Canadian Mathematics Competition

An activity of the Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario

2006 Hypatia Contest

Thursday, April 20, 2006

Solutions

1. (a) *Solution 1*

The first odd positive integer is 1. The second odd positive integer is 3, which is 2 larger than the first. The third odd positive integer is 5, which is 2 larger than the second.
Therefore, the 25th odd positive integer will be $24 \times 2 = 48$ larger than the first odd positive integer, since we must add 2 to get to each successive odd number.
Thus, the 25th odd positive integer is $1 + 48 = 49$.
There are $1 + 2 + 3 + 4 + 5 + 6 = 21$ integers in the first six rows of the pattern, so 49 must appear in the 7th row.

Solution 2

The first odd positive integer is 1, which is 1 less than the first even positive integer, namely 2.
The second odd positive integer is 3, which is 1 less than the second even positive integer, namely 4.
This pattern continues, with the 25th odd positive integer being 1 less than the 25th even positive integer, which is $25 \times 2 = 50$.
Therefore, the 25th odd positive integer is 49.
There are $1 + 2 + 3 + 4 + 5 + 6 = 21$ integers in the first six rows of the pattern, so 49 must appear in the 7th row.

(b) In the triangular pattern, the first row contains 1 number, the second row 2 numbers, the third row 3 numbers, and so on.
Thus, the first twenty rows contain $1 + 2 + 3 + \cdots + 19 + 20$ numbers in total. This total equals $\frac{20(21)}{2} = 210$ (using the fact that $1 + 2 + \cdots + (n - 1) + n = \frac{n(n + 1)}{2}$).
Therefore, the 19th integer in the 21st row is the $210 + 19 = 229$th odd positive integer.
Using either of the methods of part (a), this integer is $1 + 228(2) = 229(2) - 1 = 457$.

(c) To get from 1 to 1001, we must add 2 a total of 500 times, so 1001 is the 501st odd positive integer.
From part (b), we know that the first 20 rows contain 210 integers, so the row number is larger than 20.
How many integers do the first 30 rows contain? They contain $1 + 2 + \cdots + 29 + 30 = \frac{30(31)}{2} = 465$ integers.
This tells us that the first 31 rows contain $465 + 31 = 496$ integers.
Since 1001 is the 501st odd positive integer, it must be the 5th integer in the 32nd row.

2. (a) We find CE by first finding BE.
Since $AE = 24$ and $\angle AEB = 60^\circ$, then $BE = 24\cos(60^\circ) = 24\left(\frac{1}{2}\right) = 12$.
(We could have also used the fact that $\triangle ABE$ is a 30°-60°-90° triangle and the appropriate ratios.)
Since $BE = 12$ and $\angle BEC = 60^\circ$, then $CE = 12\cos(60^\circ) = 12\left(\frac{1}{2}\right) = 6$.

(b) Using the same strategy as in (a),

$$\begin{aligned} AB &= 24\sin(60^\circ) = 24\left(\tfrac{\sqrt{3}}{2}\right) = 12\sqrt{3} \\ BC &= 12\sin(60^\circ) = 12\left(\tfrac{\sqrt{3}}{2}\right) = 6\sqrt{3} \\ CD &= 6\sin(60^\circ) = 6\left(\tfrac{\sqrt{3}}{2}\right) = 3\sqrt{3} \\ ED &= 6\cos(60^\circ) = 6\left(\tfrac{1}{2}\right) = 3 \end{aligned}$$

The perimeter of quadrilateral $ABCD$ is equal to $AB + BC + CD + DA$ and $DA = DE + EA$, so the perimeter is $12\sqrt{3} + 6\sqrt{3} + 3\sqrt{3} + 3 + 24 = 27 + 21\sqrt{3}$.

