Canadian
Mathematics
Competition
An activity of the Centre for Education
in Mathematics and Computing,
University of Waterloo, Waterloo, Ontario

2005 Cayley Contest

(Grade 10)

Wednesday, February 23, 2005

Solutions

1. Simplifying, $a+1+a-2+a+3+a-4 = a+a+a+a+1-2+3-4 = 4a-2$.

Answer: (C)

2. Cancelling common factors in the numerators and denominators,

$$\left(\frac{4}{5}\right)\left(\frac{5}{6}\right)\left(\frac{6}{7}\right)\left(\frac{7}{8}\right)\left(\frac{8}{9}\right) = \left(\frac{4}{\cancel{5}}\right)\left(\frac{\cancel{5}}{\cancel{6}}\right)\left(\frac{\cancel{6}}{\cancel{7}}\right)\left(\frac{\cancel{7}}{\cancel{8}}\right)\left(\frac{\cancel{8}}{9}\right) = \frac{4}{9}$$

Answer: (A)

3. The largest multiple of 17 less than 70 is 68. Therefore, $70 = 4(17)+2$, so the remainder is 2.

Answer: (D)

4. Since $\dfrac{3}{x+10} = \dfrac{1}{2x}$, then cross-multiplying, we get $6x = x+10$ or $5x = 10$ or $x = 2$.

Answer: (D)

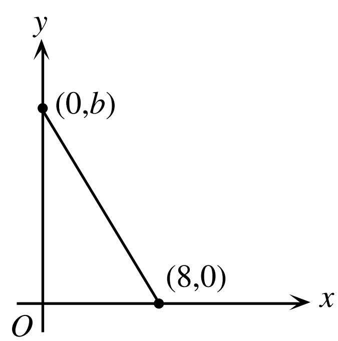
5. Calculating, $(5^2-4^2)^3 = (25-16)^3 = 9^3 = 729$.

Answer: (E)

6. 8 volunteers who each work 40 hours and each raise \$18 per hour will raise $8 \times 40 \times 18 = \$5760$. If 12 volunteers each work 32 hours and raise a total of \$5760, then they each raise $\dfrac{5760}{12 \times 32}$ or \$15 per hour.

Answer: (C)

7. *Solution 1*
Since the slope is $-\frac{3}{2}$, then for every 2 units we move to the right along the line, we must move 3 units down.



To get from $(0,b)$ to $(8,0)$, we move 8 units right, or 2 units right four times.
Therefore, we must move 3 units down four times, or 12 units down in total.
Therefore, $b = 12$.

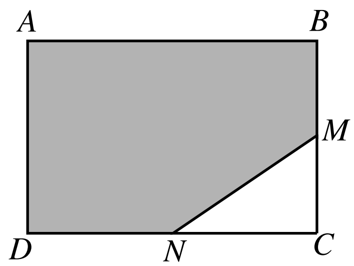
Solution 2
Since the slope of the line is $-\frac{3}{2}$, then $\frac{b-0}{0-8} = -\frac{3}{2}$ or $-\frac{b}{8} = -\frac{3}{2}$ or $b = 8 \times \frac{3}{2} = 12$.

Answer: (B)

8. Jack ran a total of 24 km.
Since he ran the first 12 km at 12 km/h, then it took him 1 hour to run the first 12 km.
Since he ran the second 12 km at 6 km/h, then it took him 2 hours to run the second 12 km.
Therefore, his run took a total of 3 hours.
Thus, Jill ran 24 km in 3 hours, so her speed was 8 km/h.

ANSWER: (A)

9. *Solution 1*
Since M is the midpoint of BC and $CM = 4$, then $BC = 8$.

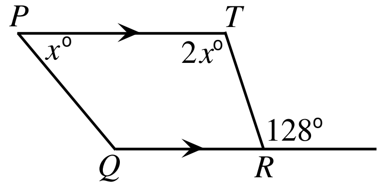


Since N is the midpoint of CD and $NC = 5$, then $CD = 10$.
Since $ABCD$ is a rectangle, its area is $10 \times 8 = 80$.
Also, the area of $\triangle NCM$ is $\frac{1}{2}(4)(5) = 10$, so the shaded area of the rectangle is the area of the whole rectangle minus the area of $\triangle NCM$, or 70.
Thus, the fraction of the rectangle that is shaded is $\dfrac{70}{80} = 0.875$, so 87.5% of the area is shaded.

