)$.
Thus, the point $(1,3)$ is translated p units to the right and q units down to reach $(4,-1)$, so $p=3$ and $q=4$.

(b) First, we determine the coordinates of A.
The area of $\triangle ABC$ is 4. We can think of AC as its base, and its height being the distance from B to the x-axis.
If the coordinates of A are $(a,0)$, then the base has length $4-a$ and the height is 4.
Thus, $\frac{1}{2}(4-a)(4)=4$, so $4-a=2$ and so $a=2$.
Therefore, the coordinates of A are $(2,0)$.

Next, we determine the equation of the parabola.
The parabola has x-intercepts 2 and 4, so has equation $y=k(x-2)(x-4)$.
Since the parabola passes through $(0,-4)$ as well, then $-4=k(-2)(-4)$ so $k=-\frac{1}{2}$.
Therefore, the parabola has equation $y=-\frac{1}{2}(x-2)(x-4)$.

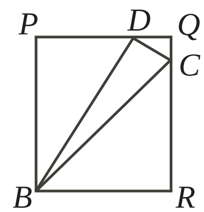
Next, we determine the coordinates of D, the vertex of the parabola.
Since the x-intercepts are 2 and 4, then the x-coordinate of the vertex is the average of these, or 3.

The y-coordinate of D can be obtained from the equation of the parabola; we obtain $y = -\frac{1}{2}(3-2)(3-4) = -\frac{1}{2}(1)(-1) = \frac{1}{2}$.
Thus, the coordinates of D are $(3, \frac{1}{2})$.

Lastly, we determine the area of $\triangle BDC$, whose vertices have coordinates $B(0,-4)$, $D(3, \frac{1}{2})$, and $C(4,0)$.

Method 1

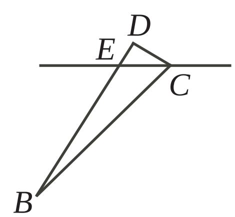
We proceed be "completing the rectangle". That is, we draw the rectangle with horizontal sides along the lines $y = \frac{1}{2}$ and $y = -4$ and vertical sides along the lines $x = 0$ and $x = 4$. We label this rectangle as $BPQR$.



The area of $\triangle BDC$ equals the area of the rectangle minus the areas of $\triangle BPD$, $\triangle DQC$ and $\triangle CRB$.
Rectangle $BPQR$ has height $4 + \frac{1}{2} = \frac{9}{2}$ and width 4.
$\triangle BPD$ has height $\frac{9}{2}$ and base 3.
$\triangle DQC$ has height $\frac{1}{2}$ and base 1.
$\triangle CRB$ has height 4 and base 4.
Therefore, the area of $\triangle BDC$ is $4(\frac{9}{2}) - \frac{1}{2}(\frac{9}{2})(3) - \frac{1}{2}(\frac{1}{2})(1) - \frac{1}{2}(4)(4) = 18 - \frac{27}{4} - \frac{1}{4} - 8 = 3$.

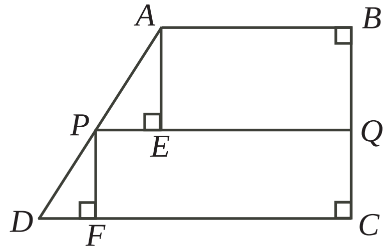
Method 2

We determine the coordinates of E, the point where BD crosses the x-axis.



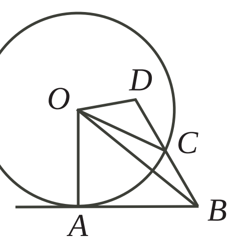
Once we have done this, then the area of $\triangle BDC$ equals the sum of the areas of $\triangle ECB$ and $\triangle ECD$.
Since B has coordinates $(0,-4)$ and D has coordinates $(3, \frac{1}{2})$, then the slope of BD is $\dfrac{\frac{1}{2} - (-4)}{3 - 0} = \dfrac{\frac{9}{2}}{3} = \dfrac{3}{2}$.
Since B is on the y-axis, then the equation of the line through B and D is $y = \frac{3}{2}x - 4$.
To find the x-coordinate of E, we set $y = 0$ to obtain $0 = \frac{3}{2}x - 4$ or $\frac{3}{2}x = 4$ or $x = \frac{8}{3}$.
We think of EC as the base of each of the two smaller triangles. Note that $EC = 4 - \frac{8}{3} = \frac{4}{3}$.
Thus, the area of $\triangle ECD$ is $\frac{1}{2}(\frac{4}{3})(\frac{1}{2}) = \frac{1}{3}$.
Also, the area of $\triangle ECB$ is $\frac{1}{2}(\frac{4}{3})(4) = \frac{8}{3}$.
Therefore, the area of $\triangle BDC$ is $\frac{1}{3} + \frac{8}{3} = 3$.

