Canadian Mathematics Competition
An activity of the Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario

2006 Fryer Contest

Thursday, April 20, 2006

Solutions

1. (a) Her average mark in the seven courses is

$$\frac{94+93+84+81+74+83+79}{7} = \frac{588}{7} = 84$$

(b) Samantha's highest possible average would be if she obtained a mark of 100 in French. In this case, her average in the eight courses would be

$$\frac{94+93+84+81+74+83+79+100}{8} = \frac{688}{8} = 86$$

(c) *Solution 1*
If Samantha's final average over all eight courses is 85, then the sum of her marks in the eight courses is $8 \times 85 = 680$.
In part (a), we saw that the sum of her first seven marks is 588, so her mark in French is $680 - 588 = 92$.

Solution 2
Samantha's average in the first seven courses is 84, so if her French mark was 84, then her average would remain as 84.
If Samantha's mark in French was 100 (as in (b)), her average would be 86.
Since her average is 85 (half-way between these two averages), then her French mark is half-way between 84 and 100, or 92.

2. (a) The bottom layer of the cube is a 7 by 7 square of cubes, so uses $7 \times 7 = 49$ cubes.
The next layer of the cube is a 5 by 5 square of cubes, so uses $5 \times 5 = 25$ cubes.
The next layer of the cube is a 3 by 3 square of cubes, so uses $3 \times 3 = 9$ cubes.
The top layer consists of a single cube.
Therefore, the total number of cubes used is $49 + 25 + 9 + 1 = 84$.

(b) *Solution 1*
The cube in the top layer has 5 visible faces (only the bottom face is hidden).
In the second layer from the top, the 4 corner cubes each have 3 visible faces (for 12 faces in total), and there is 1 cube on each of the four sides of the layer with 2 visible faces (another 8 visible faces).
In the second layer from the bottom, the 4 corner cubes each have 3 visible faces (for 12 faces in total), and there are 3 cubes on each of the four sides of the layer with 2 visible faces (another $4 \times 3 \times 2 = 24$ visible faces).
In the bottom layer, the 4 corner cubes each have 3 visible faces (for 12 faces in total), and there are 5 cubes on each of the four sides of the layer with 2 visible faces (another $4 \times 5 \times 2 = 40$ visible faces).
Therefore, the total number of visible faces is $5 + 12 + 8 + 12 + 24 + 12 + 40 = 113$.

Solution 2
When we look at the pyramid from the top, we see a 7×7 square of visible faces, or 49 visible faces. (This square is composed of faces from all of the levels of the pyramid.)
When we look at the pyramid from each of the four sides, we see $1+3+5+7 = 16$ visible faces, so there are $4(16) = 64$ visible faces on the sides.
Therefore, there are $49 + 64 = 113$ visible faces in total.

(c) *Solution 1*
To make the total of all of the visible numbers as large as possible, we should position the cubes so that the largest possible two, three or five numbers are visible, depending on its position.

For the top cube (with 5 visible faces), we position this cube with the "1" on the bottom face (and so is hidden).
The total of the numbers visible on this cube is $2 + 3 + 4 + 5 + 6 = 20$.

For the 4 corner cubes on each layer (each with 3 visible faces), we position these cubes with the 4, 5 and 6 all visible (this is possible since the faces with the 4, 5 and 6 share a vertex) and the 1, 2 and 3 hidden.
There are 12 of these cubes, so the total of the numbers visible on these cubes is $12 \times (4 + 5 + 6) = 180$.

For the cubes on the sides (that is, not at the corner) of each layer, we position the cubes with the 5 and 6 visible (this is possible since the faces with 5 and 6 share an edge) and the 1, 2, 3, and 4 hidden.
There are $4 + 12 + 20 = 36$ of these cubes, so the total of the numbers visible on these cubes is $36 \times (5 + 6) = 396$.

Therefore, the overall largest possible total is $20 + 180 + 396 = 596$.

Solution 2
To make the total of all of the visible numbers as large as possible, we should position the cubes so that the largest possible two, three or five numbers are visible, depending on its position.

As in Solution 2 to (b), view the pyramid from the top. Position the cubes so that each top faces which is visible is 6 (for a total of $49 \times 6 = 294$).

Consider next the top cube. It has only one hidden face, which will be the 1, since the 6 is on top. (This does maximize the sum of the visible faces.) This adds $2+3+4+5 = 14$ to the total of the visible faces thus far.

