Canadian Mathematics Competition

An activity of the Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario

2009 Cayley Contest

(Grade 10)

Wednesday, February 18, 2009

Solutions

1. Calculating, $\dfrac{10^2 - 10}{9} = \dfrac{100 - 10}{9} = \dfrac{90}{9} = 10$.

   Answer: (A)

2. On Saturday, Deepit worked 6 hours. On Sunday, he worked 4 hours.
   Therefore, he worked $6 + 4 = 10$ hours in total on Saturday and Sunday.

   Answer: (E)

3. Since $3(-2) = \nabla + 2$, then $-6 = \nabla + 2$ so $\nabla = -6 - 2 = -8$.

   Answer: (C)

4. Since $\sqrt{5+n} = 7$ and $7 = \sqrt{49}$, then $5 + n = 49$, so $n = 44$.

   Answer: (D)

5. *Solution 1*
   Calculating, $3^2 + 4^2 + 12^2 = 9 + 16 + 144 = 25 + 144 = 169 = 13^2$.

   Solution 2
   From our work with the Pythagorean Theorem, we might remember that $3^2 + 4^2 = 5^2$.
   (This comes from the "3-4-5" right-angled triangle.)
   Thus, $3^2 + 4^2 + 12^2 = 5^2 + 12^2$.
   We might also remember, from the "5-12-13" triangle that $5^2 + 12^2 = 13^2$.
   Therefore, $3^2 + 4^2 + 12^2 = 5^2 + 12^2 = 13^2$.

   Answer: (A)

6. Since the shaded area is 20% of the area of the circle, then the central angle should be 20% of the total possible central angle.
   Thus, $x^\circ = \frac{20}{100}(360^\circ)$ or $x = \frac{1}{5}(360) = 72$.

   Answer: (D)

7. Since the sum of the angles in any triangle is 180°, then looking at $\triangle QSR$, we have

   $$\angle SQR = 180^\circ - \angle QSR - \angle SRQ = 180^\circ - 90^\circ - 65^\circ = 25^\circ$$

   Since $PQ = PR$, then $\angle PQR = \angle PRQ$.
   Thus, $x^\circ + 25^\circ = 65^\circ$ or $x + 25 = 65$, and so $x = 40$.

   Answer: (E)

8. According to the problem, we want to find the choice that works no matter what three consecutive positive integers we choose.
   Thus, we try $1(2)(3)$ to see if we can eliminate any of the given choices.
   Now, $1(2)(3) = 6$. 6 is not a multiple of 4, 5 or 12, and is not odd.
   Therefore, only "a multiple of 6" is possible.
   (In fact, the product of three consecutive positive integers will always be a multiple of 6, because at least one of the three integers is even and one of the three must be a multiple of 3.)

   Answer: (B)

9. *Solution 1*
   Since there are 24 hours in a day, Francis spends $\frac{1}{3} \times 24 = 8$ hours sleeping.
   Also, he spends $\frac{1}{4} \times 24 = 6$ hours studying, and $\frac{1}{8} \times 24 = 3$ hours eating.
   The number of hours that he has left is $24 - 8 - 6 - 3 = 7$ hours.

   Solution 2
   Francis spends $\frac{1}{3} + \frac{1}{4} + \frac{1}{8} = \frac{8+6+3}{24} = \frac{17}{24}$ either sleeping, studying or eating.
   This leaves him $1 - \frac{17}{24} = \frac{7}{24}$ of his day.
   Since there are 24 hours in a full day, then he has 7 hours left.

   Answer: (D)

10. *Solution 1*
Suppose that the height is h cm, the width is w cm, and the depth is d cm.
Using the given areas of the faces, $wh = 12$, $dw = 8$ and $dh = 6$.
Multiplying these together, we get $whdwdh = 12(8)(6)$ or $d^2h^2w^2 = 576$.
Thus, $(dhw)^2 = 576$, so $dhw = \sqrt{576} = 24$, since $dhw > 0$.
The volume is dhw cm^3, so the volume is 24 cm^3.

