Canadian
Mathematics
Competition

An activity of the Centre for Education
in Mathematics and Computing,
University of Waterloo, Waterloo, Ontario

2008 Euclid Contest

Tuesday, April 15, 2008

Solutions

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1. (a) *Solution 1*

By the Pythagorean Theorem in $\triangle ADB$,

$$AB^2 = BD^2 + DA^2 = 9^2 + 12^2 = 81 + 144 = 225$$

so $AB = \sqrt{225} = 15$ since $AB > 0$.

By the Pythagorean Theorem in $\triangle ADC$,

$$DC^2 = CA^2 - AD^2 = 20^2 - 12^2 = 400 - 144 = 256$$

so $DC = \sqrt{256} = 16$ since $AD > 0$.

The perimeter of $\triangle ABC$ is

$$AB + BC + CA = AB + (BD + DC) + CA = 15 + (9 + 16) + 20 = 60$$

Solution 2

Since $BD : DA = 9 : 12 = 3 : 4$ and $\triangle BDA$ is right-angled at B, then $\triangle ADB$ is similar to a 3-4-5 triangle. Thus, $AB = \frac{5}{3}BD = 15$.

Since $CA : DA = 20 : 12 = 5 : 3$ and $\triangle ADC$ is right-angled at D, then $\triangle ADC$ is similar to a 3-4-5 triangle. Thus, $DC = \frac{4}{5}CA = 16$.

Therefore, the perimeter of $\triangle ABC$ is

$$AB + BC + CA = AB + (BD + DC) + CA = 15 + (9 + 16) + 20 = 60$$

(b) *Solution 1*

Since $P(5, 4)$ is the midpoint of $A(a, 0)$ and $B(8, b)$, then 5 is the average of the x-coordinates of A and B and 4 is the average of the y-coordinates of A and B.

Therefore, $5 = \frac{1}{2}(a + 8)$ so $10 = a + 8$ or $a = 2$.

Also, $4 = \frac{1}{2}(0 + b)$ so $8 = 0 + b$ or $b = 8$.

Thus, $a = 2$ and $b = 8$.

Solution 2

$P(5, 4)$ is the midpoint of $A(a, 0)$ and $B(8, b)$.

To get from A to P, we move 4 units up, so to get from P to B we move 4 units up. Therefore, the y-coordinate of B is $4 + 4 = 8$, so $b = 8$.

To get from P to B, we move 3 units to the right, so to get from P to A, we move 3 units to the left. Therefore, the x-coordinate of A is $5 - 3 = 2$, so $a = 2$.

Thus, $a = 2$ and $b = 8$.

(c) The line $ax + y = 30$ passes through the point $(6, 12)$, so $6a + 12 = 30$ or $6a = 18$ or $a = 3$.

This tells us that the line $x + 3y = k$ also passes through the point $(6, 12)$, so $6 + 3(12) = k$ or $k = 42$. Therefore, $k = 42$.

2. (a) *Solution 1*

Since $(c,7)$ lies on the parabola, then $7=(c-2)(c-8)+7$ so $(c-2)(c-8)=0$.
Thus, $c=2$ or $c=8$. Since $c\neq 2$, then $c=8$.

Solution 2

The parabola has equation $y=(x-2)(x-8)+7=x^2-10x+16+7=x^2-10x+23$.
Completing the square,

$$y=x^2-10x+25-25+23=(x-5)^2-2$$

so the axis of symmetry of the parabola is $x=5$.
Since $(2,7)$ lies on the parabola and is 3 units to the left of the axis of symmetry, then $(5+3,7)=(8,7)$ is also on the parabola.
Thus, $c=8$.

(b) *Solution 1*

Since $(2,7)$ and $(8,7)$ lie on the parabola, the axis of symmetry lies halfway between these two points, so has equation $x=5$.
The vertex lies on the axis of symmetry, so has x-coordinate equal to 5.
The y-coordinate of the vertex is thus $y=(5-2)(5-8)+7=-9+7=-2$.
Therefore, the vertex has coordinates $(5,-2)$.

Solution 2

The parabola has equation $y=(x-2)(x-8)+7=x^2-10x+16+7=x^2-10x+23$.
Completing the square,

$$y=x^2-10x+25-25+23=(x-5)^2-2$$

so the vertex has coordinates $(5,-2)$.

(c) Since the line passes through $A(5,0)$ and $B(4,-1)$, it has slope $m=\dfrac{0-(-1)}{5-4}=\dfrac{1}{1}=1$.
Thus, the equation of the line is $y=x+b$ for some number b.
Since $A(5,0)$ lies on the line, then $0=5+b$ so $b=-5$, so the line has equation $y=x-5$.
To find the intersection of the line with parabola, we use the two equations and equate:

$$\begin{aligned}(x-2)(x-8)+7 &= x-5\\ x^2-10x+16+7 &= x-5\\ x^2-11x+28 &= 0\\ (x-4)(x-7) &= 0\end{aligned}$$

Therefore, $x=4$ or $x=7$.
However, we already have the point where $x=4$, so we consider $x=7$.
We need to find the y-coordinate of this point. To do so, it is easier to use the equation

of the line than the equation of the parabola. Thus, $y = 7 - 5 = 2$.
Therefore, the other point is $(7, 2)$.