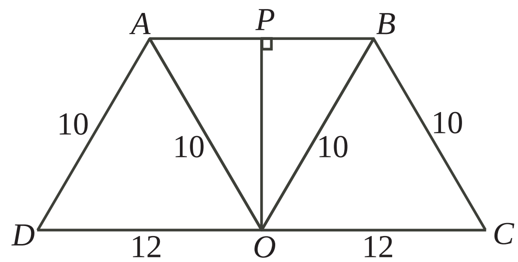
Consider lastly the faces visible on the sides (a total of $4(3 + 5 + 7) = 60$ faces). To maximize the sum of the numbers on these faces, we would like to make them all 5 (since we have already used the 6s). This would give a total of $5 \times 60 = 300$. However, there are 12 corner cubes to which we have assigned two 5s, so we must change one of the 5s on each to a 4, decreasing the total by 12. (This is possible, since the 4, 5 and 6 meet at a vertex on each cube.)

Therefore, the overall largest possible total is $294 + 14 + 300 - 12 = 596$.

3. (a) Since $\triangle AOB$ is isosceles with $AO = OB$ and OP is perpendicular to AB, then P is the midpoint of AB, so $AP = PB = \frac{1}{2}AB = \frac{1}{2}(12) = 6$.
By the Pythagorean Theorem, $OP = \sqrt{AO^2 - AP^2} = \sqrt{10^2 - 6^2} = \sqrt{64} = 8$.

(b) *Solution 1*

Trapezoid $ABCD$ is formed from three congruent triangles, so its area is three times the area of one of these triangles.



Each triangle has a base of length 12 and a height of length 8 (from (a), since OP is one of these heights), so has area $\frac{1}{2}(12)(8) = 48$.
Therefore, the area of the trapezoid is $3 \times 48 = 144$.

Solution 2

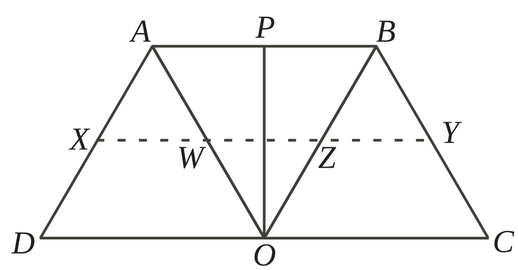
Since $ABCD$ is a trapezoid with height of length 8 (OP is the height of $ABCD$) and parallel sides (AB and DC) of length 12 and 24, then its area is

$$\frac{1}{2} \times \text{ Height } \times \text{ Sum of parallel sides} = \frac{1}{2}(8)(12 + 24) = 144$$

(c) *Solution 1*

Since XY cuts AD and BC each in half, then it also cuts the height PO in half:

> Since XY is parallel to AB and DC and cuts each of AD and BC at its midpoint, then it must cut each of AO and BO at its midpoint (W and Z, respectively).
> Therefore, $\triangle WZO$ is similar to $\triangle ABO$ and its side lengths are half of those of $\triangle ABO$.
> Thus, its height is half of PO.
> In a similar way, the dimensions of $\triangle AXW$ and $\triangle BYZ$ are half of those of $\triangle ADO$ and $\triangle BCO$.



Thus, each of the two smaller trapezoids has height 4.
Also, since $XW = \frac{1}{2}DO$, $WZ = \frac{1}{2}AB$ and $ZY = \frac{1}{2}OC$, then $XY = 3(6) = 18$.
Using the formula for the area of a trapezoid from (b), the area of trapezoid $ABYX$ is $\frac{1}{2}(4)(12 + 18) = 60$ and the area of trapezoid $XYCD$ is $\frac{1}{2}(4)(18 + 24) = 84$.
Thus, the ratio of their areas is $60 : 84 = 5 : 7$.

Solution 2

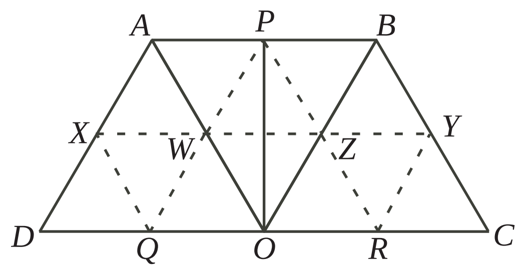
Since XY cuts AD and BC each in half, then it also cuts the height PO in half.
Thus, each of the two smaller trapezoids has height 4.
Next, we find the length of XY.