Solution 2
We try to find dimensions that work to give the three face areas that we know.
After some trial and error, we see that if the width is 4 cm, the height is 3 cm, and the depth is 2 cm, then the given face areas are correct.
Thus, the volume is 4 cm $\times$ 3 cm $\times$ 2 cm = 24 cm^3.

Solution 3
Suppose that the height is h cm, the width is w cm, and the depth is d cm.
Using the given areas of the faces, $wh = 12$, $dw = 8$ and $dh = 6$.
From the first equation, $w = \dfrac{12}{h}$.
Substituting into the second equation, we obtain $\dfrac{12d}{h} = 8$.
Multiplying this equation by the equation $dh = 6$, we obtain $12d^2 = 48$ which gives $d^2 = 4$ and so $d = 2$, since $d > 0$.
Since $dh = 6$, then $2h = 6$ so $h = 3$.
Since $dw = 8$, then $2w = 8$ so $w = 4$.
Thus, the volume is 4 cm $\times$ 3 cm $\times$ 2 cm = 24 cm^3.

ANSWER: (A)

11. To maximize the number of songs used, Gillian should use as many of the shortest length songs as possible. (This is because she can always trade a longer song for a shorter song and shorten the total time used.)
If Gillian uses all 50 songs of 3 minutes in length, this takes 150 minutes.
There are $180 - 150 = 30$ minutes left, so she can play an additional $30 \div 5 = 6$ songs that are 5 minutes in length.
In total, she plays $50 + 6 = 56$ songs.

ANSWER: (C)

12. Since there are 6 columns and each term in the sequence is 3 greater than the previous term, then the number in each box in the final column is 18 more than the number above it. (Each number in the final column is 6 terms further along in the sequence than the number above it.)
Thus, the number in the bottom right box should be $17 + 5(18) = 17 + 90 = 107$, since it is 5 rows below the 17.

ANSWER: (C)

13. The coin starts on square number 8, counting from left to right.
After the first roll, it moves 1 square to the left, since 1 is odd.
After the second roll, it moves 2 squares to the right, since 2 is even.
The coin continues to move, ending on square $8-1+2-3+4-5+6 = 11$, and so is 3 squares to the right of where it started.

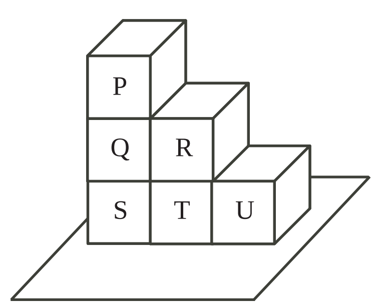
ANSWER: (E)

14. Of the integers from 3 to 20, the numbers 3, 5, 7, 11, 13, 17, and 19 are prime.
Thus, the integers 4, 6, 8, 9, 10, 12, 14, 15, 16, 18, and 20 are composite.
The sum of three different composite numbers has to be at least $4 + 6 + 8 = 18$.
Thus, the smallest prime number that could be the sum of three different composite numbers is 19.
Can we write 19 as the sum of three different composite numbers? Yes, because $19 = 4+6+9$.
ANSWER: (D)

15. We write the list of five numbers in increasing order.
We know that the number 8 occurs at least twice in the list.
Since the median of the list is 9, then the middle number (that is, the third number) in the list is 9.
Thus, the list can be written as a, b, 9, d, e.
Since 8 occurs more than once and the middle number is 9, then 8 must occur twice only with $a = b = 8$.
Thus, the list can be written as 8, 8, 9, d, e.
Since the average is 10 and there are 5 numbers in the list, then the sum of the numbers in the list is $5(10) = 50$.
Therefore, $8 + 8 + 9 + d + e = 50$ or $25 + d + e = 50$ or $d + e = 25$.
Since 8 is the only integer that occurs more than once in the list, then $d > 9$.
Thus, $10 \leq d < e$ and $d + e = 25$.
To make e as large as possible, we make d as small as possible, so we make $d = 10$, and so $e = 15$.
The list 8, 8, 9, 10, 15 has the desired properties, so the largest possible integer that could appear in the list is 15.
ANSWER: (A)