3. (a) *Solution 1*

Suppose that the middle number inside the square frame is x.
Then the other numbers in the middle row inside the frame are $x - 1$ and $x + 1$.
Also, the other numbers in the left column are $(x-1)-7 = x-8$ and $(x-1)+7 = x+6$, since there are 7 numbers in each row of the large grid. This tells us that the other numbers in the first and third rows are $x - 7$, $x - 6$, $x + 7$, and $x + 8$.
Therefore, the sum of the numbers in the frame is
$x + x - 1 + x + 1 + x - 8 + x - 7 + x - 6 + x + 6 + x + 7 + x + 8 = 9x$
Thus, the sum of numbers in the frame is 9 times the middle number.
(We can check this using the given example. We might also have been able to see initially that the average of the numbers inside the square frame is always the middle number.)
For the sum of the numbers inside the frame to be 279, the middle number in the frame must be $\frac{1}{9}(279) = 31$.
(We can check that this is correct by placing the frame in this position and adding.)

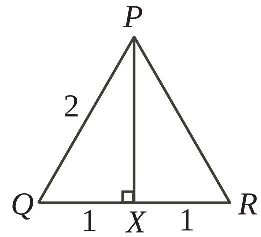
Solution 2

If we start the frame in a given position and slide it one column to the left, the sum of the numbers inside the frame decreases by 9 because the original first and second columns stay inside the frame and the original third column is replaced by a column 3 to the left. In this case, the three numbers entering the frame are each 3 less than the three numbers leaving the frame, so the overall sum decreases by $3 \times 3 = 9$.
Similarly, if the frame slides one column to the right, the sum inside the frame increases by 9.
Also, if the frame slides one row down, the sum inside the frame increases by 63, as the three numbers entering the frame are each 21 larger than the three numbers leaving the frame.
Similarly, if the frame slides one row up, the sum inside the frame decreases by 63.
The original sum inside the frame is 108. We want the new sum to be 279, so we want the sum to increase by $279 - 108 = 171$.
Now $171 = 3(63) - 2(9)$, so if we slide the frame 3 rows down and then 2 columns to the left, the sum will increase by 171 and become 279.
In this case, the new middle number is $12 + 3(7) - 2 = 31$.

(b) In Figure A, the circle has diameter of length 2, so has radius 1, and thus area πr^2 which is $\pi 1^2 = \pi \approx 3.14$.

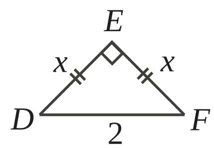
In Figure B, the figure has area equal to twice the area of an equilateral triangle of side length 2.

Consider a single equilateral triangle, $\triangle PQR$, with side length 2.
Drop a perpendicular from P to X on QR.
Since $\triangle PQR$ is equilateral, then $QX = XR = \frac{1}{2}QR = 1$.



Since $\angle PQR = 60^\circ$, then $\triangle PQX$ is 30°-60°-90° triangle, so $PX = \sqrt{3}QX = \sqrt{3}(1) = \sqrt{3}$.
Therefore, the area of $\triangle PQR$ is $\frac{1}{2}(2)(\sqrt{3}) = \sqrt{3}$.
Thus, the area of Figure B is $2\sqrt{3} \approx 3.46$.

In Figure C, the figure has area equal to twice the area of an isosceles right-angled triangle with hypotenuse of length 2.



Suppose that $DE = EF = x$.
Since the triangle is isosceles right-angled, then $x = \frac{1}{\sqrt{2}}(2) = \sqrt{2}$.
Thus, the area of this triangle is $\frac{1}{2}(\sqrt{2})(\sqrt{2}) = \frac{1}{2}(2) = 1$.
Thus, the area of Figure C is 2.

Since $2 < \pi < 2\sqrt{3}$ (because $2 < 3.14 < 3.46$), then the figure with the smallest area is Figure C and the figure with the largest area is Figure B.

4. (a) Since $PF = 20$ m and $\angle PAF = 40^\circ$, then $\dfrac{PF}{AF} = \tan(40^\circ)$ so $AF = \dfrac{20 \text{ m}}{\tan(40^\circ)}$.
Since B is halfway from A to F, then

$$BF = \tfrac{1}{2}AF = \frac{10 \text{ m}}{\tan(40^\circ)}$$

and so

$$\tan(\angle FBP) = \frac{PF}{BF} = \frac{20 \text{ m}}{\left(\frac{10 \text{ m}}{\tan(40^\circ)}\right)} = 2\tan(40^\circ) \approx 1.678$$

Thus, $\angle FBP \approx 59.21^\circ$, and so $\angle FBP$ is 59°, to the nearest degree.

(b) By the cosine law in $\triangle CBA$,

$$\begin{aligned} CA^2 &= CB^2 + BA^2 - 2(CB)(BA)\cos(\angle CBA) \\ CA^2 &= 16^2 + 21^2 - 2(16)(21)\cos(60^\circ) \\ CA^2 &= 256 + 441 - 2(16)(21)(\tfrac{1}{2}) \\ CA^2 &= 256 + 441 - (16)(21) \\ CA^2 &= 361 \\ CA &= \sqrt{361} = 19 \qquad (\text{since } CA > 0) \end{aligned}$$

In $\triangle CAD$, $\angle CDA = 180^\circ - \angle DCA - \angle DAC = 180^\circ - 45^\circ - 30^\circ = 105^\circ$.
By the sine law in $\triangle CDA$,

$$\begin{aligned} \frac{CD}{\sin(\angle DAC)} &= \frac{CA}{\sin(\angle CDA)} \\ CD &= \frac{19\sin(30^\circ)}{\sin(105^\circ)} \\ CD &= \frac{19(\frac{1}{2})}{\sin(105^\circ)} \\ CD &= \frac{19}{2\sin(105^\circ)} \\ CD &\approx 9.835 \end{aligned}$$

so, to the nearest tenth, CD equals 9.8.
(Note that we could have used

$$\begin{aligned} \sin(105^\circ) &= \sin(60^\circ + 45^\circ) = \sin(60^\circ)\cos(45^\circ) + \cos(60^\circ)\sin(45^\circ) \\ &= \frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{2}} + \frac{1}{2}\cdot\frac{1}{\sqrt{2}} = \frac{\sqrt{3}+1}{2\sqrt{2}} \end{aligned}$$

to say that $CD = \dfrac{19}{2\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)} = \dfrac{19\sqrt{2}}{\sqrt{3}+1}$ exactly, and then evaluated this expression.)