Solution 2
The calculation from Solution 1 can also be done more generally.
Suppose $BC = 2x$ and $CD = 2y$.
Since M is the midpoint of BC, then $CM = x$.
Since N is the midpoint of CD, then $NC = y$.
Since $ABCD$ is a rectangle, its area is $(2x)(2y) = 4xy$.
Also, the area of $\triangle NCM$ is $\frac{1}{2}(NC)(MC) = \frac{1}{2}xy$, so the shaded area of the rectangle is the area of the whole rectangle minus the area of $\triangle NCM$, or $4xy - \frac{1}{2}xy = \frac{7}{2}xy$.
Thus, the fraction of the rectangle that is shaded is $\dfrac{\frac{7}{2}xy}{4xy} = \dfrac{7}{8} = 0.875$, so 87.5% of the area is shaded.

ANSWER: (D)

10. *Solution 1*
Since PT and RQ are parallel, then $2x^\circ = 128^\circ$, so $x = 64$, so $\angle TPQ = 64^\circ$.



Since PT and QR are parallel, then $\angle TPQ$ and $\angle PQR$ are supplementary.
Thus, $\angle PQR + 64^\circ = 180^\circ$, so $\angle PQR = 116^\circ$.

Solution 2
Since the two angles at R add to 180°, then $\angle QRT + 128^\circ = 180^\circ$, so $\angle QRT = 52^\circ$.
Since PT and QR are parallel, then $\angle PTR$ and $\angle QRT$ are supplementary, so $2x^\circ + 52^\circ = 180^\circ$ or $2x^\circ = 128^\circ$ or $x = 64$.
Therefore, three of the angles of quadrilateral $PQRT$ are 64°, 128° and 52°.
Since the angles in a quadrilateral add to 360°, then $\angle PQR = 360^\circ - 64^\circ - 128^\circ - 52^\circ = 116^\circ$.

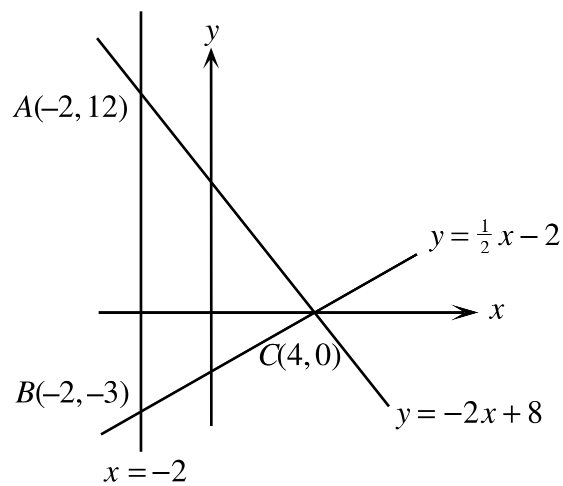
ANSWER: (A)

11. *Solution 1*
Matt's longest kick was 6 metres more than the average.
Thus, the other two kicks must be six metres less than the average when combined (that is, when we add up the difference between each of these kicks and the average, we get 6).
Since the other two kicks were the same length, then they each must have been 3 metres less than the average, or 34 metres each.

Solution 2
Since Matt's three kicks averaged 37 metres, then the sum of the lengths of the three kicks was $3 \times 37 = 111$ metres.
Let x be the length of each of the two kicks of unknown length.
Then $43 + 2x = 111$ or $x = 34$.

ANSWER: (D)

12. We first determine where the lines $y = -2x + 8$ and $y = \frac{1}{2}x - 2$ cross the line $x = -2$.
For the line $y = -2x + 8$, when $x = -2$, $y = -2(-2) + 8 = 12$, so the point of intersection is $(-2, 12)$.
For the line $y = \frac{1}{2}x - 2$, when $x = -2$, $y = \frac{1}{2}(-2) - 2 = -3$, so the point of intersection is $(-2, -3)$.