16. Suppose that the side length of each of the small squares is x. (Since the 4 small squares have the same height, they must have the same side length.)
Then the side length of the largest square is $4x$.
Since the side length of each of the shaded squares is 10, then $QR = 3(10) = 30 = PS$.
Thus, $PS = 30 = 4x + x$ or $5x = 30$ or $x = 6$.
Therefore, the side length of the largest square is $4x = 24$.
ANSWER: (B)

17. Label the six dice as shown:



The maximum overall exposed sum occurs when the sum of the exposed faces on each die is maximized.
Die P has 5 exposed faces. The sum of these faces is a maximum when the 1 is hidden, so the

maximum exposed sum on die P is $2 + 3 + 4 + 5 + 6 = 20$.
Dice Q and S each have 3 exposed faces. Two of these are opposite to each other, so have a sum of 7. Thus, to maximize the exposed sum on these dice, we position them with the 6 as the unpaired exposed face. (This is on the left face of the stack.) Each of these dice has a maximum exposed sum of $6 + 7 = 13$.
Dice R and U each have 4 exposed faces. Two of these are opposite to each other, so have a sum of 7. Thus, to maximize the exposed sum on these dice, we position them with the 6 and the 5 as the unpaired exposed faces (on the top and right of the stack). Each of these dice have a maximum exposed sum of $5 + 6 + 7 = 18$.
Die T has 2 exposed faces, which are opposite each other, so have a sum of 7.
Therefore, the maximum possible sum of the exposed faces is $20 + 13 + 13 + 18 + 18 + 7 = 89$.

ANSWER: (C)

18. The slope of line segment QP is 1. Since the "rise" of QP is 6 units, then the "run" of QP should also be 6 units. Therefore, Q is 6 units horizontally to the left of P, and so has coordinates $(-5, 0)$.
The slope of line segment RP is 2. Since the rise of RP is 6 units, then the run of RP is $\frac{1}{2}(6) = 3$ units. Therefore, R is 3 units horizontally to the left of P, and so has coordinates $(-2, 0)$.
(We could have used the coordinates of P and the slopes of the lines to find that the equations of the lines are $y = x + 5$ and $y = 2x + 4$ and used them to find the coordinates of Q and R.)
Therefore, $QR = -2 - (-5) = 3$ and P is 6 units above the x-axis.
Thus, treating QR as the base of $\triangle PQR$, we find that its area is $\frac{1}{2}(3)(6) = 9$.

ANSWER: (B)

19. *Solution 1*
The product of the digits of n is 0 if at least one of the digits of n equals 0.
Consider the integers from 5000 to 5999, inclusive, which are each of the form $5xyz$.
How many of these *do not* include a 0?
If $5xyz$ does not include a 0, there are 9 possibilities for each of x, y and z (namely 1 to 9), and so there are $9^3 = 729$ such integers.
Therefore, $1000 - 729 = 271$ of the integers from 5000 to 5999 actually do include a 0.
We must also consider 6000, which includes a 0.
Therefore, there are $271 + 1 = 272$ integers n with $5000 \leq n \leq 6000$ with the property that the product of the digits of n is 0.

Solution 2
The product of the digits of n is 0 if at least one of the digits of n equals 0.
We carefully count the number of integers with 0 in each position, being careful not to "double count" any integers.
The integer $n = 6000$ includes a 0, so contributes 1 integer to the total.
There are 100 integers of the form $50xy$ (namely, 5000 to 5099).
There are 10 integers of the form $510y$ (namely, 5100 to 5109). Similarly, there are 10 integers of each of the forms $520y$, $530y$, and so on to $590y$. This gives 9 sets of 10 integers, or 90 integers more.
There are 9 integers of the form $51x0$ where x is not 0 (namely, 5110, 5120 and so on up to 5190). Similarly, there are 9 integers of each of the forms $52x0$, $53x0$, and so on. This gives 9 sets of 9 integers, or 81 integers more.
In total, there are thus $1 + 100 + 90 + 81 = 272$ such integers.