8. (a) Since PQ is parallel to AB, then it is parallel to DC and is perpendicular to BC.
Drop perpendiculars from A to E on PQ and from P to F on DC.



Then $ABQE$ and $PQCF$ are rectangles. Thus, $EQ = x$, which means that $PE = r - x$ and $FC = r$, which means that $DF = y - r$.
Let $BQ = b$ and $QC = c$. Thus, $AE = b$ and $PF = c$.
The area of trapezoid $ABQP$ is $\frac{1}{2}(x+r)b$.
The area of trapezoid $PQCD$ is $\frac{1}{2}(r+y)c$.
Since these areas are equal, then $\frac{1}{2}(x+r)b = \frac{1}{2}(r+y)c$, which gives $\dfrac{x+r}{r+y} = \dfrac{c}{b}$.
Since AE is parallel to PF, then $\angle PAE = \angle DPF$ and $\triangle AEP$ is similar to $\triangle PFD$.
Thus, $\dfrac{AE}{PE} = \dfrac{PF}{DF}$ which gives $\dfrac{b}{r-x} = \dfrac{c}{y-r}$ or $\dfrac{c}{b} = \dfrac{y-r}{r-x}$.
Combining $\dfrac{x+r}{r+y} = \dfrac{c}{b}$ and $\dfrac{c}{b} = \dfrac{y-r}{r-x}$ gives $\dfrac{x+r}{r+y} = \dfrac{y-r}{r-x}$ or $(x+r)(r-x) = (r+y)(y-r)$.
From this, we get $r^2 - x^2 = y^2 - r^2$ or $2r^2 = x^2 + y^2$, as required.

(b) Join O to A, B and C.



Since AB is tangent to the circle at A, then $\angle OAB = 90^\circ$.
By the Pythagorean Theorem in $\triangle OAB$, we get $OA^2 + AB^2 = OB^2$ or $r^2 + p^2 = OB^2$.
In $\triangle ODC$, we have $OD = DC = q$ and $OC = r$.
By the cosine law,

$$\begin{aligned} OC^2 &= OD^2 + DC^2 - 2(OD)(DC)\cos(\angle ODC) \\ r^2 &= q^2 + q^2 - 2q^2\cos(\angle ODC) \\ \cos(\angle ODC) &= \frac{2q^2 - r^2}{2q^2} \end{aligned}$$

In $\triangle ODB$, we have $\angle ODB = \angle ODC$.
Thus, using the cosine law again,

$$\begin{aligned} OB^2 &= OD^2 + DB^2 - 2(OD)(DB)\cos(\angle ODB) \\ &= q^2 + (2q)^2 - 2(q)(2q)\left(\frac{2q^2 - r^2}{2q^2}\right) \\ &= q^2 + 4q^2 - 2(2q^2 - r^2) \\ &= q^2 + 2r^2 \end{aligned}$$

So $OB^2 = r^2 + p^2 = q^2 + 2r^2$, which gives $p^2 = q^2 + r^2$, as required.