5. (a) Consider P on AB with CP perpendicular to AB. Note that $CP = 12$.
Since the small circle with centre A is tangent to the large circle with centre C, then AC equals the sum of the radii of these circles, or $AC = 4 + 9 = 13$. Similarly, $BC = 13$.
This tells us that $\triangle APC$ is congruent to $\triangle BPC$ (they have equal hypotenuses and each is right-angled and has a common side), so $BP = AP$.
By the Pythagorean Theorem in $\triangle APC$,

$$AP^2 = AC^2 - PC^2 = 13^2 - 12^2 = 169 - 144 = 25$$

so $AP = 5$ (since $AP > 0$).
Therefore, $BP = AP = 5$ and so $AB = 10$.
Since it takes the bug 5 seconds to walk this distance, then in 1 second, the bug walks a distance of $\frac{10}{5} = 2$.

(b) For the parabola to have its vertex on the x-axis, the equation

$$y = kx^2 + (5k+3)x + (6k+5) = 0$$

must have two equal real roots.
That is, its discriminant must equal 0, and so

$$\begin{aligned}(5k+3)^2 - 4k(6k+5) &= 0\\ 25k^2 + 30k + 9 - 24k^2 - 20k &= 0\\ k^2 + 10k + 9 &= 0\\ (k+1)(k+9) &= 0\end{aligned}$$

Therefore, $k = -1$ or $k = -9$.

6. (a) Since $f(x) = f(x-1) + f(x+1)$, then $f(x+1) = f(x) - f(x-1)$, and so

$$\begin{aligned}f(1) &= 1\\ f(2) &= 3\\ f(3) &= f(2) - f(1) = 3 - 1 = 2\\ f(4) &= f(3) - f(2) = 2 - 3 = -1\\ f(5) &= f(4) - f(3) = -1 - 2 = -3\\ f(6) &= f(5) - f(4) = -3 - (-1) = -2\\ f(7) &= f(6) - f(5) = -2 - (-3) = 1 = f(1)\\ f(8) &= f(7) - f(6) = 1 - (-2) = 3 = f(2)\end{aligned}$$

Since the value of f at an integer depends only on the values of f at the two previous integers, then the fact that the first several values form a cycle with $f(7) = f(1)$ and $f(8) = f(2)$ tells us that the values of f will always repeat in sets of 6.
Since 2008 is 4 more than a multiple of 6 (as $2008 = 4 + 2004 = 4 + 6(334)$), then $f(2008) = f(2008 - 6(334)) = f(4) = -1$.

(b) *Solution 1*
Since a, b, c form an arithmetic sequence, then we can write $a = b - d$ and $c = b + d$ for some real number d.
Since $a + b + c = 60$, then $(b - d) + b + (b + d) = 60$ or $3b = 60$ or $b = 20$.
Therefore, we can write a, b, c as $20 - d, 20, 20 + d$.
(We could have written a, b, c instead as $a, a + d, a + 2d$ and arrived at the same result.)
Thus, $a - 2 = 20 - d - 2 = 18 - d$ and $c + 3 = 20 + d + 3 = 23 + d$, so we can write $a - 2, b, c + 3$ as $18 - d, 20, 23 + d$.

Since these three numbers form a geometric sequence, then

$$\begin{aligned} \frac{20}{18-d} &= \frac{23+d}{20} \\ 20^2 &= (23+d)(18-d) \\ 400 &= -d^2-5d+414 \\ d^2+5d-14 &= 0 \\ (d+7)(d-2) &= 0 \end{aligned}$$

Therefore, $d=-7$ or $d=2$.
If $d=-7$, then $a=27$, $b=20$ and $c=13$.
If $d=2$, then $a=18$, $b=20$ and $c=22$.
(We can check that, in each case, $a-2,b,c+3$ is a geometric sequence.)

Solution 2
Since a, b, c form an arithmetic sequence, then $c-b=b-a$ or $a+c=2b$.
Since $a+b+c=60$, then $2b+b=60$ or $3b=60$ or $b=20$.
Thus, $a+c=40$, so $a=40-c$.
Therefore, we can write a,b,c as $40-c,20,c$.
Also, $a-2=40-c-2=38-c$, so we can write $a-2,b,c+3$ as $38-c,20,c+3$.
Since these three numbers form a geometric sequence, then

$$\begin{aligned} \frac{20}{38-c} &= \frac{c+3}{20} \\ 20^2 &= (38-c)(c+3) \\ 400 &= -c^2+35c+114 \\ c^2-35d+286 &= 0 \\ (c-13)(c-22) &= 0 \end{aligned}$$

Therefore, $c=13$ or $c=22$.
If $c=13$, then $a=27$, so $a=27$, $b=20$ and $c=13$.
If $c=22$, then $a=18$, so $a=18$, $b=20$ and $c=22$.
(We can check that, in each case, $a-2,b,c+3$ is a geometric sequence.)