(c) The area of quadrilateral $ABCD$ is equal to the sum of the areas of triangles ABE, BCE and CDE.
Thus,

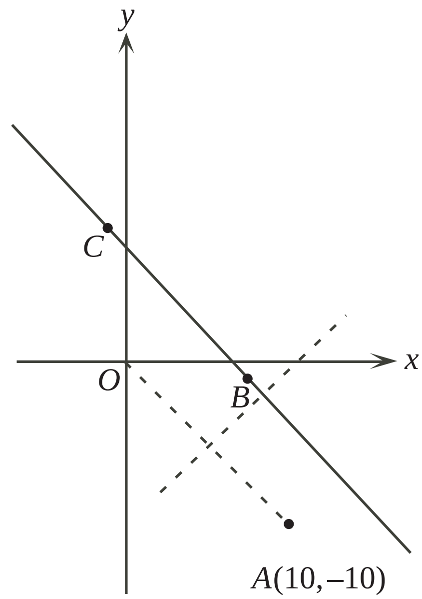
$$\begin{aligned} \text{Area} &= \tfrac{1}{2}(BE)(BA) + \tfrac{1}{2}(CE)(BC) + \tfrac{1}{2}(DE)(DC) \\ &= \tfrac{1}{2}(12)(12\sqrt{3}) + \tfrac{1}{2}(6)(6\sqrt{3}) + \tfrac{1}{2}(3)(3\sqrt{3}) \\ &= 72\sqrt{3} + 18\sqrt{3} + \tfrac{9}{2}\sqrt{3} \\ &= \tfrac{189}{2}\sqrt{3} \end{aligned}$$

3. (a) The line through points B and C has slope $\dfrac{-1-7}{7-(-1)} = -1$.
Since $(7,-1)$ lies on the line, the line has equation $y - (-1) = -1(x-7)$ or $y = -x + 6$.

(b) *Solution 1*
Suppose P has coordinates (x, y). Since P lies on ℓ, then $y = -x+6$, so P has coordinates $(x, -x+6)$.
Since A has coordinates $(10, -10)$, then $PA = \sqrt{(x-10)^2 + (-x+16)^2}$.
Since O has coordinates $(0, 0)$, then $PO = \sqrt{x^2 + (-x+6)^2}$.
For $PA = PO$, we must have $PA^2 = PO^2$ or

$$\begin{aligned} (x-10)^2 + (-x+16)^2 &= x^2 + (-x+6)^2 \\ x^2 - 20x + 100 + x^2 - 32x + 256 &= x^2 + x^2 - 12x + 36 \\ 320 &= 40x \\ x &= 8 \end{aligned}$$

so P has coordinates $(8, -2)$.



Solution 2

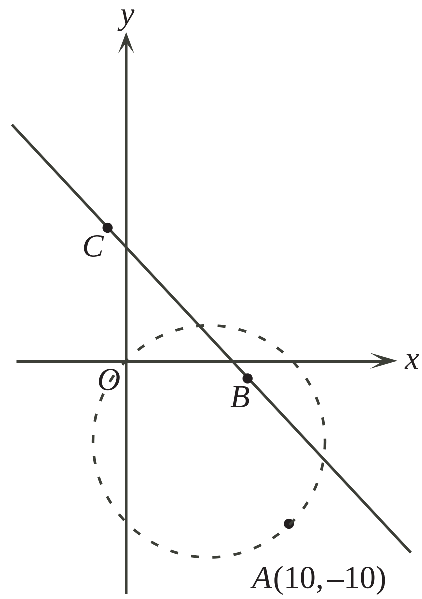
For P to be equidistant from A and O, P must lie on the perpendicular bisector of AO. Since A has coordinates $(10,-10)$ and O has coordinates $(0,0)$, then AO has slope -1, so the perpendicular bisector has slope 1 and passes through the midpoint, $(5,-5)$, of AO. Therefore, the perpendicular bisector has equation $y-(-5)=x-5$ or $y=x-10$.
Thus, P must be the point of intersection of the lines $y=x-10$ and the line ℓ which has equation $y=-x+6$, so $-x+6=x-10$ or $2x=16$ or $x=8$.
Therefore, P has coordinates $(8,-2)$.