9. (a) First, we convert each of the logarithms to a logarithm with base 2:

$$\begin{aligned} 1+\log_4 x &= 1+\frac{\log_2 x}{\log_2 4} = 1+\frac{\log_2 x}{2} = 1+\tfrac{1}{2}\log_2 x \\ \log_8 4x &= \frac{\log_2 4x}{\log_2 8} = \frac{\log_2 4+\log_2 x}{3} = \tfrac{2}{3}+\tfrac{1}{3}\log_2 x \end{aligned}$$

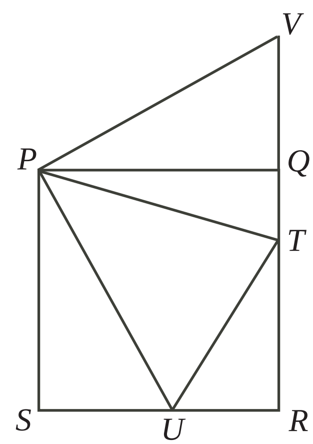
Let $y=\log_2 x$. Then the three terms are y, $1+\frac{1}{2}y$, and $\frac{2}{3}+\frac{1}{3}y$. Since these three are in geometric sequence, then

$$\begin{aligned} \frac{y}{1+\frac{1}{2}y} &= \frac{1+\frac{1}{2}y}{\frac{2}{3}+\frac{1}{3}y} \\ y(\tfrac{2}{3}+\tfrac{1}{3}y) &= (1+\tfrac{1}{2}y)^2 \\ \tfrac{2}{3}y+\tfrac{1}{3}y^2 &= 1+y+\tfrac{1}{4}y^2 \\ 8y+4y^2 &= 12+12y+3y^2 \\ y^2-4y-12 &= 0 \\ (y-6)(y+2) &= 0 \end{aligned}$$

Therefore, $y=\log_2 x=6$ or $y=\log_2 x=-2$, which gives $x=2^6=64$ or $x=2^{-2}=\frac{1}{4}$.

(b) *Solution 1*

Rotate a copy of $\triangle PSU$ by 90° counterclockwise around P, forming a new triangle PQV. Note that V lies on the extension of RQ.



Then $PV=PU$ by rotation.
Also, $\angle VPT=\angle VPQ+\angle QPT=\angle UPS+\angle QPT=90^\circ-\angle UPT=90^\circ-45^\circ$.
This tells us that $\triangle PTU$ is congruent to $\triangle PTV$, by "side-angle-side".
Thus, the perimeter of $\triangle RUT$ equals

$$\begin{aligned} UR+RT+UT &= UR+RT+TV \\ &= UR+RT+TQ+QV \\ &= UR+RQ+SU \\ &= SU+UR+RQ \\ &= SR+RQ \\ &= 8 \end{aligned}$$

That is, the perimeter of $\triangle RUT$ always equals 8, so the maximum possible perimeter is 8.

Solution 2

Let $\angle SPU = \theta$. Note that $0^\circ \leq \theta \leq 45^\circ$.

Then $\tan\theta = \dfrac{SU}{PS}$, so $SU = 4\tan\theta$.

Since $SR = 4$, then $UR = SR - SU = 4 - 4\tan\theta$.

Since $\angle UPT = 45^\circ$, then $\angle QPT = 90^\circ - 45^\circ - \theta = 45^\circ - \theta$.

Thus, $\tan(45^\circ - \theta) = \dfrac{QT}{PQ}$ and so $QT = 4\tan(45^\circ - \theta)$.

Since $QR = 4$, then $RT = 4 - 4\tan(45^\circ - \theta)$.

But $\tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$, so $\tan(45^\circ - \theta) = \dfrac{\tan(45^\circ) - \tan\theta}{1 + \tan(45^\circ)\tan\theta} = \dfrac{1 - \tan\theta}{1 + \tan\theta}$, since $\tan(45^\circ) = 1$.

This gives $RT = 4 - 4\left(\dfrac{1 - \tan\theta}{1 + \tan\theta}\right) = \dfrac{4 + 4\tan\theta}{1 + \tan\theta} - \dfrac{4 - 4\tan\theta}{1 + \tan\theta} = \dfrac{8\tan\theta}{1 + \tan\theta}$.

