



Canadian Mathematics Competition

An activity of the Centre for Education
in Mathematics and Computing,
University of Waterloo, Waterloo, Ontario

Euclid Contest

Tuesday, April 15, 2008

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Time: $2\frac{1}{2}$ hours

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Calculators are permitted, provided they are non-programmable and without graphic displays.

Do not open this booklet until instructed to do so. The paper consists of 10 questions, each worth 10 marks. Parts of each question can be of two types. **SHORT ANSWER** parts are worth 3 marks each. **FULL SOLUTION** parts are worth the remainder of the 10 marks for the question.

Instructions for SHORT ANSWER parts:


1. **SHORT ANSWER** parts are indicated like this:  .
2. **Enter the answer in the appropriate box in the answer booklet.**
For these questions, full marks will be given for a correct answer which is placed in the box. Part marks will be awarded **only if relevant work** is shown in the space provided in the answer booklet.

Instructions for FULL SOLUTION parts:


1. **FULL SOLUTION** parts are indicated like this:  .
2. **Finished solutions must be written in the appropriate location in the answer booklet.** Rough work should be done separately. If you require extra pages for your finished solutions, foolscap will be supplied by your supervising teacher. Insert these pages into your answer booklet. Be sure to write your name, school name and question number on any inserted pages.
3. Marks are awarded for completeness, clarity, and style of presentation. A correct solution poorly presented will not earn full marks.


NOTE: At the completion of the Contest, insert the information sheet inside the answer booklet.

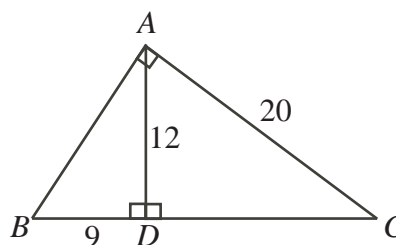
The names of some top-scoring students will be published in the Euclid Results on our Web site, <http://www.cemc.uwaterloo.ca>.


- NOTES:
1. Please read the instructions on the front cover of this booklet.
 2. Write all answers in the answer booklet provided.
 3. For questions marked “  ”, full marks will be given for a correct answer placed in the appropriate box in the answer booklet. **If an incorrect answer is given, marks may be given for work shown.** Students are strongly encouraged to show their work.
 4. All calculations and answers should be expressed as exact numbers such as 4π , $2 + \sqrt{7}$, etc., rather than as 12.566... or 4.646..., except where otherwise indicated.

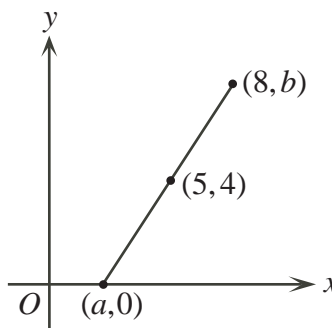
A Note about Writing Solutions


For each problem marked “  ”, a full solution is required. The solutions that you provide in the answer booklet should be well organized and contain a balance of mathematical statements and words of explanation. Working out some of the details in rough on a separate piece of paper before writing your finished solution is a good idea. Your final solution should be written so that the marker can understand your approach to the problem and all of the mathematical steps of your solution.

1.  (a) In the diagram, what is the perimeter of $\triangle ABC$?




-  (b) In the diagram, the line segment with endpoints $(a, 0)$ and $(8, b)$ has midpoint $(5, 4)$. What are the values of a and b ?





-  (c) The lines $ax + y = 30$ and $x + ay = k$ intersect at the point $P(6, 12)$. Determine the value of k .

2. Each part of this problem refers to the parabola $y = (x - 2)(x - 8) + 7$.

-  (a) The points $(2, 7)$ and $(c, 7)$, where $c \neq 2$, lie on the parabola. What is the value of c ?

-  (b) What are the coordinates of the vertex of the parabola?

-  (c) A line that passes through the point $A(5, 0)$ intersects the parabola at $B(4, -1)$. Determine the other point at which this line intersects the parabola.

3.  (a) A 3×3 square frame is placed on a grid of numbers, as shown. In the example, the sum of the numbers inside the square frame is 108, and the middle number is 12. When the square frame is moved to a new position, the sum of its numbers becomes 279. In the frame's new position, what is the middle number?

| | | | | | | |
|----|----|----|----|----|----|----|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 8 | 9 | 10 | 11 | 12 | 13 | 14 |
| 15 | 16 | 17 | 18 | 19 | 20 | 21 |
| 22 | 23 | 24 | 25 | 26 | 27 | 28 |
| 29 | 30 | 31 | 32 | 33 | 34 | 35 |
| 36 | 37 | 38 | 39 | 40 | 41 | 42 |
| 43 | 44 | 45 | 46 | 47 | 48 | 49 |



- (b) Of the three figures shown, which has the smallest area and which has the largest area? Explain how you determined your answer. (In Figure A, the circle has a diameter of length 2.)

Figure A

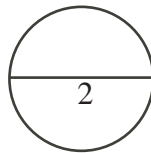


Figure B

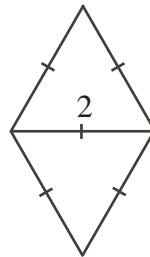
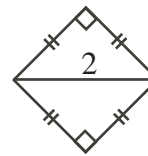

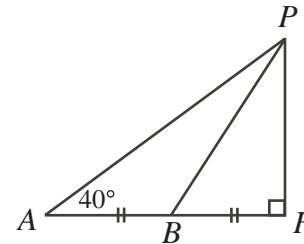


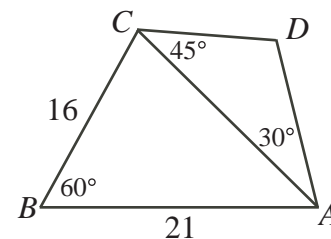
Figure C




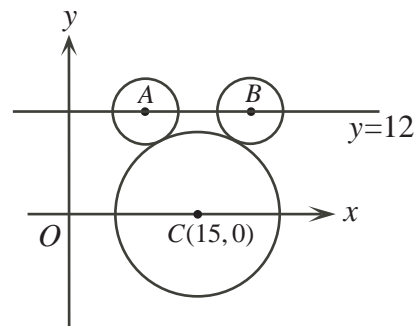
4.  (a) A flagpole FP is 20 metres tall. From point A on the flat ground, the angle of elevation to the top of the flagpole is 40° . If B is halfway from A to F , what is the measure of $\angle FBP$, to the nearest degree?