(c) *Solution 1*

Since Q lies on the line ℓ, then its coordinates are of the form $(q,-q+6)$, as in (b).
For $\angle OQA=90^\circ$, we need the slopes of OQ and QA to be negative reciprocals (that is, the product of the slopes is -1).
The slope of OQ is $\dfrac{-q+6}{q}$.
The slope of QA is $\dfrac{-q+6-(-10)}{q-10}=\dfrac{-q+16}{q-10}$.
Therefore, we must solve

$$\begin{aligned}
\frac{-q+6}{q}\cdot\frac{-q+16}{q-10} &= -1\\
(-q+6)(-q+16) &= -q(q-10)\\
q^2-22q+96 &= -q^2+10q\\
2q^2-32q+96 &= 0\\
q^2-16q+48 &= 0\\
(q-4)(q-12) &= 0
\end{aligned}$$

Therefore, $q=4$ or $q=12$, so Q has coordinates $(4,2)$ or $(12,-6)$.



Solution 2

For $\angle OQA = 90^\circ$, then Q must lie on the circle with diameter OA.
The centre of the circle with diameter OA is the midpoint M of OA, or $(5, -5)$.
The radius of the circle with diameter OA is OM or $\sqrt{5^2 + (-5)^2} = \sqrt{50}$.
Therefore, this circle has equation $(x-5)^2 + (y+5)^2 = 50$.
The points Q that we are seeking are the points on both the circle and the line ℓ, ie. the points of intersection. Since $y = -x + 6$,

$$\begin{aligned} (x-5)^2 + (-x+6+5)^2 &= 50 \\ x^2 - 10x + 25 + x^2 - 22x + 121 &= 50 \\ 2x^2 - 32x + 96 &= 0 \\ x^2 - 16x + 48 &= 0 \\ (x-4)(x-12) &= 0 \end{aligned}$$

Therefore, $x = 4$ or $x = 12$, so Q has coordinates $(4, 2)$ or $(12, -6)$.

4. (a) Suppose p is a prime number.
The only positive divisors of p are 1 and p, so $\sigma(p) = 1 + p$.
Thus,

$$I(p) = \frac{1+p}{p} = \frac{1}{p} + 1 \leq \tfrac{1}{2} + 1 = \tfrac{3}{2}$$

since $p \geq 2$.

(b) *Solution 1*

Suppose that p is an odd prime number and k is a positive integer. Note that $p \geq 3$.
The positive divisors of p^k are $1, p, p^2, \ldots, p^{k-1}, p^k$, so

$$I(p^k) = \frac{1 + p + p^2 + \cdots + p^k}{p^k} = \frac{1}{p^k}\left(\frac{1(p^{k+1}-1)}{p-1}\right) = \frac{p^{k+1}-1}{p^{k+1}-p^k}$$

Thus,

$$\begin{array}{rrcl} & I(p^k) & < & 2 \\ \iff & \dfrac{p^{k+1}-1}{p^{k+1}-p^k} & < & 2 \\ \iff & p^{k+1}-1 & < & 2(p^{k+1}-p^k) \\ \iff & 0 & < & p^{k+1}-2p^k+1 \\ \iff & 0 & < & p^k(p-2)+1 \end{array}$$

which is true since $p \geq 3$.

Solution 2

Suppose that p is an odd prime number and k is a positive integer. Note that $p \geq 3$. The positive divisors of p^k are 1, p, p^2, …, p^{k-1}, p^k, so

$$\begin{aligned} I(p^k) &= \frac{p^k+p^{k-1}+\cdots+p+1}{p^k} \\ &= 1+\frac{1}{p}+\cdots+\frac{1}{p^{k-1}}+\frac{1}{p^k} \\ &= \frac{1\left(1-\left(\dfrac{1}{p}\right)^{k+1}\right)}{1-\dfrac{1}{p}} \qquad \text{(geometric series)} \\ &< \frac{1}{1-\dfrac{1}{p}} \\ &= \frac{p}{p-1} \\ &= 1+\frac{1}{p-1} \\ &\leq 1+\frac{1}{2} \\ &< 2 \end{aligned}$$

(c) Since p is a prime number, the positive divisors of p^2 are 1, p and p^2, so $I(p^2) = \dfrac{1+p+p^2}{p^2}$.

Since q is a prime number, $I(q) = \dfrac{1+q}{q}$, as in (a).

Since p and q are prime numbers, the positive divisors of p^2q are 1, p, p^2, q, pq, and p^2q, so

$$\begin{aligned} I(p^2q) &= \frac{1+p+p^2+q+pq+p^2q}{p^2q} \\ &= \frac{(1+p+p^2)+q(1+p+p^2)}{p^2q} \\ &= \frac{(1+p+p^2)(1+q)}{p^2q} \\ &= \frac{1+p+p^2}{p^2}\cdot\frac{1+q}{q} \\ &= I(p^2)I(q) \end{aligned}$$

as required.