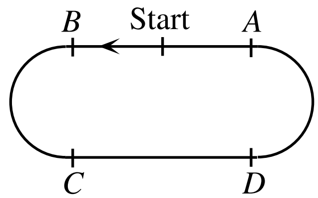


Therefore, we can think of $\triangle ABC$ as having base AB of length $12 - (-3) = 15$ and height being the distance from C to the line segment AB, or $4 - (-2) = 6$.
Therefore, the area of $\triangle ABC$ is $\frac{1}{2}(15)(6) = 45$.

ANSWER: (E)

13. If Andrew walks 1.4 metres per second, then he walks $60 \times 1.4 = 84$ metres per minute.
Since Andrew is walking for 30 minutes, then he walks a total of $30 \times 84 = 2520$ m.

Now the total length of track is 400 m, so after walking 2400 m, Andrew is back at the Start line.
Since the points A, B, C, and D are equally spaced, then consecutive points are 100 m apart.
Since the Start is half-way between A and B, then the Start is 50 m from B.



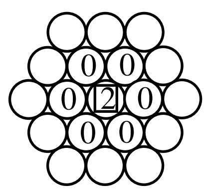
Therefore, after walking 2450 m, Andrew is at B.
After walking 70 m more to get to his total of 2520 m, Andrew will be 70 m beyond B and 30 m from C, so he will be closest to C.

Answer: (C)

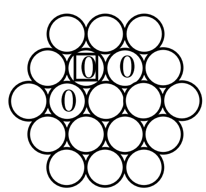
14. To make $\sqrt{1+2+3+4+x}$ an integer, we need $1+2+3+4+x = 10+x$ to be a perfect square.
So we can rephrase the question as "For how many positive integers x less than 100 is $10+x$ a perfect square?".
Since x is between 1 and 99, then $10+x$ is between 11 and 109.
There are 7 perfect squares in this interval: 16, 25, 36, 49, 64, 81, and 100, so there are 7 possible values of x: 6, 15, 26, 39, 54, 71, and 90.

Answer: (B)

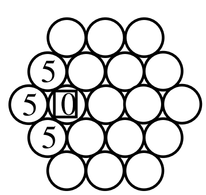
15. From the 2 in the centre, there are 6 possible 0s to which we can move.



From any 0, there are 2 possible 0s to which we can move.



From any 0, there are 3 possible 5s to which we can move.



For each of the 6 choices of the first 0, we can choose either of the 2 choices for the second 0, and from whichever second 0 is chosen we can choose any of the 3 possible 5s.
Therefore, there are $6 \times 2 \times 3 = 36$ possible paths that can be followed.

Answer: (A)

16. A good first step is to write out more terms in the sequence to see if we see a pattern:

$$88, 24, 64, 40, 24, 16, 8, 8, 0, 8, 8, 0, 8, 8, 0, 8, \ldots$$

So after some beginning terms, the sequence starts to repeat blocks of "8, 8, 0". (We can see that this will always happen: after "8, 0", the next term is $8 - 0 = 8$, so we get "8, 0, 8"; after "0, 8", the next term is $8-0 = 8$, so we get "8, 0, 8, 8"; after "8, 8", the next term is $8-8 = 0$, so we get "8, 0, 8, 8, 0", so the pattern continues.)
So in the first 100 numbers we have the first 6 terms (88, 24, 64, 40, 24, 16), and then 31 blocks of "8, 8, 0" (93 terms in total), and then the 100th term will be the beginning of a new block "8, 8, 0" (ie. the number 8).
Therefore, the sum of the first 100 terms is

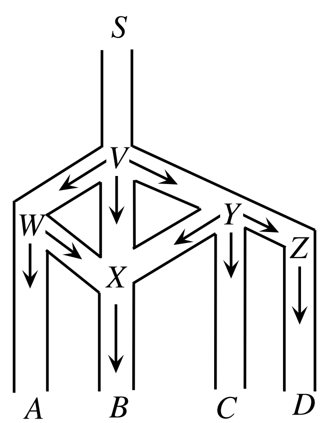
$$88 + 24 + 64 + 40 + 24 + 16 + 31(8 + 8 + 0) + 8 = 256 + 31(16) + 8 = 760$$

Answer: (B)

17. Using exponent laws, $1000^{100} = (10^3)^{100} = 10^{300} = (10^{100})^3 = \text{googol}^3$.

Answer: (E)

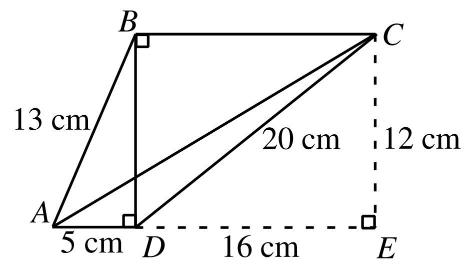
18. We label the five junctions as V, W, X, Y, and Z.



