Canadian Mathematics Competition

An activity of the Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario

Euclid Contest

Tuesday, April 17, 2007

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Time: $2\frac{1}{2}$ hours

Calculators are permitted, provided they are non-programmable and without graphic displays.

Do not open this booklet until instructed to do so. The paper consists of 10 questions, each worth 10 marks. Parts of each question can be of two types. **SHORT ANSWER** parts are worth 2 marks each (questions 1-2) or 3 marks each (questions 3-7). **FULL SOLUTION** parts are worth the remainder of the 10 marks for the question.

Instructions for SHORT ANSWER parts:

1. **SHORT ANSWER** parts are indicated like this:  .
2. **Enter the answer in the appropriate box in the answer booklet.**
   For these questions, full marks will be given for a correct answer which is placed in the box. Part marks will be awarded **only if relevant work** is shown in the space provided in the answer booklet.

Instructions for FULL SOLUTION parts:

1. **FULL SOLUTION** parts are indicated like this:  .
2. **Finished solutions must be written in the appropriate location in the answer booklet.** Rough work should be done separately. If you require extra pages for your finished solutions, foolscap will be supplied by your supervising teacher. Insert these pages into your answer booklet. Be sure to write your name, school name and question number on any inserted pages.
3. Marks are awarded for completeness, clarity, and style of presentation. A correct solution poorly presented will not earn full marks.

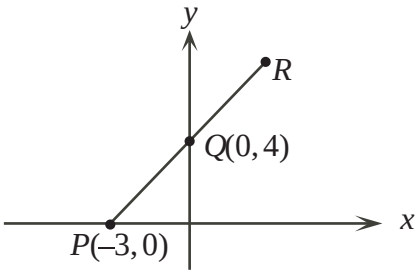
NOTE: At the completion of the Contest, insert the information sheet inside the answer booklet.

The names of some top-scoring students will be published in the Euclid Results on our Web site, http://www.cemc.uwaterloo.ca.

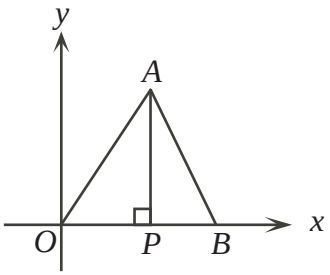
NOTES:
1. Please read the instructions on the front cover of this booklet.
2. Write all answers in the answer booklet provided.
3. For questions marked " ", full marks will be given for a correct answer placed in the appropriate box in the answer booklet. **If an incorrect answer is given, marks may be given for work shown.** Students are strongly encouraged to show their work.
4. All calculations and answers should be expressed as exact numbers such as 4π, $2+\sqrt{7}$, etc., except where otherwise indicated.

1. (a) If the point $(a-1, a+1)$ lies on the line $y = 2x - 3$, what is the value of a?

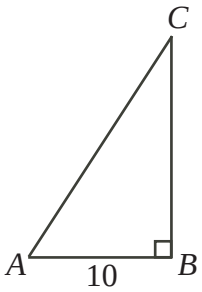
(b) In the diagram, a line is drawn through points P, Q and R. If $PQ = QR$, what are the coordinates of R?



(c) In the diagram, $OA = 15$, $OP = 9$ and $PB = 4$. Determine the equation of the line through A and B. Explain how you got your answer.

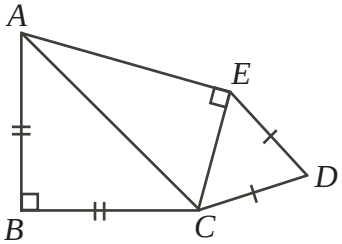


2. (a) In the diagram, $\triangle ABC$ is right-angled at B and $AB = 10$. If $\cos(\angle BAC) = \frac{5}{13}$, what is the value of $\tan(\angle ACB)$?



(b) Suppose $0° < x < 90°$ and $2\sin^2 x + \cos^2 x = \frac{25}{16}$. What is the value of $\sin x$?

(c) In the diagram, $AB = BC = 2\sqrt{2}$, $CD = DE$, $\angle CDE = 60°$, and $\angle EAB = 75°$. Determine the perimeter of figure $ABCDE$. Explain how you got your answer.

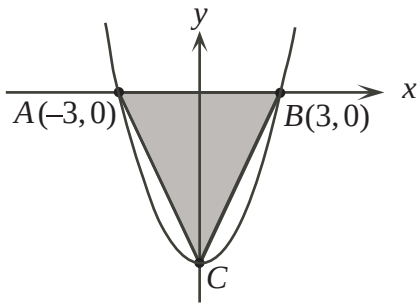


3. (a) The first term of a sequence is 2007. Each term, starting with the second, is the sum of the cubes of the digits of the previous term. What is the 2007th term?

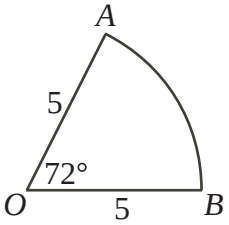
(b) Sequence A has nth term $n^2 - 10n + 70$.
(The first three terms of sequence A are 61, 54, 49.)
Sequence B is an arithmetic sequence with first term 5 and common difference 10.
(The first three terms of sequence B are 5, 15, 25.)
Determine all n for which the nth term of sequence A is equal to the nth term of sequence B. Explain how you got your answer.

4. (a) Determine all values of x for which $2 + \sqrt{x-2} = x - 2$.

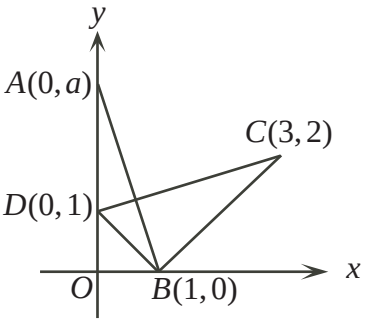
(b) In the diagram, the parabola intersects the x-axis at $A(-3, 0)$ and $B(3, 0)$ and has its vertex at C below the x-axis. The area of $\triangle ABC$ is 54. Determine the equation of the parabola. Explain how you got your answer.