ANSWER: (D)

20. Let the length of his route be d km.
Since he arrives 1 minute early when travelling at 75 km/h and 1 minute late when travelling at 70 km/h, then the difference between these times is 2 minutes, or $\frac{1}{30}$ of an hour.
The time that his trip takes while travelling at 75 km/h is $\frac{d}{75}$ hours, and at 70 km/h is $\frac{d}{70}$ hours.
Therefore,

$$\begin{aligned} \frac{d}{70} - \frac{d}{75} &= \frac{1}{30} \\ 75d - 70d &= \frac{75(70)}{30} \\ 5d &= 25(7) \\ d &= 35 \end{aligned}$$

Therefore, the route is 35 km long.

Answer: (B)

21. We count the lattice points by starting with the point closest to PS and proceeding to just below QR.
For a lattice point (a, b) on the line $y = 3x - 5$ to be above PS, we need $b \geq 0$ or $3a - 5 \geq 0$ or $3a \geq 5$ or $a \geq \frac{5}{3}$.
Since a is an integer, then $a \geq 2$. ($a = 2$ gives the point $(2, 1)$.)
For a lattice point (a, b) on the line $y = 3x-5$ to be below QR, we need $b \leq 2009$ or $3a-5 \leq 2009$ or $3a \leq 2014$ or $a \leq \frac{2014}{3}$.
Since a is an integer, then $a \leq 671$. ($a = 671$ gives the point $(671, 2008)$.)
Thus, for a lattice point (a, b) to be on the line and inside the square, it must have $2 \leq a \leq 671$.
In fact, every such integer a gives a lattice point, because it gives an integer value of $b = 2a - 5$.
The number of such integers a is $671 - 1 = 670$, so there are 670 such lattice points.

Answer: (E)

22. *Solution 1*
If we add the second and third equations, we obtain

$$\begin{aligned} ac + b + bc + a &= 18 + 6 \\ c(a + b) + (a + b) &= 24 \\ (c + 1)(a + b) &= 24 \end{aligned}$$

From the first given equation, $a + b = 3$ and so we get $(c + 1)(3) = 24$ or $c + 1 = 8$.
Thus, $c = 7$.

Solution 2
From the first equation, $b = 3 - a$. (We could solve for a in terms of b instead.)
The second equation becomes $ac + (3 - a) = 18$ or $ac - a = 15$.
The third equation becomes $(3 - a)c + a = 6$ or $-ac + 3c + a = 6$.
Adding these two equations, we obtain $3c = 15 + 6 = 21$ and so $c = 7$.

Answer: (E)

23. *Solution 1*
Suppose that Angela and Barry share 100 hectares of land. (We may assume any convenient total area, since this is a multiple choice problem.)

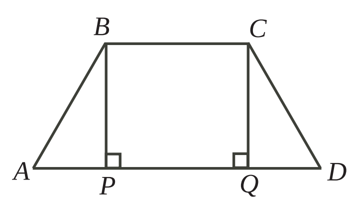
Since the ratio of the area of Angela's land to the area of Barry's land is $3:2$, then Angela has $\frac{3}{5}$ of the 100 hectares, or 60 hectares. Barry has the remaining 40 hectares.
Since the land is covered by corn and peas in the ratio $7:3$, then $\frac{7}{10}$ of the 100 hectares (that is, 70 hectares) is covered with corn and the remaining 30 hectares with peas.
On Angela's land, the ratio is $4:1$ so $\frac{4}{5}$ of her 60 hectares, or 48 hectares, is covered with corn and the remaining 12 hectares with peas.
Since there are 70 hectares of corn in total, then Barry has $70-48=22$ hectares of corn.
Since there are 30 hectares of peas in total, then Barry has $30-12=18$ hectares of peas.
Therefore, the ratio of corn to peas on Barry's land is $22:18=11:9$.