- (b) In the diagram, $AB = 21$ and $BC = 16$. Also, $\angle ABC = 60^\circ$, $\angle CAD = 30^\circ$, and $\angle ACD = 45^\circ$. Determine the length of CD , to the nearest tenth.



5.  (a) In the diagram, the large circle has radius 9 and centre $C(15,0)$. The small circles have radius 4 and centres A and B on the horizontal line $y = 12$. Each of the two small circles is tangent to the large circle. It takes a bug 5 seconds to walk at a constant speed from A to B along the line $y = 12$. How far does the bug walk in 1 second?





- (b) Determine all values of k , with $k \neq 0$, for which the parabola

$$y = kx^2 + (5k + 3)x + (6k + 5)$$

has its vertex on the x -axis.

6.



- (a) The function $f(x)$ satisfies the equation $f(x) = f(x - 1) + f(x + 1)$ for all values of x . If $f(1) = 1$ and $f(2) = 3$, what is the value of $f(2008)$?



- (b) The numbers a, b, c , in that order, form a three term arithmetic sequence (see below) and $a + b + c = 60$.
The numbers $a - 2, b, c + 3$, in that order, form a three term geometric sequence. Determine all possible values of a, b and c .

(An *arithmetic sequence* is a sequence in which each term after the first is obtained from the previous term by adding a constant. For example, 3, 5, 7 is an arithmetic sequence with three terms.

A *geometric sequence* is a sequence in which each term after the first is obtained from the previous term by multiplying it by a constant. For example, 3, 6, 12 is a geometric sequence with three terms.)

7.



- (a) The average of three consecutive multiples of 3 is a .
The average of four consecutive multiples of 4 is $a + 27$.
The average of the smallest and largest of these seven integers is 42.
Determine the value of a .

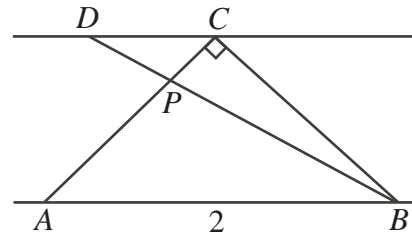


- (b) Billy and Crystal each have a bag of 9 balls. The balls in each bag are numbered from 1 to 9. Billy and Crystal each remove one ball from their own bag. Let b be the sum of the numbers on the balls remaining in Billy's bag. Let c be the sum of the numbers on the balls remaining in Crystal's bag. Determine the probability that b and c differ by a multiple of 4.

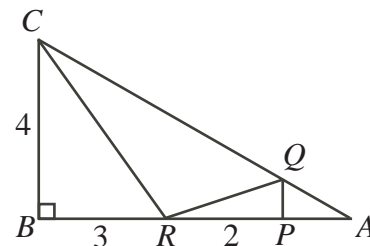
8.




- (a) Points A, B, C , and D are arranged, as shown, with AB parallel to DC and P the point of intersection of AC and BD . Also, $\angle ACB = 90^\circ$, $AC = CB$, $AB = BD = 2$. Determine the measure of $\angle DBC$.



- (b) In the diagram, ABC is a right-angled triangle with P and R on AB . Also, Q is on AC , and PQ is parallel to BC . If $RP = 2$, $BR = 3$, $BC = 4$, and the area of $\triangle QRC$ is 5, determine the length of AP .




9.  (a) The equation $2^{x+2}5^{6-x} = 10^{x^2}$ has two real solutions. Determine these two solutions.



- (b) Determine all real solutions to the system of equations

$$\begin{aligned}x + \log_{10} x &= y - 1 \\y + \log_{10}(y - 1) &= z - 1 \\z + \log_{10}(z - 2) &= x + 2\end{aligned}$$

and prove that there are no more solutions.

10.  Suppose that n is a positive integer. Consider an upward-pointing equilateral triangle of side length n , cut up into unit triangles, as shown.

$$n = 3$$



$$n = 4$$



For each n , let $f(n)$ represent the total number of downward-pointing equilateral triangles of *all* sizes. For example, $f(3) = 3$ and $f(4) = 6 + 1 = 7$, as illustrated below.

$$n = 3$$



$$f(3) = 3$$

$$n = 4$$



$$f(4) = 6 + 1 = 7$$

- (a) Determine the values of $f(5)$ and $f(6)$.
 (b) Prove that $f(2k) = f(2k - 1) + k^2$ for each positive integer $k \geq 1$.
 (c) Determine, with justification, all positive integers n for which $f(n)$ is divisible by n .



Canadian Mathematics Competition



For students...

Thank you for writing the 2008 Euclid Contest!

In 2007, more than 14 000 students from around the world registered to write the Euclid Contest.

If you are graduating from secondary school, good luck in your future endeavours!

If you will be returning to secondary school next year, encourage your teacher to register you for the 2008 Sun Life Financial Canadian Open Mathematics Challenge, which will be written in late November.

Visit our website

www.cemc.uwaterloo.ca

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- More information about the Sun Life Financial Canadian Open Mathematics Challenge
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