7. (a) Since the average of three consecutive multiples of 3 is a, then a is the middle of these three integers, so the integers are $a-3,a,a+3$.
Since the average of four consecutive multiples of 4 is $a+27$, then $a+27$ is halfway in between the second and third of these multiples (which differ by 4), so the second and third of the multiples are $(a+27)-2=a+25$ and $(a+27)+2=a+29$, so the four integers are $a+21,a+25,a+29,a+33$.
(We have used in these two statements the fact that if a list contains an odd number of integers, then there is a middle integer in the list, and if the list contains an even number

of integers, then the "middle" integer is between two integers from the list.)

The smallest of these seven integers is $a-3$ and the largest is $a+33$.

The average of these two integers is $\frac{1}{2}(a-3+a+33)=\frac{1}{2}(2a+30)=a+15$.

Since $a+15=42$, then $a=27$.

(b) Suppose that Billy removes the ball numbered x from his bag and that Crystal removes the ball numbered y from her bag.

Then $b=1+2+3+4+5+6+7+8+9-x=45-x$.

Also, $c=1+2+3+4+5+6+7+8+9-y=45-y$.

Hence, $b-c=(45-x)-(45-y)=y-x$.

Since $1\leq x\leq 9$ and $1\leq y\leq 9$, then $-8\leq y-x\leq 8$.

(This is because $y-x$ is maximized when y is largest (that is, $y=9$) and x is smallest (that is, $x=1$), so $y-x\leq 9-1=8$. Similarly, $y-x\geq -8$.)

Since $b-c=y-x$ is between -8 and 8, then for it to be a multiple of 4, $b-c=y-x$ can be -8, -4, 0, 4, or 8.

Since each of Billy and Crystal chooses 1 ball from 9 balls and each ball is equally likely to be chosen, then the probability of any specific ball being chosen from one of their bags is $\frac{1}{9}$. Thus, the probability of any specific pair of balls being chosen (one from each bag) is $\frac{1}{9}\times\frac{1}{9}=\frac{1}{81}$.

Therefore, to compute the desired probability, we must count the number of pairs (x,y) where $y-x$ is -8, -4, 0, 4, 8, and multiply this result by $\frac{1}{81}$.

<u>Method 1</u>

If $y-x=-8$, then (x,y) must be $(9,1)$.

If $y-x=8$, then (x,y) must be $(1,9)$.

If $y-x=-4$, then (x,y) can be $(5,1),(6,2),(7,3),(8,4),(9,5)$.

If $y-x=4$, then (x,y) can be $(1,5),(2,6),(3,7),(4,8),(5,9)$.

If $y-x=0$, then (x,y) can be $(1,1),(2,2),(3,3),(4,4),(5,5),(6,6),(7,7),(8,8),(9,9)$.

There are thus 21 pairs (x,y) that work, so the desired probability is $\frac{21}{81}=\frac{7}{27}$.

<u>Method 2</u>

If $x=9$, then for $y-x$ to be a multiple of 4, y could be 9, 5 or 1.

If $x=8$, then for $y-x$ to be a multiple of 4, y could be 8 or 4.

If $x=7$, then for $y-x$ to be a multiple of 4, y could be 7 or 3.

If $x=6$, then for $y-x$ to be a multiple of 4, y could be 6 or 2.

If $x=5$, then for $y-x$ to be a multiple of 4, y could be 9, 5 or 1.

If $x=4$, then for $y-x$ to be a multiple of 4, y could be 8 or 4.

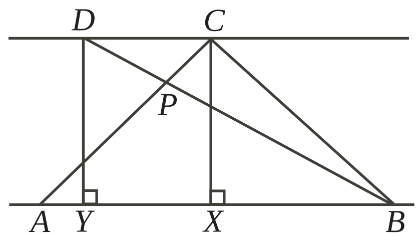
If $x=3$, then for $y-x$ to be a multiple of 4, y could be 7 or 3.

If $x=2$, then for $y-x$ to be a multiple of 4, y could be 6 or 2.

If $x=1$, then for $y-x$ to be a multiple of 4, y could be 9, 5 or 1.

There are thus 21 pairs (x, y) that work, so the desired probability is $\frac{21}{81} = \frac{7}{27}$.

8. (a) Since $AC = CB$, then $\triangle ACB$ is isosceles and right-angled, so $\angle CAB = \angle CBA = 45^\circ$.
Drop perpendiculars from C and D to X and Y on AB.



Since $\triangle ACB$ is isosceles, then $AX = XB = \frac{1}{2}AB = 1$.
Since $\angle CAX = 45^\circ$, then $\triangle AXC$ is isosceles and right-angled, so $CX = AX = 1$.
Since AB is parallel to DC, then $DY = CX = 1$.
Consider $\triangle BDY$. We know that $\angle DYB = 90^\circ$, $DB = 2$ and $DY = 1$.
This tells us that $\triangle BDY$ is a 30°-60°-90° triangle, as one of its legs and its hypotenuse are in the ratio $1 : 2$.
Therefore, $\angle DBY = 30^\circ$, and so $\angle DBC = \angle CBA - \angle DBY = 45^\circ - 30^\circ = 15^\circ$.