Solution 2
Suppose that the total combined area of land is x.
Since the ratio of the area of Angela's land to the area of Barry's land is $3:2$, then Angela has $\frac{3}{5}$ of the land, or $\frac{3}{5}x$, while Barry has the remaining $\frac{2}{5}x$.
Since the land is covered by corn and peas in the ratio $7:3$, then $\frac{7}{10}x$ is covered with corn and the remaining $\frac{3}{10}x$ with peas.
On Angela's land, the ratio is $4:1$ so $\frac{4}{5}$ of her $\frac{3}{5}x$, or $\frac{4}{5}(\frac{3}{5}x)=\frac{12}{25}x$, is covered with corn and the remaining $\frac{3}{5}x-\frac{12}{25}x=\frac{3}{25}x$ with peas.
Since the area of corn is $\frac{7}{10}x$ in total, then Barry's area of corn is $\frac{7}{10}x-\frac{12}{25}x=\frac{35}{50}x-\frac{24}{50}x=\frac{11}{50}x$.
Since the area of peas is $\frac{3}{10}x$ in total, then Barry's area of peas is $\frac{3}{10}x-\frac{3}{25}x=\frac{15}{50}x-\frac{6}{50}x=\frac{9}{50}x$.
Therefore, the ratio of corn to peas on Barry's land is $\frac{11}{50}x:\frac{9}{50}x=11:9$.

ANSWER: (A)

24. We first note that the given quadrilateral is a trapezoid, because $60^\circ+120^\circ=180^\circ$, and so the top and bottom sides are parallel.
We need to determine the total area of the trapezoid and then what fraction of that area is closest to the longest side.

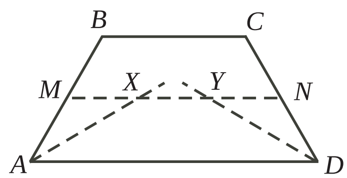
<u>Area of trapezoid</u>
Label the trapezoid as $ABCD$ and drop perpendiculars from B and C to P and Q on AD.



Since $\triangle ABP$ is right-angled at P and $\angle BAP=60^\circ$, then $AP=100\cos(60^\circ)=100(\frac{1}{2})=50$ m and $BP=100\sin(60^\circ)=100(\frac{\sqrt{3}}{2})=50\sqrt{3}$ m. (We could have used the ratios in a 30°-60°-90° triangle to do these calculations.)
By symmetry, $QD=50$ m as well.
Also, since BC is parallel to PQ, and BP and CQ are perpendicular to PQ, then $BPQC$ is a rectangle, so $PQ=BC=100$ m.
Thus, the area of trapezoid $ABCD$ is $\frac{1}{2}(BC+AD)(BP)=\frac{1}{2}(100+(50+100+50))(50\sqrt{3})$ or $7500\sqrt{3}$ square metres.

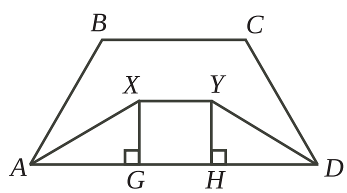
<u>Determination of region closest to AD</u>
Next, we need to determine what region of the trapezoid is closest to side AD.

To be closest to side AD, a point inside the trapezoid must be closer to AD than to each of BC, AB, and DC.
For a point in the trapezoid to be closer to AD than to BC, it must be below the "half-way mark", which is the midsegment MN.
Thus, such a point must be below the parallel line that is $\frac{1}{2}(50\sqrt{3}) = 25\sqrt{3}$ m above AD.
For a point in the trapezoid to be closer to AD than to AB, it must be below the angle bisector of $\angle BAD$. (See the end of this solution for a justification of this.)
Similarly, for a point in the trapezoid to be closer to AD than to DC, it must be below the angle bisector of $\angle CDA$.
Define points X and Y to be the points of intersection between the angle bisectors of $\angle BAD$ and $\angle CDA$, respectively, with the midsegment MN. We will confirm later in the solution that the angle bisectors intersect above MN, not below MN.