From the arrows which Harry can follow, we see that in order to get to B, he must get to X. So we calculate the probability that he gets to X.
To get to X, Harry can go S to V to W to X, or S to V to Y to X, or S to V to X directly.
At V, the probability that Harry goes down any of the three paths (that is, towards W, X or Y) is $\frac{1}{3}$.
So the probability that Harry goes directly from V to X to $\frac{1}{3}$.
At W, the probability that Harry turns to X is $\frac{1}{2}$, so the probability that he goes from V to W to X is $\frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$.
At Y, the probability that Harry turns to X is $\frac{1}{3}$, so the probability that he goes from V to Y to X is $\frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$.
Therefore, the probability that Harry gets to X (and thus to B) is $\frac{1}{3} + \frac{1}{6} + \frac{1}{9} = \frac{6+3+2}{18} = \frac{11}{18}$.

Answer: (C)

19. We extend AD to the point E where it intersects the perpendicular to BC from C.



By the Pythagorean Theorem in $\triangle ADB$, $BD^2 = BA^2 - AD^2 = 13^2 - 5^2 = 144$, so $BD = 12$ cm.
By the Pythagorean Theorem in $\triangle DBC$, $BC^2 = DC^2 - BD^2 = 20^2 - 12^2 = 256$, so $BC = 16$ cm.
Since $BCED$ has three right angles (and in fact, a fourth right angle at E), then it is a rectangle, so $DE = BC = 16$ cm and $CE = BD = 12$ cm.
Therefore, if we look at $\triangle AEC$, we see that $AE = 16 + 5 = 21$ cm, so by the Pythagorean Theorem, $AC^2 = 21^2 + 12^2 = 585$, so $AC \approx 24.2$ cm, to the nearest tenth of a centimetre.

Answer: (A)

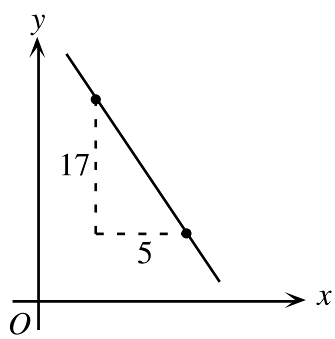
20. Let B be the total number of Beetles in the parking lot.
Then the number of Acuras is $\frac{1}{2}B$.
Also, the number of Camrys is 80% of $\frac{1}{2}B + B$, so the number of Camrys is $\frac{4}{5} \times \frac{3}{2}B = \frac{6}{5}B$.
Therefore, since the total number of cars in the parking lot is 81, then $B + \frac{1}{2}B + \frac{6}{5}B = 81$ or $\frac{27}{10}B = 81$, or $B = \frac{10}{27} \times 81 = 30$.
Therefore, the number of Beetles is 30.

Answer: (B)

21. We start by determining the combination of these bills totalling 453 Yacleys which uses the fewest 17 Yacley bills.
To do this, we notice that since we can use as many 5 Yacley bills as we'd like, then any multiple of 17 less than 453 which ends in a 5 or an 8 can be "topped up" to 453 Yacleys using 5 Yacley bills.
The first few multiples of 17 are 17, 34, 51, 68.
So if we use four 17 Yacley bills, we have 68 Yacleys, leaving $453 - 68 = 385$ Yacleys to get to 453. For 385 Yacleys, we need 77 of the 5 Yacley bills.
So 4 of the 17 Yacley bills and 77 of the 5 Yacley bills works.
To get other combinations, we use the fact that 5 of the 17 Yacley bills are worth the same as 17 of the 5 Yacley bills, so we can subtract 17 of the 5 Yacley bills and add 5 of the 17 Yacley bills and keep the total the same.
Thus, $77 - 17 = 60$ of the 5 Yacley bills and $4 + 5 = 9$ of the 17 Yacley bills make 453 Yacleys. (Check this!)
Also, 43 and 14 of the two types of bills, and 26 and 19 of the two types of bills, and 9 and 24 of the two types of bills.
Since we are down to 9 of the 5 Yacley bills, we can no longer use this exchanging process (since we need at least 17 of the 5 Yacley bills to be able to do this).
Therefore, there are 5 combinations that work.