By the Pythagorean Theorem in $\triangle URT$, we obtain

$$\begin{aligned}
UT &= \sqrt{UR^2 + RT^2} \\
&= \sqrt{(4 - 4\tan\theta)^2 + \left(\frac{8\tan\theta}{1 + \tan\theta}\right)^2} \\
&= 4\sqrt{(1 - \tan\theta)^2 + \left(\frac{2\tan\theta}{1 + \tan\theta}\right)^2} \\
&= 4\sqrt{\left(\frac{1 - \tan^2\theta}{1 + \tan\theta}\right)^2 + \left(\frac{2\tan\theta}{1 + \tan\theta}\right)^2} \\
&= 4\sqrt{\frac{1 - 2\tan^2\theta + \tan^4\theta + 4\tan^2\theta}{(1 + \tan\theta)^2}} \\
&= 4\sqrt{\frac{1 + 2\tan^2\theta + \tan^4\theta}{(1 + \tan\theta)^2}} \\
&= 4\sqrt{\frac{(1 + \tan^2\theta)^2}{(1 + \tan\theta)^2}} \\
&= 4\left(\frac{1 + \tan^2\theta}{1 + \tan\theta}\right)
\end{aligned}$$

Therefore, the perimeter of $\triangle URT$ is

$$\begin{aligned}
UR + RT + UT &= 4 - 4\tan\theta + \frac{8\tan\theta}{1 + \tan\theta} + 4\left(\frac{1 + \tan^2\theta}{1 + \tan\theta}\right) \\
&= 4\left(\frac{1 - \tan^2\theta}{1 + \tan\theta} + \frac{2\tan\theta}{1 + \tan\theta} + \frac{1 + \tan^2\theta}{1 + \tan\theta}\right) \\
&= 4\left(\frac{2 + 2\tan\theta}{1 + \tan\theta}\right) \\
&= 8
\end{aligned}$$

Thus, the perimeter is always 8, regardless of the value of θ, so the maximum possible perimeter is 8.

10. Throughout this problem, we represent the states of the n plates as a string of 0's and 1's (called a *binary string*) of length n of the form $p_1p_2\cdots p_n$, with the rth digit from the left (namely p_r) equal to 1 if plate r contains a gift and equal to 0 if plate r does not. We call a binary string of length n *allowable* if it satisfies the requirements – that is, if no two adjacent digits both equal 1. Note that digit p_n is also "adjacent" to digit p_1, so we cannot have $p_1 = p_n = 1$.

(a) Suppose that $p_1 = 1$.
Then $p_2 = p_7 = 0$, so the string is of the form $10p_3p_4p_5p_60$.
Since $k = 3$, then 2 of p_3, p_4, p_5, p_6 equal 1, but in such a way that no two adjacent digits are both 1.
The possible strings in this case are 1010100, 1010010 and 1001010.

Suppose that $p_1 = 0$. Then p_2 can equal 1 or 0.
If $p_2 = 1$, then $p_3 = 0$ as well. This means that the string is of the form $010p_4p_5p_6p_7$, which is the same as the general string in the first case, but shifted by 1 position around the circle, so there are again 3 possibilities.
If $p_2 = 0$, then the string is of the form $00p_3p_4p_5p_6p_7$ and 3 of the digits p_3, p_4, p_5, p_6, p_7 equal 1 in such a way that no 2 adjacent digits equal 1.
There is only 1 way in which this can happen: 0010101.
Overall, this gives 7 possible configurations, so $f(7, 3) = 7$.

(b) *Solution 1*
An allowable string $p_1p_2\cdots p_{n-1}p_n$ has $(p_1, p_n) = (1, 0)$, $(0, 1)$, or $(0, 0)$.
Define $g(n, k, 1, 0)$ to be the number of allowable strings of length n, containing k 1's, and with $(p_1, p_n) = (1, 0)$.
We define $g(n, k, 0, 1)$ and $g(n, k, 0, 0)$ in a similar manner.
Note that $f(n, k) = g(n, k, 1, 0) + g(n, k, 0, 1) + g(n, k, 0, 0)$.