(b) *Solution 1*
Let $AP = x$ and $QP = h$.
Since QP is parallel to CB, then QP is perpendicular to BA.
Consider trapezoid $CBPQ$. We can think of this as having parallel bases of lengths 4 and h and height 5. Thus, its area is $\frac{1}{2}(4 + h)(5)$.
However, we can also compute its area by adding the areas of $\triangle CBR$ (which is $\frac{1}{2}(4)(3)$), $\triangle CRQ$ (which is given as 5), and $\triangle RPQ$ (which is $\frac{1}{2}(2)(h)$).
Thus,

$$\begin{aligned}
\tfrac{1}{2}(4 + h)(5) &= \tfrac{1}{2}(4)(3) + 5 + \tfrac{1}{2}(2)(h) \\
20 + 5h &= 12 + 10 + 2h \\
3h &= 2 \\
h &= \tfrac{2}{3}
\end{aligned}$$

Now, $\triangle APQ$ is similar to $\triangle ABC$, as each has a right angle and they share a common angle at A. Thus,

$$\begin{aligned}
\frac{AP}{PQ} &= \frac{AB}{BC} \\
(AP)(BC) &= (PQ)(AB) \\
4x &= \tfrac{2}{3}(x + 5) \\
4x &= \tfrac{2}{3}x + \tfrac{10}{3} \\
\tfrac{10}{3}x &= \tfrac{10}{3} \\
x &= 1
\end{aligned}$$

Therefore, $AP = x = 1$.

Solution 2

Let $AP = x$ and $QP = h$.
Since QP is parallel to CB, then QP is perpendicular to BA.
Since $\triangle ABC$ is right-angled at B, its area is $\frac{1}{2}(4)(5+x) = 10 + 2x$.
However, we can look at the area of the $\triangle ABC$ in terms of its four triangular pieces: $\triangle CBR$ (which has area $\frac{1}{2}(4)(3)$), $\triangle CRQ$ (which has area 5), $\triangle QPR$ (which has area $\frac{1}{2}h(2)$), and $\triangle QPA$ (which has area $\frac{1}{2}xh$).
Therefore, $10 + 2x = 6 + 5 + h + \frac{1}{2}xh$ so $xh - 4x + 2h + 2 = 0$.

Now, $\triangle APQ$ is similar to $\triangle ABC$, as each has a right angle and they share a common angle at A. Thus,

$$\begin{aligned}
\frac{AP}{PQ} &= \frac{AB}{BC} \\
(AP)(BC) &= (PQ)(AB) \\
x(4) &= h(x+5) \\
4x &= hx + 5h \\
-5h &= hx - 4x
\end{aligned}$$

Substituting this into the equation above, $xh + 2h - 4x + 2 = 0$ becomes $-5h + 2h + 2 = 0$ or $3h = 2$ or $h = \frac{2}{3}$.
Lastly, we solve for x by subsituting our value for h: $-5(\frac{2}{3}) = \frac{2}{3}x - 4x$ or $-\frac{10}{3} = -\frac{10}{3}x$ and so $x = 1$.
Therefore, $AP = x = 1$.

9. (a) *Solution 1*

Rewriting the equation, we obtain

$$\begin{aligned}
2^{x+2}5^{6-x} &= 2^{x^2}5^{x^2} \\
1 &= 2^{x^2}2^{-2-x}5^{x^2}5^{x-6} \\
1 &= 2^{x^2-x-2}5^{x^2+x-6} \\
0 &= (x^2-x-2)\log_{10}2 + (x^2+x-6)\log_{10}5 \\
0 &= (x-2)(x+1)\log_{10}2 + (x-2)(x+3)\log_{10}5 \\
0 &= (x-2)[(x+1)\log_{10}2 + (x+3)\log_{10}5] \\
0 &= (x-2)[(\log_{10}2 + \log_{10}5)x + (\log_{10}2 + 3\log 105)] \\
0 &= (x-2)[(\log_{10}10)x + \log_{10}(2\cdot 5^3)] \\
0 &= (x-2)(x + \log_{10}250)
\end{aligned}$$

Therefore, $x = 2$ or $x = -\log_{10}250$.

Solution 2

We take base 10 logarithms of both sides:

$$\begin{aligned}
\log_{10}(2^{x+2}5^{6-x}) &= \log_{10}(10^{x^2}) \\
\log_{10}(2^{x+2}) + \log_{10}(5^{6-x}) &= x^2 \\
(x+2)\log_{10} 2 + (6-x)\log_{10} 5 &= x^2 \\
x(\log_{10} 2 - \log_{10} 5) + (2\log_{10} 2 + 6\log_{10} 5) &= x^2 \\
x^2 - x(\log_{10} 2 - \log_{10} 5) - (2\log_{10} 2 + 6\log_{10} 5) &= 0
\end{aligned}$$

Now, $\log_{10} 2 + \log_{10} 5 = \log_{10} 10 = 1$ so $\log_{10} 5 = 1 - \log_{10} 2$, so we can simplify the equation to

$$x^2 - x(2\log_{10} 2 - 1) - (6 - 4\log_{10} 2) = 0$$

This is a quadratic equation in x, so should have at most 2 real solutions.