(An alternate approach would be to let x be the number of 17 Yacley bills and y the number of 5 Yacley bills used.

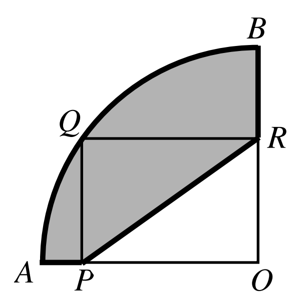
We then would like to consider the equation $17x + 5y = 453$, and find the number of pairs (x, y) which satisfy this equation and where both x and y are positive integers.
Geometrically, we are trying to find the number of points (x, y), with x and y both positive integers, lying on the line $17x + 5y = 453$.
In a similar way to above, we can find that $(x, y) = (4, 77)$ is such a point.
Since the slope of the line is $-\frac{17}{5}$, then to get a another point with integer coordinates on the line, we can move 5 units right and 17 units down.



We can repeat this process as above, and get 5 combinations that work.)

Answer: (C)

22. In order to determine the perimeter of the shaded region, we need to determine the total combined length of arc AQB and segments AP, PR and RB.



Since AOB is a quarter circle of radius 10, then arc AQB has length $\frac{1}{4}(2\pi(10)) = 5\pi$.
Since $PQRO$ is a rectangle, then $PR = QO$ and QO is a radius of the quarter circle, so $PR = QO = 10$. So we now need to calculate $AP + RB$.
But $AP + RB = (AO - PO) + (BO - RO) = AO + BO - (PO + RO)$. We know that $AO = BO = 10$ since each is a radius of the quarter circle.
Also, $PO + RO$ is half of the perimeter of the rectangle (which has total perimeter 26), so $PO + RO = 13$.
Therefore, $AP + RB = 10 + 10 - 13 = 7$.
Thus, the perimeter of the shaded region is $5\pi + 10 + 7 = 17 + 5\pi$.

Answer: (C)

23. We solve this problem by systematically keeping track of the distance from home of each of Anna, Bill and Dexter. At 12:00 noon, each is 0 km from home.
At 12:15:

Dexter is 0 km from home (since he hasn't started running)
Anna is $\frac{1}{4} \times 4 = 1$ km from home (since she has walked at 4 km/h for $\frac{1}{4}$ of an hour)
Bill is $\frac{1}{4} \times 3 = \frac{3}{4}$ km from home (since he has walked at 3 km/h for $\frac{1}{4}$ of an hour)

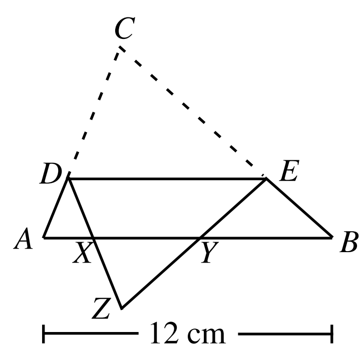
At 12:15, Dexter leaves and runs until he catches up to Anna.
How long does this take? Since Anna walks at 4 km/h and Dexter runs at 6 km/h in the same direction, then Dexter gains 2 km on Anna every hour. Since Anna starts 1 km ahead of Dexter, then it takes Dexter $\frac{1}{2}$ hour to catch Anna, so he catches her at 12:45.
At 12:45:

Dexter is 3 km from home (since he has run at 6 km/h for $\frac{1}{2}$ hour)
Anna is $\frac{3}{4} \times 4 = 3$ km from home (since she has walked at 4 km/h for $\frac{3}{4}$ of an hour)
Bill is $\frac{3}{4} \times 3 = \frac{9}{4}$ km from home (since he has walked at 3 km/h for $\frac{3}{4}$ of an hour)