Area of trapezoid $AXYD$

Lastly, we need to determine the area of trapezoid $AXYD$.
Note that $\angle XAD = \angle YDA = \frac{1}{2}(60^\circ) = 30^\circ$.
Drop perpendiculars from X and Y to G and H, respectively, on AD.



We know that $AD = 200$ m and $XG = YH = 25\sqrt{3}$ m.
Since each of $\triangle AXG$ and $\triangle DYH$ is a 30°-60°-90° triangle, then

$$AG = DH = \sqrt{3}XG = \sqrt{3}(25\sqrt{3}) = 75$$

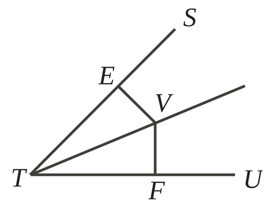
This tells us that the angle bisectors must intersect above MN, since $AG + HD = 150$ and $AD = 200$, so $AG + HD < AD$.
Since $XGHY$ is a rectangle (by similar reasoning as for $BPQC$), then

$$XY = GH = AD - AG - DH = 200 - 75 - 75 = 50$$

Therefore, the area of trapezoid $AXYD$ is $\frac{1}{2}(AD + XY)(XG) = \frac{1}{2}(50 + 200)(25\sqrt{3})$ or $3125\sqrt{3}$ square metres.
This tells us that the fraction of the crop that is brought to AD is $\frac{3125\sqrt{3}}{7500\sqrt{3}} = \frac{25}{60} = \frac{5}{12}$.

Property of angle bisectors

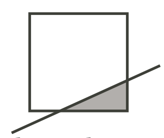
We must still justify the fact about angle bisectors above.
Consider $\angle STU$ and point V on its angle bisector.
Drop perpendiculars from V to E and F on ST and UT, respectively.



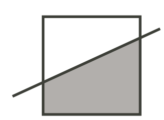
Now $\triangle VET$ is congruent to $\triangle VFT$ since each is right-angled, the two have equal angles at T, and the two share a common hypotenuse TV.
This tells us that $EV = FV$; that is, V is the same distance from ST and from UT.
This also tells us that any point on the angle bisector will be equidistant from ST and UT.
We can also deduce from this that any point below the angle bisector will be closer to UT, as moving from a point on the angle bisector to such a point moves closer to UT and further from ST.

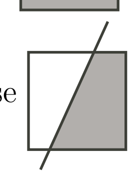
ANSWER: (B)

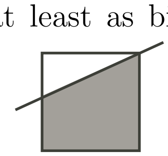
25. We add coordinates to the diagram, with the bottom left corner at $(0,0)$, the bottom right at $(m,0)$, the top right at (m,n), and the top left at $(0,n)$.
Thus, the slope of the diagonal is $\dfrac{n}{m}$.
This tells us that the equation of the diagonal is $y = \dfrac{n}{m}x$.
Since $2n < m < 3n$, then $\dfrac{1}{3} < \dfrac{n}{m} < \dfrac{1}{2}$; that is, the slope is between $\frac{1}{3}$ and $\frac{1}{2}$.
There are three possible configurations of shading in these partially shaded squares:

- A small triangle is shaded, while the rest is unshaded: 

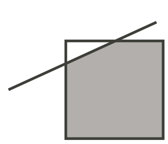
  Here, the maximum possible length of the base is 1 and so the maximum possible height is when the slope is as large as possible, so is $\frac{1}{2}$.
  Thus, in this case, the maximum shaded area is $\frac{1}{2}(1)(\frac{1}{2}) = \frac{1}{4}$.
  Since we want a shaded area of more than 0.999, then this is not the case we need.

- A trapezoid is shaded and a trapezoid is unshaded: 

  (Note that since the slope is less than 1, then the case  is not possible.)