From (b), we know how to compute the area of a trapezoid and we know that the sum of the areas of trapezoids $ABYX$ and $XYCD$ must equal that of trapezoid $ABCD$.
Therefore,

$$\begin{aligned} \tfrac{1}{2}(4)(AB+XY)+\tfrac{1}{2}(4)(XY+DC) &= 144 \\ 2(12+XY)+2(XY+24) &= 144 \\ 4(XY) &= 72 \\ XY &= 18 \end{aligned}$$

Thus, the area of trapezoid $ABYX$ is $\frac{1}{2}(4)(12+18)=60$ and the area of trapezoid $XYCD$ is $\frac{1}{2}(4)(18+24)=84$.
Thus, the ratio of their areas is $60:84=5:7$.

Solution 3
Let Q and R be the midpoints of DO and OC, respectively, and W and Z the points where XY crosses AO and BO, respectively.
Join X and W to Q, Z and Y to R, and W and Z to P.



We have now divided trapezoid $ABCD$ into 12 congruent triangles:

> To show this, we consider the partitioning of $\triangle AOD$ into four small triangles.
> Because X, W and Q are the midpoints of AD, AO and OD, respectively, then XW, WQ and QX are parallel to DO, AD and OA, respectively.
> Therefore, triangles AXW, XDQ, WQO, and QWX are congruent, because each has sides of length 5, 5 and 6.
> The same is true for the partitioning of $\triangle ABO$ and $\triangle BOC$.

Since $ABYX$ is made up of 5 of these triangles and $XYCD$ is made up of 7 of these triangles and each of these 12 triangles has equal area, then the ratio of the area of $ABYX$ to the area of $XYCD$ is $5:7$.

4. (a) *Solution 1*
There are 100 integers from 1 to 100.
Of these, 10 end with the digit 7: 7, 17, 27, 37, 47, 57, 67, 77, 87, 97.
Also, 10 begin with the digit 7: 70, 71, 72, 73, 74, 75, 76, 77, 78, 79.
But 77 is included in both lists, so we need to be careful not to count it twice. Therefore, we need to subtract $10+10-1=19$ of the 100 integers.
Therefore, the number of these integers which do not contain 7 is $100-19=81$.

Solution 2
The integer 100 does not contain the digit 7, nor does the integer 0, so we replace 100 with 0 in the list and count the number of integers from 0 to 99 which do not contain the

digit 7.
Each of these integers can be written as a two-digit integer (where we allow the first digit to be 0): 00, 01, 02, . . ., 98, 99.
Since we would like all integers not containing the digit 7, there are 9 possibilities for the first digit (0 through 9, excluding 7) and for each of these possibilities, there are 9 possibilities for the second digit.
Therefore, there are $9 \times 9 = 81$ integers in this range which do not contain the digit 7.

(b) *Solution 1*
From part (a), there are 81 integers not containing the digit 7 between 1 and 100.
Similarly, there will be 81 such integers in each of the ranges 101 to 200, 201 to 300, 301 to 400, 401 to 500, and 501 to 600. (We can conclude this since the first digit is not a 7, and the tens and units digits follow the same pattern as in (a).)
From 601 to 700, there will be 80 such integers (since 700 includes 7).
From 701 to 800, there is only 1 such integer (namely, 800 – all others contain a 7).
In each of the intervals 801 to 900, 901 to 1000, 1001 to 1100, 1101 to 1200, 1201 to 1300, 1301 to 1400, 1401 to 1500, and 1501 to 1600, there are 81 such integers.
From 1601 to 1700, there are 80; from 1701 to 1800, there is 1; from 1801 to 1900 and 1901 to 2000, there are 81.
Therefore, the total number of such integers is $16(81) + 2(80) + 2(1) = 18(81) = 1458$.

Solution 2
The integer 2000 does not contain the digit 7, nor does the integer 0, so we can replace 2000 with 0 in the list and count the number of integers from 0 to 1999 which do not contain the digit 7.
Each of these integers can be written as a four-digit integer $\underline{a}\,\underline{b}\,\underline{c}\,\underline{d}$, where the integer is allowed to begin with one or more zeros, with a allowed to be 0 or 1, and each of b, c and d allowed to be 0, 1, 2, 3, 4, 5, 6, 8, or 9.
Thus, there are 2 possibilities for the first digit; for each of these choices, there are 9 choices for the second digit; for each of these choices, there are 9 choices for the third digit; for each of these choices, there are 9 choices for the fourth digit. The total number of such integers is thus $2 \times 9 \times 9 \times 9 = 1458$.