Consider the strings counted by $g(n, k, 0, 1)$.
Since $p_n = 1$, then $p_{n-1} = 0$. Since $p_1 = 0$, then p_2 can equal 0 or 1.
We remove the first and last digits of these strings.
We obtain strings $p_2p_3\cdots p_{n-2}p_{n-1}$ that is strings of length $n-2$ containing $k-1$ 1's.
Since $p_{n-1} = 0$, then the first and last digits of these strings are not both 1. Also, since the original strings did not contain two consecutive 1's, then these new strings does not either.
Therefore, $p_2p_3\cdots p_{n-2}p_{n-1}$ are allowable strings of length $n-2$ containing $k-1$ 1's, with $p_{n-1} = 0$ and $p_2 = 1$ or $p_2 = 0$.
The number of such strings with $p_2 = 1$ and $p_{n-1} = 0$ is $g(n-2, k-1, 1, 0)$ and the number of such strings with $p_2 = 0$ and $p_{n-1} = 0$ is $g(n-2, k-1, 0, 0)$.
Thus, $g(n, k, 0, 1) = g(n-2, k-1, 1, 0) + g(n-2, k-1, 0, 0)$.

Consider the strings counted by $g(n, k, 0, 0)$.
Since $p_1 = 0$ and $p_n = 0$, then we can remove p_n to obtain strings $p_1p_2\cdots p_{n-1}$ of length $n-1$ containing k 1's. These strings are allowable since $p_1 = 0$ and the original strings were allowable.
Note that we have $p_1 = 0$ and p_{n-1} is either 0 or 1.
So the strings $p_1p_2\cdots p_{n-1}$ are allowable strings of length $n-1$ containing k 1's, starting with 0, and ending with 0 or 1.
The number of such strings with $p_1 = 0$ and $p_{n-1} = 0$ is $g(n-1, k, 0, 0)$ and the number of such strings with $p_1 = 0$ and $p_{n-1} = 1$ is $g(n-1, k, 0, 1)$.
Thus, $g(n, k, 0, 0) = g(n-1, k, 0, 0) + g(n-1, k, 0, 1)$.

Consider the strings counted by $g(n,k,1,0)$.
Here, $p_1 = 1$ and $p_n = 0$. Thus, p_{n-1} can equal 0 or 1. We consider these two sets separately.
If $p_{n-1} = 0$, then the string $p_1p_2\cdots p_{n-1}$ is an allowable string of length $n-1$, containing k 1's, beginning with 1 and ending with 0.
Therefore, the number of strings counted by $g(n,k,1,0)$ with $p_{n-1} = 0$ is equal to $g(n-1,k,1,0)$.
If $p_{n-1} = 1$, then the string $p_2p_3\cdots p_{n-1}$ is of length $n-2$, begins with 0 and ends with 1. Also, it contains $k-1$ 1's (having removed the original leading 1) and is allowable since the original string was.
Therefore, the number of strings counted by $g(n,k,1,0)$ with $p_{n-1} = 1$ is equal to $g(n-2,k-1,0,1)$.

Therefore,

$$\begin{aligned}
f(n,k) &= g(n,k,1,0) + g(n,k,0,1) + g(n,k,0,0) \\
&= (g(n-1,k,1,0) + g(n-2,k-1,0,1)) \\
&\quad +(g(n-2,k-1,1,0) + g(n-2,k-1,0,0)) \\
&\quad +(g(n-1,k,0,0) + g(n-1,k,0,1)) \\
&= (g(n-1,k,1,0) + g(n-1,k,0,1) + g(n-1,k,0,0)) \\
&\quad +(g(n-2,k-1,0,1) + g(n-2,k-1,1,0) + g(n-2,k-1,0,0)) \\
&= f(n-1,k) + f(n-2,k-1)
\end{aligned}$$

as required.

Solution 2
We develop an explicit formula for $f(n,k)$ by building these strings.
Consider the allowable strings of length n that include k 1's. Either $p_n = 0$ or $p_n = 1$.