By the quadratic formula,

$$\begin{aligned}
x &= \frac{(2\log_{10} 2 - 1) \pm \sqrt{(2\log_{10} 2 - 1)^2 - 4(1)(-(6 - 4\log_{10} 2))}}{2(1)} \\
&= \frac{(2\log_{10} 2 - 1) \pm \sqrt{4(\log_{10} 2)^2 - 4(\log_{10} 2) + 1 + 24 - 16\log_{10} 2}}{2} \\
&= \frac{(2\log_{10} 2 - 1) \pm \sqrt{4(\log_{10} 2)^2 - 20(\log_{10} 2) + 25}}{2} \\
&= \frac{(2\log_{10} 2 - 1) \pm \sqrt{(2\log_{10} 2 - 5)^2}}{2} \\
&= \frac{(2\log_{10} 2 - 1) \pm (5 - 2\log_{10} 2)}{2}
\end{aligned}$$

since $5 - 2\log_{10} 2 > 0$.

Therefore,

$$x = \frac{(2\log_{10} 2 - 1) + (5 - 2\log_{10} 2)}{2} = \frac{4}{2} = 2$$

or

$$x = \frac{(2\log_{10} 2 - 1) - (5 - 2\log_{10} 2)}{2} = \frac{4\log_{10} 2 - 6}{2} = 2\log_{10} 2 - 3$$

(Note that at any point, we could have used a calculator to convert to decimal approximations and solve.)

(b) First, we rewrite the system as

$$\begin{array}{ccccc}
x & + & \log_{10} x & = & y-1 \\
(y-1) & + & \log_{10}(y-1) & = & z-2 \\
(z-2) & + & \log_{10}(z-2) & = & x
\end{array}$$

Second, we make the substitution $a = x$, $b = y - 1$ and $c = z - 2$, allowing us to rewrite the system as

$$\begin{aligned} a + \log_{10} a &= b && (1) \\ b + \log_{10} b &= c && (2) \\ c + \log_{10} c &= a && (3) \end{aligned}$$

Third, we observe that $(a,b,c) = (1,1,1)$ is a solution, since $1 + \log_{10} 1 = 1 + 0 = 1$.
Next, if $a > 1$, then $\log_{10} a > 0$, so from (1),

$$b = a + \log_{10} a > a + 0 = a > 1$$

so $\log_{10} b > 0$, so from (2),

$$c = b + \log_{10} b > b + 0 = b > a > 1$$

so $\log_{10} c > 0$, so from (3),

$$a = c + \log_{10} c > c + 0 = c > b > a > 1$$

But this says that $a > c > b > a$, which is a contradiction.
Therefore, a cannot be larger than 1.
Lastly, if $0 < a < 1$ (a cannot be negative), then $\log_{10} a < 0$, so from (1),

$$b = a + \log_{10} a < a + 0 = a < 1$$

so $\log_{10} b < 0$, so from (2),

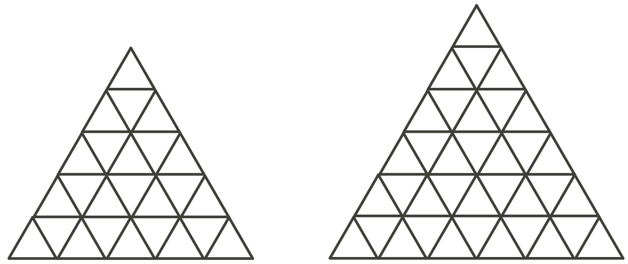
$$c = b + \log_{10} b < b + 0 = b < a < 1$$

so $\log_{10} c < 0$, so from (3),

$$a = c + \log_{10} c > c + 0 = c < b < a < 1$$

But this says that $a < c < b < a$, which is a contradiction.
Therefore, a cannot be smaller than 1 either.
Thus, a must equal 1.
If $a = 1$, then $b = a + \log_{10} a = 1 + \log_{10} 1 = 1 + 0 = 1$ from (1), which will similarly give $c = 1$ from (2).
Thus, the only solution to the system is $(a,b,c) = (1,1,1) = (x, y-1, z-2)$ since a cannot be either larger than or smaller than 1, so $(x,y,z) = (1,2,3)$.

10. (a) If $n = 5$, there are 10 downward-pointing triangles with side length 1 and 3 downward-pointing triangles with side length 2, so $f(5) = 10 + 3 = 13$.



If $n = 6$, there are 15 downward-pointing triangles with side length 1, 6 downward-pointing triangles with side length 2, and 1 downward-pointing triangle with side length 3, so $f(6) = 15 + 6 + 1 = 22$.

(b) *Solution 1*

We determine explicit formulas for $f(2k)$ and $f(2k-1)$ in terms of k and use these formulas to show that $f(2k) - f(2k-1) = k^2$.

Start with a large triangle of side length n. (We will split into cases with n even and odd later.)

We call the ith horizontal line from the top "row i". Note that row i has length i. We label the unit points along the row, starting with 0 at the left-hand end and ending with i at the right-hand end. (These are the points where the diagonal lines intersect row i. We will refer to a generic such point with the variable j.)

Consider first downward-pointing triangles with side length $m = 1$. We count these by counting the possible locations for their bottom vertex.

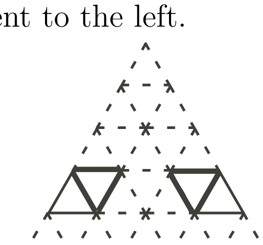
There are no such bottom vertices in row 1.

There is one such bottom vertex in row 2, at $j = 1$.

There are two such bottom vertices in row 3, at $j = 1$ and $j = 2$.

This continues, with $n - 1$ such bottom vertices in row n, at $j = 1$ to $j = n - 1$.

