Canadian Mathematics Competition

An activity of the Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario

2007 Fryer Contest

Wednesday, April 18, 2007

Solutions

1. (a) Since Rectangle 3 has 4 rows of squares, then Rectangle 4 has 5 rows.
Since Rectangle 3 has 7 columns of squares, then Rectangle 4 has 9 columns.
Therefore, Rectangle 4 has $5 \times 9 = 45$ squares.

(b) Since Rectangle 4 has 5 rows, its height is 5.
Since Rectangle 4 has 9 columns, its width is 9.
Therefore, the perimeter of Rectangle 4 is $2(5) + 2(9) = 28$.

(c) Since Rectangle 4 has 5 rows of squares, then Rectangle 7 has $5 + 3 = 8$ rows.
Since Rectangle 4 has 9 columns of squares, then Rectangle 7 has $9 + 2(3) = 15$ columns.
Since Rectangle 7 has 8 rows and 15 columns, it is 8 by 15, so has perimeter $2(8)+2(15) = 46$.

(d) *Solution 1*
Rectangle 7 is 8 by 15 and so has perimeter $2(8) + 2(15) = 46$.
Let us try a bigger rectangle.
Rectangle 17 has $8 + 10 = 18$ rows and $15 + 2(10) = 35$ columns, so is 18 by 35 and has perimeter $2(18) + 2(35) = 106$.
Let us try a rectangle that is still bigger.
Rectangle 27 has $18 + 10 = 28$ rows and $35 + 2(10) = 55$ columns, so is 28 by 55 and has perimeter $2(28) + 2(55) = 166$.
We are getting close!
Rectangle 28 has 29 rows and 57 columns, so has perimeter $2(29) + 2(57) = 172$.
Rectangle 29 has 30 rows and 59 columns, so has perimeter $2(30) + 2(59) = 178$.
Therefore, $n = 29$.
(Notice that this is the only answer as moving more steps along in the sequence makes the rectangles larger.)

Solution 2
Rectangle 7 is 8 by 15 and so has perimeter $2(8) + 2(15) = 46$.
When we move to Rectangle 8, the height increases by 1 and the width increases by 2. This increase the perimeter by $2(1) + 2(2) = 6$.
The same increase in perimeter occurs at each step in the sequence.
Since $178 - 46 = 132 = 22(6)$, then we must move 22 steps beyond Rectangle 7 to get from a perimeter of 46 to a perimeter of 178.
Thus, Rectangle 29 has a perimeter of 178, so $n = 29$.

Solution 3
Rectangle 1 has 1 more row than its number, and the number of rows increases by 1 from each step to the next.
Therefore, Rectangle n has $n + 1$ rows.
Rectangle 1 has 3 columns (which is 1 more than twice its number), and the number of columns increases by 2 from each step to the next.
Therefore, Rectangle n has $2n + 1$ columns.
A formula for the perimeter of Rectangle n is $2(n+1)+2(2n+1) = 2n+2+4n+2 = 6n+4$.
For a perimeter of 178, we must have $6n + 4 = 178$ or $6n = 174$ or $n = 29$.
Therefore, Rectangle 29 has a perimeter of 178.

2. (a) Jim buys $5 + 2 + 3 = 10$ tickets.
The total cost of the tickets is $5(\$25) + 2(\$10) + 3(\$5) = \$160$.

Therefore, the average cost of the tickets was $\dfrac{\$160}{10} = \$16$.

(b) Since Mike buys 8 tickets at an average cost of \$12, their total cost is $8 \times \$12 = \$96$.
When he buys 5 more platinum tickets, he pays $5 \times \$25 = \$125$.
In total, Mike thus pays $\$96 + \$125 = \$221$ for 13 tickets.
Therefore, the new average cost of the tickets is $\dfrac{\$221}{13} = \$17$.

(c) *Solution 1*
The first 10 tickets that Ophelia buys at an average cost of \$14 have a total cost of $10 \times \$14 = \$140$.
When she buys n more platinum tickets, she pays $25n$ dollars for these additional tickets.
In total, she has now paid $140 + 25n$ dollars for $10 + n$ tickets.
Since the average price of the tickets that she has bought is \$20, then

$$\begin{aligned}
\frac{140 + 25n}{10 + n} &= 20 \\
140 + 25n &= 20(10 + n) \\
140 + 25n &= 200 + 20n \\
5n &= 60 \\
n &= 12
\end{aligned}$$

so she buys 12 more platinum tickets.

Solution 2
For the first ten tickets that Ophelia buys, the average cost is \$6 less per ticket than the final average cost of \$20.
Therefore, she has paid $10 \times \$6 = \$60$ less in total than she would have if she had paid \$20 on average for these tickets.
For a final average of \$20 per ticket, the new platinum tickets must cost in total \$60 more than an average of \$20.
Since each platinum ticket costs \$5 more on average than the final average, then she buys $\$60 \div \$5 = 12$ more platinum tickets.