Consider first the case when $p_n = 0$. (Here, p_1 can equal 0 or 1.)
These strings are all of the form $p_1p_2p_3\cdots p_{n-1}0$.
In this case, since a 1 is always followed by a 0 and the strings end with 0, we can build these strings using blocks of the form 10 and 0. Any combination of these blocks will be an allowable string, as each 1 will always be both preceded and followed by a 0.
Thus, these strings can all be built using k 10 blocks and $n-2k$ 0 blocks. This gives k 1's and $k + (n-2k) = n-k$ 0's. Note that any string built with these blocks will be allowable and will end with a 0, and any such allowable string can be built in this way.
The number of ways of arranging k blocks of one kind and $n-2k$ blocks of another kind is $\binom{k+(n-2k)}{k}$, which simplifies to $\binom{n-k}{k}$.
Consider next the case when $p_n = 1$.
Here, we must have $p_{n-1} = p_1 = 0$, since these are the two digits adjacent to p_n.
Thus, these strings are all of the form $0p_2p_3\cdots 01$.
Consider the strings formed by removing the first and last digits.
These strings are allowable, are of length $n-2$, include $k-1$ 1's, end with 0, and can begin with 0 or 1.
Again, since a 1 is always followed by a 0 and the strings end with 0, we can build these strings using blocks of the form 10 and 0. Any combination of these blocks will be an allowable string, as each 1 will always be both preceded and followed by a 0.
Translating our method of counting from the first case, there are $\binom{(n-2)-(k-1)}{k-1}$ or

$\binom{n-k-1}{k-1}$ such strings.

Thus, $f(n,k) = \binom{n-k}{k} + \binom{n-k-1}{k-1}$ such strings.

To prove the desired fact, we will use the fact that $\binom{m}{r} = \binom{m-1}{r} + \binom{m-1}{r-1}$, which we prove at the end.

Now

$$\begin{aligned}
&f(n-1,k) + f(n-2,k-1) \\
&= \binom{(n-1)-k}{k} + \binom{(n-1)-k-1}{k-1} + \binom{(n-2)-(k-1)}{k-1} + \binom{(n-2)-(k-1)-1}{(k-1)-1} \\
&= \binom{n-k-1}{k} + \binom{n-k-2}{k-1} + \binom{n-k-1}{k-1} + \binom{n-k-2}{k-2} \\
&= \binom{n-k-1}{k} + \binom{n-k-1}{k-1} + \binom{n-k-2}{k-1} + \binom{n-k-2}{k-2} \\
&= \binom{n-k}{k} + \binom{n-k-1}{k-1} \qquad \text{(using the identity above)} \\
&= f(n,k)
\end{aligned}$$

as required.

To prove the identity, we expand the terms on the right side:

$$\begin{aligned}
\binom{m-1}{r} + \binom{m-1}{r-1} &= \frac{(m-1)!}{r!(m-r-1)!} + \frac{(m-1)!}{(r-1)!(m-r)!} \\
&= \frac{(m-1)!(m-r)}{r!(m-r-1)!(m-r)} + \frac{r(m-1)!}{r(r-1)!(m-r)!} \\
&= \frac{(m-1)!(m-r)}{r!(m-r)!} + \frac{r(m-1)!}{r!(m-r)!} \\
&= \frac{(m-1)!(m-r+r)}{r!(m-r)!} \\
&= \frac{(m-1)!m}{r!(m-r)!} \\
&= \frac{m!}{r!(m-r)!} \\
&= \binom{m}{r}
\end{aligned}$$

as required.

(c) We use the formula for $f(n,k)$ developed in Solution 2 to (b). In order to look at divisibility, we need to first simplify the formula:

$$
\begin{aligned}
f(n,k) &= \binom{n-k}{k} + \binom{n-k-1}{k-1} \\
&= \frac{(n-k)!}{k!(n-k-k)!} + \frac{(n-k-1)!}{(k-1)!((n-k-1)-(k-1))!} \\
&= \frac{(n-k)!}{k!(n-2k)!} + \frac{(n-k-1)!}{(k-1)!(n-2k)!} \\
&= \frac{(n-k-1)!(n-k)}{k!(n-2k)!} + \frac{(n-k-1)!k}{k!(n-2k)!} \\
&= \frac{(n-k-1)!(n-k+k)}{k!(n-2k)!} \\
&= \frac{n(n-k-1)!}{k!(n-2k)!} \\
&= \frac{n(n-k-1)(n-k-2)\cdots(n-2k+2)(n-2k+1)}{k!}
\end{aligned}
$$

Now that we have written $f(n,k)$ as a product, it is significantly easier to look at divisibility.

Note that $2009 = 41 \times 49 = 7^2 \times 41$, so we need $f(n,k)$ to be divisible by 41 and by 7 twice. For this to be the case, the numerator of $f(n,k)$ must have at least one more factor of 41 and at least two more factors of 7 than the denominator.