(c) *Solution 1*
In this solution, we will repeatedly use the fact that the sum of the integers from 1 to n is $\frac{1}{2}n(n + 1)$.
Consider first the integers from 1 to 100.
The sum of these integers is $\frac{1}{2}(100)(101) = 5050$.
The integers in this set which do contain the digit 7 are 7, 17, 27, 37, 47, 57, 67, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 87, 97, whose sum is 1188.
Therefore, the sum of the integers from 1 to 100 which do not contain the digit 7 is $5050 - 1188 = 3862$.

There are 81 numbers from 101 to 200 not containing the digit 7 as well. Each of these is 100 more than a corresponding number between 1 and 100 which does not contain the digit 7, so the sum of these 81 numbers is $3862 + 81(100)$.
We can use this approach to determine the sum of the appropriate numbers in each range of 100, as shown in the table:

Range	Number of integers not containing 7	Sum
1 to 100	81	3862
101 to 200	81	$3862 + 81(100)$
201 to 300	81	$3862 + 81(200)$
301 to 400	81	$3862 + 81(300)$
401 to 500	81	$3862 + 81(400)$
501 to 600	81	$3862 + 81(500)$
601 to 700	80	$3862 + 81(600) - 700$
701 to 800	1	800
801 to 900	81	$3862 + 81(800)$
901 to 1000	81	$3862 + 81(900)$
1001 to 1100	81	$3862 + 81(1000)$
1101 to 1200	81	$3862 + 81(1100)$
1201 to 1300	81	$3862 + 81(1200)$
1301 to 1400	81	$3862 + 81(1300)$
1401 to 1500	81	$3862 + 81(1400)$
1501 to 1600	81	$3862 + 81(1500)$
1601 to 1700	80	$3862 + 81(1600) - 1700$
1701 to 1800	1	1800
1801 to 1900	81	$3862 + 81(1800)$
1901 to 2000	81	$3862 + 81(1900)$
2001 to 2006	6	$2001 + 2002 + 2003 + 2004 + 2005 + 2006$

Therefore, the overall sum is

$$\begin{aligned}
& 18(3862) \\
& \quad +81(100)(1+2+3+4+5+6+8+9+10+11+12+13+14+15+16+18+19) \\
& \quad -700+800-1700+1800+(2001+2002+2003+2004+2005+2006) \\
= \; & 69516 + 8100(166) + 200 + 12\,021 \\
= \; & 1\,426\,337
\end{aligned}$$

Solution 2

Consider first the integers from 000 to 999 that do not contain the digit 7. (We can include 000 in this list as it will not affect the sum.)

Since each of the three digits has 9 possible values, there are $9 \times 9 \times 9 = 729$ such integers.

If we fix any specific digit in any of the three positions, there will be exactly 81 integers with that digit in that position, as there are 9 possibilities for each of the remaining digits. (For example, there are 81 such integers ending in 0, 81 ending in 1, etc.)

We sum these integers by first summing the units digits, then summing the tens digits, and then summing the hundreds digits.

Since each of the 9 possible units digits occurs 81 times, the sum of the units digits column is

$$81(0) + 81(1) + 81(2) + 81(3) + 81(4) + 81(5) + 81(6) + 81(8) + 81(9) = 81(38)$$

Since each of the 9 possible tens digits occurs 81 times, the sum of the tens digits column is $81(0 + 10 + 20 + 30 + 40 + 50 + 60 + 80 + 90) = 81(380)$.

Similarly, the sum of the hundreds digits column is 81(3800).
Thus, the sum of the integers from 0 to 999 that do not contain the digit 7 is

$$81(38) + 81(380) + 81(3800) = 81(38)(1 + 10 + 100) = 81(38)(111) = 341\,658$$

Each of the 729 integers from 1000 to 1999 which do not contain 7 is 1000 more than such an integer between 0 and 999. (There are again 729 of these integers as the first digit is fixed at 1, and each of the remaining three digits has 9 possible values.)
Thus, the sum of these integers from 1000 to 1999 is equal to the sum of the corresponding ones from 0 to 999 plus 729(1000), or $729\,000 + 341\,658 = 1\,070\,658$.
Each of the integers from 2000 to 2006 does not contain a 7, and their sum is
$2000 + 2001 + 2002 + 2003 + 2004 + 2005 + 2006 = 7(2003) = 14\,021$.
Therefore, the overall sum is $341\,658 + 1\,070\,658 + 14\,021 = 1\,426\,337$.