  Consider the unshaded area. For the shaded area to be more than 0.999, the unshaded area is less than 0.001.
  But the unshaded trapezoid is at least as big as the triangle that is cut off when the diagonal passes through a vertex: 

  Such a triangle has base 1 and height at least $\frac{1}{3}$ (since the slope is at least $\frac{1}{3}$).
  Thus, the area of such a trapezoid is at least $\frac{1}{2}(1)(\frac{1}{3}) = \frac{1}{6}$ and so cannot be less than 0.001.

- A triangle is unshaded: 

It is this last case upon which we need to focus.
Suppose that the coordinates of the top left corner of such a unit square are (p,q).
The point where the diagonal ($y = \frac{n}{m}x$) crosses the top edge of the square ($y = q$) has coordinates $\left(\frac{m}{n}q, q\right)$, since if $y = q$, then $q = \frac{n}{m}x$ gives $x = \frac{m}{n}q$.
Similarly, the point where the diagonal crosses the left edge ($x = p$) of the square has coordinates $\left(p, \frac{n}{m}p\right)$.
Thus, the triangle has (horizontal) base of length $\frac{m}{n}q - p$ and (vertical) height of length $q - \frac{n}{m}p$.
We also know that neither the base nor the height is 0, since there is some unshaded area.
Since the area of the unshaded triangle is less than 0.001, then

$$\begin{array}{rcccll}
0 & < & \frac{1}{2}\left(\frac{m}{n}q - p\right)\left(q - \frac{n}{m}p\right) & < & 0.001 & \\
0 & < & \left(\frac{m}{n}q - p\right)\left(q - \frac{n}{m}p\right) & < & 0.002 & \\
0 & < & (mq - pn)(mq - pn) & < & 0.002mn & \text{(multiplying by } mn\text{)} \\
0 & < & 500(mq - pn)^2 & < & mn &
\end{array}$$

Now m, n, p and q are integers and $mq - pn$ is not zero. In fact, $mq - pn = n\left(\frac{m}{n}q - p\right) > 0$.
Thus, $(mq - pn)^2 \geq 1$ because m, q, p, n are all integers and $(mq - pn)^2 > 0$.
Thus, $mn > 500(1) = 500$.
Note that if $(mq - pn)^2 > 1$, then mn would be much bigger.
So, since we want the smallest value of mn, we try to see if we can find a solution with $(mq - pn)^2 = 1$.

So we need to try to find m and n with $2n < m < 3n$, with the product mn as close to 500 as possible, and so that we can also find p and q with $mq - pn = 1$.
We consider the restriction that $2n < m < 3n$ first and look at the integers from 501 to 510 to see if we can find m and n with a product equal to one of these numbers and with $2n < m < 3n$.

- $501 = 3(167)$ and 167 is prime, so this is not possible
- $502 = 2(251)$ and 251 is prime, so this is not possible
- 503 is prime, so this is not possible
- $504 = 8(7)(9)$ so we can choose $n = 14$ and $m = 36$ (this is the only such way)
- $505 = 5(101)$ and 101 is prime, so this is not possible
- $506 = 11(2)(23)$ which cannot be written in this way
- $507 = 3(13)(13)$ which cannot be written in this way
- $508 = 4(127)$ and 127 is prime, so this is not possible
- 509 which is prime, so this is not possible
- $510 = 2(3)(5)(17)$, so we can choose $n = 15$ and $m = 34$ (this is only such way)

This gives two possible pairs m and n to consider so far. If one of them works, then this pair will give the smallest possible value of mn.
In order to verify if one of these works, we do need to determine if we can find an appropriate p and q.
Consider $n = 14$ and $m = 36$. In this case, we want to find integers p and q with $36q - 14p = 1$.

This is not possible since the left side is even and the right side is odd.
Consider $n = 15$ and $m = 34$. In this case, we want to find integers p and q with $34q - 15p = 1$.
The integers $q = 4$ and $p = 9$ satisfy this equation.
Therefore, $(m, n) = (34, 15)$ is a pair with the smallest possible value of mn that satisfies the given conditions, and so $mn = 510$.

Answer: (C)