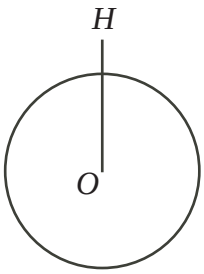
5. (a) In the diagram, a sector of a circle with centre O and radius 5 is shown. What is the perimeter of the sector?



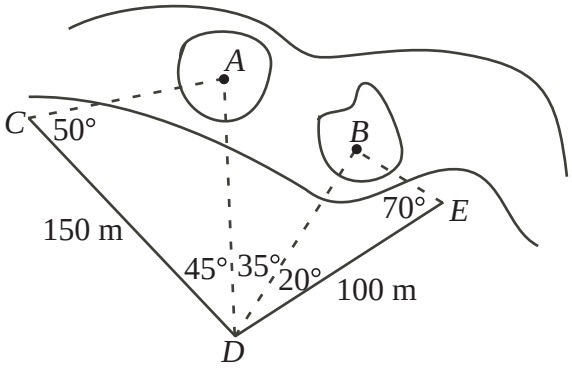
(b) In the diagram, $A(0, a)$ lies on the y-axis above D. If the triangles AOB and BCD have the same area, determine the value of a. Explain how you got your answer.



6. (a) The Little Prince lives on a spherical planet which has a radius of 24 km and centre O. He hovers in a helicopter (H) at a height of 2 km above the surface of the planet. From his position in the helicopter, what is the distance, in kilometres, to the furthest point on the surface of the planet that he can see?

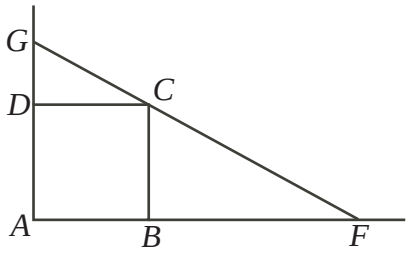


(b) In the diagram, points A and B are located on islands in a river full of rabid aquatic goats. Determine the distance from A to B, to the nearest metre. (Luckily, someone has measured the angles shown in the diagram as well as the distances CD and DE.)

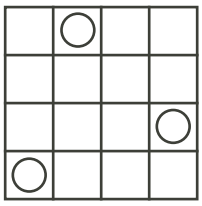


7. (a) Determine all values of x for which $(\sqrt{x})^{\log_{10} x} = 100$.

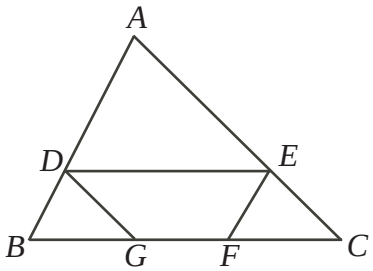
(b) In the diagram, line segment FCG passes through vertex C of square $ABCD$, with F lying on AB extended and G lying on AD extended. Prove that $\dfrac{1}{AB} = \dfrac{1}{AF} + \dfrac{1}{AG}$.



8. (a) In the 4×4 grid shown, three coins are randomly placed in different squares. Determine the probability that no two coins lie in the same row or column.



(b) In the diagram, the area of $\triangle ABC$ is 1. Trapezoid $DEFG$ is constructed so that G is to the left of F, DE is parallel to BC, EF is parallel to AB and DG is parallel to AC. Determine the maximum possible area of trapezoid $DEFG$.

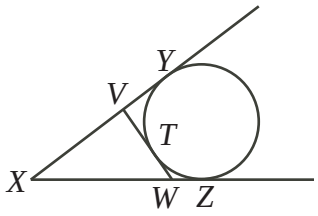


9. The parabola $y = f(x) = x^2 + bx + c$ has vertex P and the parabola $y = g(x) = -x^2 + dx + e$ has vertex Q, where P and Q are distinct points. The two parabolas also intersect at P and Q.

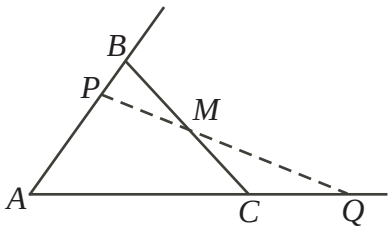
(a) Prove that $2(e - c) = bd$.

(b) Prove that the line through P and Q has slope $\frac{1}{2}(b+d)$ and y-intercept $\frac{1}{2}(c+e)$.

10. (a) In the diagram, the circle is tangent to XY at Y and to XZ at Z. Point T is chosen on the minor arc YZ and a tangent to the circle is drawn at T, cutting XY at V and XZ at W. Prove that the perimeter of $\triangle VXW$ is independent of the position of T.



(b) In the diagram, $AB = 10$, $BC = 14$, $AC = 16$, and M is the midpoint of BC. Various lines can be drawn through M, cutting AB (possibly extended) at P and AC (possibly extended) at Q. Determine, with proof, the minimum possible perimeter of $\triangle APQ$.



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For students...

Thank you for writing the 2007 Euclid Contest!
In 2006, more than 15 000 students around the world registered to write the Euclid Contest.

If you are graduating from secondary school, good luck in your future endeavours!
If you will be returning to secondary school next year, encourage your teacher to register you for the 2007 Canadian Open Mathematics Challenge, which will be written in late November.
Visit our website

www.cemc.uwaterloo.ca

to find

- **More information about the Canadian Open Mathematics Challenge**
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- **Workshops to help you prepare for future Contests**
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