(d) We start by listing a number of facts:

- From part (b), if n is a prime number, then $I(n) < 2$. Thus, to obtain an n with $I(n) > 2$, we must combine multiple primes.
- If $n = {p_1}^{e_1}{p_2}^{e_2}\cdots {p_m}^{e_m}$ where the p_i's are distinct primes and the e_i's are positive integers, then by extending the ideas of (c), we can see that $I(n) = I({p_1}^{e_1})I({p_2}^{e_2})\cdots I({p_m}^{e_m})$.
- If p and q are primes with $p < q$, then $I(p^k) > I(q^k)$.
(This is because $I(p^k) = \dfrac{p^k + p^{k-1} + \cdots + p + 1}{p^k} = 1 + \dfrac{1}{p} + \cdots + \dfrac{1}{p^{k-1}} + \dfrac{1}{p^k}$ and $I(q^k) = 1 + \dfrac{1}{q} + \cdots + \dfrac{1}{q^{k-1}} + \dfrac{1}{q^k}$. If $p < q$, then $\dfrac{1}{p} > \dfrac{1}{q}$, $\dfrac{1}{p^2} > \dfrac{1}{q^2}$, etc., so $I(p^k) > I(q^k)$.)
- The previous point tells us that smaller primes are more efficient than larger primes at increasing $I(n)$. In other words, given an n, we can find an m with $I(m) > I(n)$ by replacing some of the prime factors of n with smaller primes to obtain m. (For example, if $n = 5^2 7^3 11$, then $m = 5^2 3^3 7$ would gives $I(m) > I(n)$.)
- From (b), $I(p^k) < \dfrac{p}{p-1}$ so in particular $I(3^a) < \frac{3}{2}$ and $I(5^b) < \frac{5}{4}$. Therefore, $I(3^a 5^b) = I(3^a)I(5^b) < \frac{3}{2}\cdot\frac{5}{4} = \frac{15}{8} < 2$.
- The previous two points tell us that no odd integer n with at most two prime factors can have $I(n) > 2$. Therefore, to get $I(n) > 2$, n must have at least three prime factors.

Let us consider integers of the form $n = 3^a 5^b 7^c$ and try to find one with $I(n) > 2$:

a	b	c	n	$I(n)$
1	1	1	105	$I(n) = I(3)I(5)I(7) = \frac{4}{3}\cdot\frac{6}{5}\cdot\frac{8}{7} = \frac{64}{35} < 2$
2	1	1	315	$I(n) = I(3^2)I(5)I(7) = \frac{13}{9}\cdot\frac{6}{5}\cdot\frac{8}{7} = \frac{208}{105} < 2$
3	1	1	945	$I(n) = I(3^3)I(5)I(7) = \frac{40}{27}\cdot\frac{6}{5}\cdot\frac{8}{7} = \frac{128}{63} > 2$

So $I(945) > 2$.

Why is $n = 945$ the smallest odd integer with $I(n) > 2$?

- We notice that any positive odd integer with at least 4 prime factors is at least $3(5)(7)(11)$ or 1155, which is larger than 945, so we can restrict to looking at integers with at most 3 prime factors.

- By our opening remarks, we can restrict our search even further, looking at only those odd integers with 3, 5 and 7 as possible prime factors, since we can decrease the primes and make $I(n)$ larger at the same time.
- Furthermore, we only need to consider those integers $n = 3^{e_1}5^{e_2}7^{e_3}$ with $e_1 \geq e_2 \geq e_3$, since otherwise we could reassign the exponents in this order and obtain a smaller integer. (For example, $3^2 5^3 7$ is larger than $3^3 5^2 7$.)
- Since n must have at least three prime factors and we can assume that these prime factors are 3, 5 and 7 and that the exponents have the property from the previous point, then there are no integers smaller than 945 left to check other than the ones in the table above (none of which have $I(n) > 2$).

Therefore, $n = 945$ is the smallest odd positive integer with $I(n) > 2$.