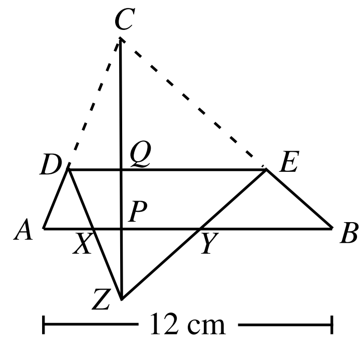
At 12:45, Dexter turns around instantaneously and runs back to Bill from Anna.
How long does this take? Since Bill walks at 3 km/h and Dexter runs at 6 km/h in the opposite direction, then Dexter and Bill are getting closer at a rate of 9 km/h.
Since Dexter and Bill start $3 - \frac{9}{4} = \frac{3}{4}$ km apart, then it takes Dexter $\frac{1}{9} \times \frac{3}{4} = \frac{1}{12}$ hour (or 5 minutes) to meet Bill.
Therefore, Bill and Dexter meet at 12:50 p.m.

ANSWER: (E)

24. *Solution 1*
Let X and Y be the points where the folded portion of the triangle crosses AB, and Z be the location of the original vertex C after folding.

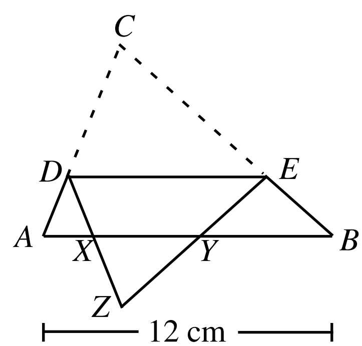


We are told that the area of $\triangle XYZ$ is 16% that of the area of $\triangle ABC$.
Now $\triangle ACB$ is similar to $\triangle XZY$, since $\angle XZY$ is the folded over version of $\angle ACB$ and since $\angle XYZ = \angle EYB = \angle DEY = \angle CED = \angle CBA$ by parallel lines and folds.
Since $\triangle XZY$ is similar to $\triangle ACB$ and its area is $0.16 = (0.4)^2$ that of $\triangle ACB$, then the sides of $\triangle XZY$ are 0.4 times as long as the sides of $\triangle ACB$.
Draw the altitude of $\triangle ACB$ from C down to P on AB (crossing DE at Q) and extend it through to Z.



Now $CP = CQ + QP = ZQ + QP = ZP + 2PQ$.
Since the sides of $\triangle XZY$ are 0.4 times as long as the sides of $\triangle ACB$, then $ZP = 0.4CP$.
Since $CP = ZP + 2PQ$, then $PQ = 0.3CP$, and so $CQ = CP - PQ = 0.7CP$.
Since CQ is 0.7 times the length of CP, then DE is 0.7 times the length of AB, again by similar triangles, so $DE = 0.7(12) = 8.4$.

Solution 2
Let X and Y be the points where the folded portion of the triangle crosses AB, and Z be the location of the original vertex C after folding.



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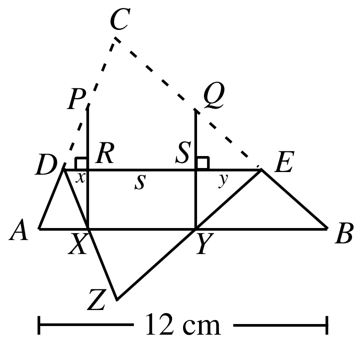
Since $\triangle XZY$ is similar to $\triangle ACB$ and its area is $0.16 = (0.4)^2$ that of $\triangle ACB$, then the sides of $\triangle XZY$ are 0.4 times as long as the sides of $\triangle ACB$.
Draw perpendiculars to AB at X and Y, intersecting AC and BC and P and Q, respectively, and DE at R and S, respectively.
By symmetry, PQ and RS are parallel to XY and the same length, so let $PQ = RS = XY = s$.
Since the sides of $\triangle XZY$ are 0.4 times as long as the sides of $\triangle ACB$, then $s = 0.4 \times 12 = 4.8$.
Since $\triangle CDE$ is congruent to $\triangle ZDE$ (since one is the folded over version of the other), then by symmetry, $PR = RX$ and $QS = SY$.
Let $DR = x$ and $ES = y$.