In general, there are $i - 1$ such bottom vertices in row i:

> To see this, we show that the leftmost such vertex is at $j = 1$ and the rightmost is at $j = i - 1$. We obtain the other bottom vertices by translating the downward-pointing triangle.
>
> The leftmost such vertex is at $j = 1$ because the downward-pointing triangle can be completed to form a parallelogram starting at this leftmost vertex and drawing a 1 unit horizontal line segment to the left.
>
> 
>
> Since the downward-pointing triangle is inside the large triangle, this parallelogram must also be, so $j = 1$. (This tells us that we cannot go any further to the left.)

Similarly, the rightmost such vertex is at $j = i - 1$, which we can see by drawing a parallelogram to the right.

Therefore, there are $1 + 2 + \cdots + (n-2) + (n-1) = \frac{1}{2}(n-1)(n)$ downward-pointing triangles of side length $m = 1$, as there are $i-1$ such triangles for each row from $i = 2$ to $i = n$.

For a general side length m of a downward-pointing triangle, we can argue as above that in row i, the leftmost bottom vertex will occur at $j = m$ and the rightmost at $j = i - m$. For there to be any such triangles, we need $m \leq i - m$ (or $2m \leq i$) so the leftmost vertex is not to the right of the rightmost vertex.
At row i, there are thus $(i - m) - m + 1 = i + 1 - 2m$ possible locations for the bottom vertex.
(When $i = 2m$ (the smallest possible value of i), there is $2m + 1 - 2m = 1$ location for the vertex.
When $i = n$ (the largest possible value of i), there are $n + 1 - 2m$ locations for the vertex.)

For a fixed positive integer n, what are the permissible values of m? Certainly, $m \geq 1$.
If $n = 2k$ for some positive integer k, then $2m \leq n = 2k$ since the largest possible downward-triangle that can be fit has its bottom vertex on the bottom row, so $m \leq k$.
If $n = 2k - 1$ for some positive integer k, then $2m \leq n = 2k - 1$ so $m \leq k - 1$.

Thus, for a fixed permissible value of m, the total number of downward-pointing triangles of side length m is

$$1 + 2 + \cdots + (n + 1 - 2m) = \tfrac{1}{2}(n + 1 - 2m)(n + 2 - 2m) \qquad (*)$$

which is the sum of $i + 1 - 2m$ from $i = 2m$ to $i = n$, because we look at all possible locations for the bottom vertex.

For $n = 2k$, the permissible values of m are $m = 1$ to $m = k$, so we add up the formula in $(*)$ for $m = 1$ to $m = k$:

$$\begin{aligned}
f(2k) &= \sum_{m=1}^{k} \tfrac{1}{2}(2k+1-2m)(2k+2-2m) \\
&= \sum_{l=1}^{k} \tfrac{1}{2}(2l-1)(2l) \qquad (\text{letting } l = k+1-m) \\
&= \sum_{l=1}^{k} (2l-1)(l) \\
&= \sum_{l=1}^{k} (2l^2 - l) \\
&= 2\sum_{l=1}^{k} l^2 - \sum_{l=1}^{k} l \\
&= 2\left(\tfrac{1}{6}k(k+1)(2k+1)\right) - \tfrac{1}{2}k(k+1) \\
&= k(k+1)\left(\tfrac{1}{3}(2k+1) - \tfrac{1}{2}\right) \\
&= k(k+1)\left(\tfrac{2}{3}k - \tfrac{1}{6}\right) \\
&= \frac{k(k+1)(4k-1)}{6}
\end{aligned}$$

For $n = 2k-1$, the permissible values of m are $m = 1$ to $m = k-1$, so we add up the formula in $(*)$ for $m = 1$ to $m = k-1$:

$$\begin{aligned}
f(2k-1) &= \sum_{m=1}^{k-1} \tfrac{1}{2}(2k-2m)(2k+1-2m) \\
&= \sum_{l=1}^{k-1} \tfrac{1}{2}(2l)(2l+1) \qquad (\text{letting } l = k-m) \\
&= \sum_{l=1}^{k-1} l(2l+1) \\
&= \sum_{l=1}^{k-1} (2l^2 + l) \\
&= 2\sum_{l=1}^{k-1} l^2 + \sum_{l=1}^{k-1} l \\
&= 2\left(\tfrac{1}{6}(k-1)(k)(2k-1)\right) + \tfrac{1}{2}(k-1)(k) \\
&= k(k-1)\left(\tfrac{1}{3}(2k-1) + \tfrac{1}{2}\right) \\
&= k(k-1)\left(\tfrac{2}{3}k + \tfrac{1}{6}\right) \\
&= \frac{k(k-1)(4k+1)}{6}
\end{aligned}$$

Therefore,

$$\begin{aligned} f(2k) - f(2k-1) &= \frac{k(k+1)(4k-1)}{6} - \frac{k(k-1)(4k+1)}{6} \\ &= \tfrac{1}{6}k((k+1)(4k-1) - (k-1)(4k+1)) \\ &= \tfrac{1}{6}k((4k^2+3k-1) - (4k^2-3k-1)) \\ &= \tfrac{1}{6}k(6k) \\ &= k^2 \end{aligned}$$

as required.

Solution 2

As in Solution 1, we can show that if $n = 2k$, we can fit in downward-pointing triangles of sizes $m = 1$ to $m = k$ and if $n = 2k-1$, we can fit in downward-pointing triangles of sizes $m = 1$ to $m = k-1$.

Consider $f(2k) - f(2k-1)$. To calculate this quantity, we must determine how many *additional* downward-pointing triangles can be put into the large triangle of size $n = 2k$ instead of that of size $n = 2k-1$.