3. (a) For $992\,466\,1A6$ to be divisible by 8, we must have $1A6$ divisible by 8.
We check each of the possibilities, using a calculator or by checking by hand:

106 is not divisible by 8, 116 is not divisible by 8, 126 is not divisible by 8,
136 is divisible by 8,
146 is not divisible by 8, 156 is not divisible by 8, 166 is not divisible by 8,
176 is divisible by 8,
186 is not divisible by 8, 196 is not divisible by 8

Therefore, the possible values of A are 3 and 7.

(b) For $D\,767\,E89$ to be divisible by 9, we must have $D + 7 + 6 + 7 + E + 8 + 9 = 37 + D + E$ divisible by 9.
Since D and E are each a single digit then each is between 0 and 9, so $D + E$ is between 0 and 18.
Therefore, $37 + D + E$ is between 37 and 55.
The numbers between 37 and 55 that are divisible by 9 are 45 and 54.

If $37 + D + E = 45$, then $D + E = 8$.
If $37 + D + E = 54$, then $D + E = 17$.
Therefore, the possible values of $D + E$ are 8 and 17.

(c) For $5\,41G\,507\,2H6$ to be divisible by 72, it must be divisible by 8 and by 9.
It is easier to check for divisibility by 8 first, since this will allow us to determine a small number of possibilities for H.
For $5\,41G\,507\,2H6$ to be divisible by 8, we must have $2H6$ divisible by 8.
Going through the possibilities as in part (a), we can find that $2H6$ is divisible by 8 when $H = 1, 5, 9$ (that is, 216, 256 and 296 are divisible by 8 while 206, 226, 236, 246, 266, 276, 286 are not divisible by 8).
We must now use each possible value of H to find the possible values of G that make $5\,41G\,507\,2H6$ divisible by 9.
First, $H = 1$. What value(s) of G make $5\,41G\,507\,216$ divisible by 9?
In this case, we need $5 + 4 + 1 + G + 5 + 0 + 7 + 2 + 1 + 6 = 31 + G$ divisible by 9.
Since G is between 0 and 9, then $31 + G$ is between 31 and 40, so must be equal to 36 if it is divisible by 9. Thus, $G = 5$.
Next, $H = 5$. What value(s) of G make $5\,41G\,507\,256$ divisible by 9?
In this case, we need $5 + 4 + 1 + G + 5 + 0 + 7 + 2 + 5 + 6 = 35 + G$ divisible by 9.
Since G is between 0 and 9, then $35 + G$ is between 35 and 44, so must be equal to 36 if it is divisible by 9. Thus, $G = 1$.
Last, $H = 9$. What value(s) of G make $5\,41G\,507\,296$ divisible by 9?
In this case, we need $5 + 4 + 1 + G + 5 + 0 + 7 + 2 + 9 + 6 = 39 + G$ divisible by 9.
Since G is between 0 and 9, then $39 + G$ is between 39 and 48, so must be equal to 45 if it is divisible by 9. Thus, $G = 6$.
Therefore, the possible pairs of values are $H = 1$ and $G = 5$, $H = 5$ and $G = 1$, and $H = 9$ and $G = 6$.
(Note that we could have combined the analysis of these last three cases.)

4. (a) *Solution 1*
By the Pythagorean Theorem, $YZ^2 = YX^2 + XZ^2 = 60^2 + 80^2 = 3600 + 6400 = 10000$, so $YZ = 100$.
(We could also have found YZ without using the Pythagorean Theorem by noticing that $\triangle XYZ$ is a right-angled triangle with its right-angle at X and $XY = 60 = 3(20)$ and $XZ = 80 = 4(20)$. This means that $\triangle XYZ$ is similar to a 3-4-5 triangle, so has $YZ = 5(20) = 100$.)
Since $\triangle YXZ$ is right-angled at X, its area is $\frac{1}{2}(60)(80) = 2400$.
Since XW is perpendicular to YZ, then the area of $\triangle YXZ$ is also equal to $\frac{1}{2}(100)(XW) = 50XW$.
Therefore, $50XW = 2400$, so $XW = 48$.
By the Pythagorean Theorem, $WZ^2 = 80^2 - 48^2 = 6400 - 2304 = 4096$.
Thus, $WZ = \sqrt{4096} = 64$.