Also, we want to minimize $n+k$, so we work to keep n and k as small as possible.

If $n = 49$ and $k = 5$, then

$$f(49,5) = \frac{49(43)(42)(41)(40)}{5!} = \frac{49(43)(42)(41)(40)}{5(4)(3)(2)(1)} = 49(43)(14)(41)$$

which is divisible by 2009.

We show that this pair minimizes the value of $n+k$ with a value of 54.

We consider the possible cases by looking separately at the factors of 41 and 7 that must occur. We focus on the factor of 41 first.

For the numerator to contain a factor of 41, either n is divisible by 41 or one of the terms in the product $(n-k-1)(n-k-2)\cdots(n-2k+1)$ is divisible by 41.

Case 1: n is divisible by 41

We already know that $n = 82$ is too large, so we consider $n = 41$. From the original interpretation of $f(n,k)$, we see that $k \leq 20$, as there can be no more than 20 gifts placed on 41 plates.

Here, the numerator becomes 41 times the product of $k-1$ consecutive integers, the largest of which is $40-k$.

Now the numerator must also contain at least two factors of 7 more than the denominator. But the denominator is the product of k consecutive integers. Since the numerator contains the product of $k-1$ consecutive integers and the denominator contains the product of k consecutive integers, then the denominator will always include at least as many multiples of 7 as the numerator (since there are more consecutive integers in the product in the denominator). Thus, it is impossible for the numerator to contain even one more

additional factor of 7 than the denominator.
Therefore, if $n = 41$, then $f(n, k)$ cannot be divisible by 2009.

Case 2: n is not divisible by 41
This means that the factor of 41 in the numerator must occur in the product

$$(n-k-1)(n-k-2)\cdots(n-2k+1)$$

In this case, the integer 41 must occur in this product, since an occurrence of 82 would make n greater than 82, which does not minimize $n+k$.
So we try to find values of n and k that include the integer 41 in this list.
Note that $n-k-1$ is the largest factor in the product and $n-2k+1$ is the smallest.
Since 41 is contained somewhere in the product, then $n-2k+1 \le 41$ (giving $n \le 40+2k$) and $41 \le n-k-1$ (giving $n \ge 42+k$).
Combining these restrictions, we get $42+k \le n \le 40+2k$.

Now, we focus on the factors of 7.
Either n is not divisible by 7 or n is divisible by 7.

* If n is not divisible by 7, then at least two factors of 7 must be included in the product

  $$(n-k-1)(n-k-2)\cdots(n-2k+1)$$

  which means that either $k \ge 8$ (to give two multiples of 7 in this list of $k-1$ consecutive integers) or one of the factors is divisible by 49.
  - If $k \ge 8$, then $n \ge 42+k \ge 50$ so $n+k \ge 58$, which is not minimal.
  - If one of the factors is a multiple of 49, then 49 must be included in the list so $n-2k+1 \le 49$ (giving $n \le 48+2k$) and $49 \le n-k-1$ (giving $n \ge 50+k$).
    In this case, we already know that $42+k \le n \le 40+2k$ and now we also have $50+k \le n \le 48+2k$.
    For these ranges to overlap, we need $50+k \le 40+2k$ and so $k \ge 10$, which means that $n \ge 50+k \ge 60$, and so $n+k \ge 70$, which is not minimal.
* Next, we consider the case where n is a multiple of 7.
  Here, $42+k \le n \le 40+2k$ (to include 41 in the product) and n is a multiple of 7.
  Since k is at least 2 by definition, then $n \ge 42+k \ge 44$, so n is at least 49.
  If n was 56 or more, we do not get a minimal value for $n+k$.
  Thus, we need to have $n = 49$. In this case, we do not need to look for another factor of 7 in the list.
  To complete this case, we need to find the smallest value of k for which 49 is in the range from $42+k$ to $40+2k$ because we need to have $42+k \le n \le 40+2k$.
  This value of k is $k = 5$, which gives $n+k = 49+5 = 54$.

Since $f(49, 5)$ is divisible by 2009, as determined above, then this is the case that minimizes $n+k$, giving a value of 54.