Since the triangle of size $2k-1$ can be put inside the triangle of size $2k$ with their top vertices coinciding, then any new downward-pointing triangles inside the triangle of size $n = 2k$ all have their bottom vertices in row $n = 2k$.

We count the number of such triangles by considering the possible values of m.

If $m = 1$, Solution 1 tells us that there are $2k+1-2(1) = 2k-1$ such triangles.

If $m = 2$, there are $2k+1-2(2) = 2k-3$ such triangles.

For a general m, there are $2k+1-2m$ such triangles.

The value of $f(2k) - f(2k-1)$ is equal to the sum of $2k+1-2m$ over all possible values of m.

Therefore,

$$\begin{aligned} f(2k) - f(2k-1) &= \sum_{m=1}^{k}(2k+1-2m) \\ &= \sum_{m=1}^{k}(2k+1) - 2\sum_{m=1}^{k} m \\ &= k(2k+1) - 2\left(\tfrac{1}{2}k(k+1)\right) \\ &= 2k^2 + k - (k^2+k) \\ &= k^2 \end{aligned}$$

as required.

(c) From Solution 1 to (b), we know that

$$f(2k) = \frac{k(k+1)(4k-1)}{6} \qquad \text{and} \qquad f(2k-1) = \frac{k(k-1)(4k+1)}{6}$$

We rewrite each of these in terms of n.

In the case of n even, since $n = 2k$, then $k = \frac{1}{2}n$, so

$$f(n) = f(2k) = \frac{\frac{1}{2}n(\frac{1}{2}n+1)(4(\frac{1}{2}n)-1)}{6} = \frac{\frac{1}{2}n(\frac{1}{2}n+1)(2n-1)}{6} = \frac{n(n+2)(2n-1)}{24}$$

In the case of n odd, since $n = 2k-1$, then $k = \frac{1}{2}(n+1)$, so

$$\begin{aligned} f(n) &= f(2k-1) = \frac{\frac{1}{2}(n+1)(\frac{1}{2}(n+1)-1)(4(\frac{1}{2}(n+1))+1)}{6} \\ &= \frac{\frac{1}{2}(n+1)(\frac{1}{2}(n+1)-1)(2n+3)}{6} = \frac{(n+1)(n-1)(2n+3)}{24} \end{aligned}$$

<u>Case 1: n is even</u>

If $f(n)$ is divisible by n, then $f(n) = nq$ for some integer q.

Thus,

$$\begin{aligned} nq &= \frac{n(n+2)(2n-1)}{24} \\ 24nq &= n(n+2)(2n-1) \\ 24q &= (n+2)(2n-1) \qquad \text{(since } n \neq 0\text{)} \end{aligned}$$

Thus, we need $(n+2)(2n-1)$ to be a multiple of 24.

Since $2n-1$ is odd, then $n+2$ must be a multiple of 8, so $n+2 = 8a$ for some integer a, so $n = 8a-2$.

Therefore, $24q = 8a(2(8a-2)-1)$ or $3q = a(16a-5)$.

Therefore, we still need $a(16a-5)$ to be a multiple of 3.

Since 3 is a prime number, then either a is divisible by 3 or $16a-5 = 3(5a-2)+(a+1)$ is divisible by 3.

If a is divisible by 3, then $a = 3b$ for some integer b.

If $16a-5$ is divisible by 3, then since $16a-5 = 3(5a-2)+(a+1)$, we have that $a+1 = (16a-5)-3(5a-2)$ is divisible by 3, as it is the difference of two multiples of 3. Therefore, $a+1 = 3b$ for some integer b. Therefore, $n = 8(3b)-2 = 24b-2$ or $n = 8(3b-1)-2 = 24b-10$ for some integer b.

We have proven that if $f(n)$ is divisible by n, then $n = 24b-2$ or $n = 24b-10$. We need to verify that each of these forms for n works for all b.

If $n = 24b-2$, then

$$f(n) = f(24b-2) = \frac{(24b-2)(24b)(48b-5)}{24} = b(24b-2)(48b-5)$$

which is divisible by $24b-2$, and so $f(n)$ is always divisible by n in this case.

If $n = 24b-10$, then

$$f(n) = f(24b-10) = \frac{(24b-10)(24b-8)(48b-21)}{24} = (24b-10)(3b-1)(16b-7)$$

which is divisible by $24b - 10$, and so $f(n)$ is always divisible by n in this case.

Thus, each of these forms for n works for all positive integers b.

<u>Case 2: n is odd</u>

If $f(n)$ is divisible by n, then $f(n) = nq$ for some integer q.

Thus,

$$\begin{aligned} nq &= \frac{(n+1)(n-1)(2n+3)}{24} \\ 24nq &= (n+1)(n-1)(2n+3) \\ 24nq &= (n^2-1)(2n+3) \\ 24nq &= 2n^3+3n^2-2n-3 \\ 3 &= 2n^3+3n^2-2n-24nq \\ 3 &= n(2n^2+3n-2-24q) \end{aligned}$$

Since the right side is divisible by n, then the left side must be as well, so n divides into 3 which gives us that $n = 1$ or $n = 3$.

(Note that $f(1) = 0$ which is divisible by 1 and $f(3) = 3$ which is divisible by 3, so both of these cases do work.)

Therefore, $f(n)$ is divisible by n when $n = 1$, $n = 3$, $n = 24b - 10$ for all positive integers b or $n = 24b - 2$ for all positive integers b.