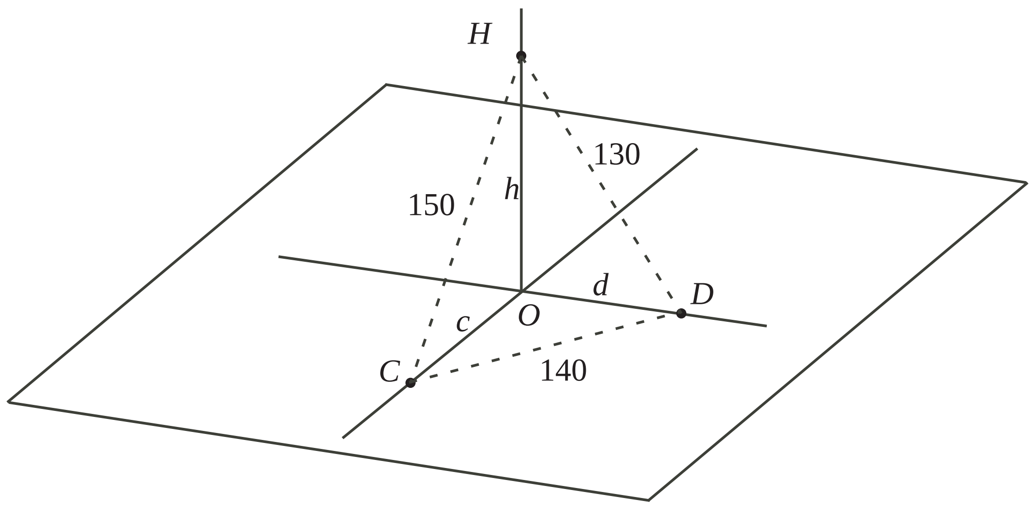
Solution 2
By the Pythagorean Theorem, $YZ^2 = YX^2 + XZ^2 = 60^2 + 80^2 = 3600 + 6400 = 10000$, so $YZ = 100$.
Let $WZ = a$. Then $YW = 100 - a$.
Let $XW = h$.
By the Pythagorean Theorem in $\triangle XWY$, we have $(100 - a)^2 + h^2 = 60^2$.

By the Pythagorean Theorem in $\triangle XWZ$, we have $a^2 + h^2 = 80^2$.
Subtracting the first of these equations from the second, we obtain

$$\begin{aligned} a^2 - (100-a)^2 &= 80^2 - 60^2 \\ a^2 - (10000 - 200a + a^2) &= 6400 - 3600 \\ 200a - 10000 &= 2800 \\ 200a &= 12800 \\ a &= 64 \end{aligned}$$

Therefore, $WZ = 64$.

(b) Let $OC = c$, $OD = d$ and $OH = h$.

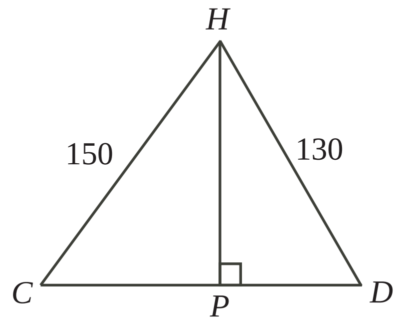


Note that OH is perpendicular to the field, so OH is perpendicular to OC and to OD.
Also, since OD points east and OC points south, then OD is perpendicular to OC.
Since $HC = 150$, then $h^2 + c^2 = 150^2$ by the Pythagorean Theorem.
Since $HD = 130$, then $h^2 + d^2 = 130^2$.
Since $CD = 140$, then $c^2 + d^2 = 140^2$.
Adding the first two equations, we obtain $2h^2 + c^2 + d^2 = 150^2 + 130^2$.
Since $c^2 + d^2 = 140^2$, then

$$\begin{aligned} 2h^2 + 140^2 &= 150^2 + 130^2 \\ 2h^2 &= 150^2 + 130^2 - 140^2 \\ 2h^2 &= 19800 \\ h^2 &= 9900 \\ h &= \sqrt{9900} = 30\sqrt{11} \end{aligned}$$

Therefore, the height of the balloon above the field is $30\sqrt{11} \approx 99.5$ m.

(c) To save the most rope, we must have HP having minimum length.
For HP to have minimum length, HP must be perpendicular to CD.



(Among other things, we can see from this diagram that sliding P away from the perpendicular position does make HP longer.)
In the diagram, $HC = 150$, $HD = 130$ and $CD = 140$.
Let $HP = x$ and $PD = a$. Then $CP = 140 - a$.
By the Pythagorean Theorem in $\triangle HPC$, $x^2 + (140 - a)^2 = 150^2$.
By the Pythagorean Theorem in $\triangle HPD$, $x^2 + a^2 = 130^2$.
Subtracting the second equation from the first, we obtain

$$\begin{aligned}(140 - a)^2 - a^2 &= 150^2 - 130^2 \\ (19600 - 280a + a^2) - a^2 &= 5600 \\ 19600 - 280a &= 5600 \\ 280a &= 14000 \\ a &= 50\end{aligned}$$

Therefore, $x^2 + 90^2 = 150^2$ or $x^2 = 150^2 - 90^2 = 22500 - 8100 = 14400$ so $x = 120$.
So the shortest possible rope that we can use is 120 m, which saves $130+150-120 = 160$ m of rope.