Then $AX = 2x$, since $\triangle PXA$ is similar to $\triangle PRD$ and has sides twice as long (since $PX = 2PR$. Similarly, $BY = 2y$.
Now looking at AB as a whole, we have $AB = 2x + s + 2y = 12$, so $x + y = \frac{1}{2}(12 - s) = 3.6$.
Looking at DE, we have $DE = s + x + y = 4.8 + 3.6 = 8.4$.

ANSWER: (B)

25. The first challenge in this problem is to find one set of numbers a, b, c that actually works.
Since this looks a bit similar to the Pythagorean Theorem, we can start with $3^2 + 4^2 = 5^2$ and try to manipulate this.
If we divide both sides by the least common multiple of 3^2, 4^2 and 5^2, which is $(3\times4\times5)^2 = 60^2$, we then obtain $\frac{3^2}{60^2} + \frac{4^2}{60^2} = \frac{5^2}{60^2}$ or $\frac{1}{20^2} + \frac{1}{15^2} = \frac{1}{12^2}$.
This gives us two possible triples: $(a, b, c) = (20, 15, 12)$ and $(a, b, c) = (15, 20, 12)$ (so two possible values for a so far).
How can we get more? We can multiply the equation $\frac{1}{20^2} + \frac{1}{15^2} = \frac{1}{12^2}$ by reciprocals of perfect squares.
Multiplying by $\frac{1}{2^2}$, we get $\frac{1}{40^2} + \frac{1}{30^2} = \frac{1}{24^2}$.
Multiplying by $\frac{1}{3^2}$, we get $\frac{1}{60^2} + \frac{1}{45^2} = \frac{1}{36^2}$.
Multiplying by $\frac{1}{4^2}$, we get $\frac{1}{80^2} + \frac{1}{60^2} = \frac{1}{48^2}$.
Multiplying by $\frac{1}{5^2}$, we get $\frac{1}{100^2} + \frac{1}{75^2} = \frac{1}{60^2}$.
Multiplying by $\frac{1}{6^2}$, we get $\frac{1}{120^2} + \frac{1}{90^2} = \frac{1}{72^2}$.
Multiplying by $\frac{1}{7^2}$, we get $\frac{1}{140^2} + \frac{1}{105^2} = \frac{1}{84^2}$.
At this point, the strategy will no longer work, since we are only looking for values of $a \leq 100$.
So far, the possible values of a are (from looking at each denominator of the left side of each the equations here): 20, 15, 40, 30, 60, 45, 80, 100, 75, 90. (Notice that 60 doesn't appear twice in the list!) The sum of these numbers is 555.

Can we find more starting equations by beginning with a different Pythagorean triple?
If we start with $5^2 + 12^2 = 13^2$ and divide both sides by the least common multiple of 5^2, 12^2 and 13^2 (ie. $(5 \times 12 \times 13)^2 = 780^2$), we get $\frac{1}{156^2} + \frac{1}{65^2} = \frac{1}{60^2}$ which gives us 65 as another possible value of a.
Therefore, our running total for values of a is $555 + 65 = 620$.
We can't generate more possible values for a using $\frac{1}{156^2} + \frac{1}{65^2} = \frac{1}{60^2}$ since multiplying both sides

by the reciprocal of any perfect square will make both of a and b at least 130, so bigger than 100.

Can we use $6^2 + 8^2 = 10^2$? Here, the least common multiple of 6^2, 8^2 and 10^2 is 120^2, and dividing by 120^2 gives us $\frac{1}{20^2} + \frac{1}{15^2} = \frac{1}{12^2}$, which we have already used.
Can we use any other Pythagorean triple? No, since any other Pythagorean triple is at least as big as 7-24-25, and so the smallest possible denominator that we will get on the left side by using this technique is $(7 \times 25)^2 = 175^2$, which would give an a larger than 100.
Also, any triple (a, b, c) that actually works does come from a Pythagorean triple, since we can multiply both sides of $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c^2}$ by $(abc)^2$ to get $(bc)^2 + (ac)^2 = (ab)^2$.
So every possible triple (a, b, c) comes from a Pythagorean triple, and no Pythagorean triples give any more allowable values of a, so we have found them all.
Therefore, the sum of all possible values of $a \leq 100$ is 620.

ANSWER: (E)