Canadian
Mathematics
Competition

An activity of The Centre for Education
in Mathematics and Computing,
University of Waterloo, Waterloo, Ontario

2004 Solutions
Galois Contest (Grade 10)

1. (a) Since at least one of each type of prize is given out, then these four prizes account for $\$5 + \$25 + \$125 + \$625 = \$780$.
   Since there are five prizes given out which total \$905, then the fifth prize must have a value of $\$905 - \$780 = \$125$.
   Thus, the Fryer Foundation gives out one \$5 prize, one \$25 prize, two \$125 prizes, and one \$625 prize.

   (b) As in (a), giving out one of each type of prize accounts for \$780.
   The fifth prize could be a \$5 prize for a total of $\$780 + \$5 = \$785$.
   The fifth prize could be a \$25 prize for a total of $\$780 + \$25 = \$805$.
   The fifth prize could be a \$625 prize for a total of $\$780 + \$625 = \$1405$.
   (We already added an extra \$125 prize in (a).)

   (c) *Solution 1*
   Since at least one of each type of prize is given out, this accounts for \$780. So we must figure out how to distribute the remaining $\$880 - \$780 = \$100$ using at most 5 of each type of prize. We cannot use any \$125 or \$625 prizes, since these are each greater than the remaining amount.
   We could use four additional \$25 prizes to make up the \$100.
   Could we use fewer than four \$25 prizes? If we use three additional \$25 prizes, this accounts for \$75, which leaves \$25 remaining in \$5 prizes, which can be done by using five additional \$5 prizes.
   Could we use fewer than three \$25 prizes? If so, then we would need to make at least \$50 with \$5 prizes, for which we need at least ten such prizes. But we can use at most six \$5 prizes in total, so this is impossible.
   Therefore, the two ways of giving out \$880 in prizes under the given conditions are:
      i) one \$625 prize, one \$125 prize, five \$25 prizes, one \$5 prize
      ii) one \$625 prize, one \$125 prize, four \$25 prizes, six \$5 prizes
   We can check by addition that each of these totals \$880.

   Solution 2
   We know that the possible total values using at least one of each type of prize and exactly five prizes are \$785, \$805, \$905 and \$1405.
   We try starting with \$785 and \$805 to get to \$880. (Since \$905 and \$1405 are already larger than \$880, we do not need to try these.)

   Starting with \$785, we need to give out an additional \$95 to get to \$880. Using three \$25 prizes accounts for \$75, leaving \$20 to be split among four \$5 prizes. (Using fewer than three \$25 prizes will mean we need more than six \$5 prizes in total.) So in this way, we need one \$625 prize, one \$125 prize, four \$25 prizes, and six \$5 prizes (since there were already two included in the \$785).

Starting with \$805, we need to give out an additional \$75 to get to \$880. Using three \$25 prizes will accomplish this, for a total of one \$625 prize, one \$125 prize, five \$25 prizes, and one \$5 prize. We could also use two \$25 prizes and five \$5 prizes to make up the \$75, for a total of one \$625 prize, one \$125 prize, four \$25 prizes, and six \$5 prizes (which is the same as we obtained above starting with \$785). If we use fewer than two additional \$25 prizes, we would need too many \$5 prizes.

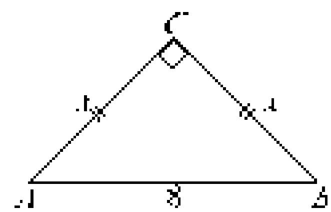
Therefore, the two ways of giving out \$880 in prizes under the given conditions are:
- i) one \$625 prize, one \$125 prize, five \$25 prizes, one \$5 prize
- ii) one \$625 prize, one \$125 prize, four \$25 prizes, six \$5 prizes

We can check by addition that each of these totals \$880.

2. (a) *Solution 1*

Let $AC = x$. Then $BC = x$ since triangle ABC is isosceles.
Since triangle ABC is right-angled, then, by the Pythagorean Theorem,

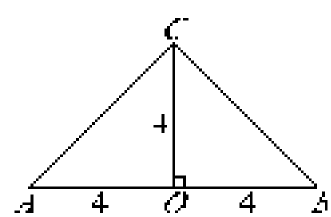
$$\begin{aligned} x^2 + x^2 &= 8^2 \\ 2x^2 &= 64 \\ x^2 &= 32 \\ x &= \sqrt{32} = 4\sqrt{2} \end{aligned}$$



(In fact, we won't actually need to know x – we will only need to know x^2.)
Therefore, the area of the triangle is $\frac{1}{2}(AC)(BC) = \frac{1}{2}x^2 = 16$.

Solution 2

Let O be the midpoint AB. Thus, O is the centre of the semi-circle, and the semi-circle has radius 4.
Join O to C. Since triangle ABC is isosceles, OC is perpendicular to AB. Since C is on the circle, then OC is a radius, and so $OC = 4$.
Therefore, the area of the triangle is
$\frac{1}{2}(AB)(OC) = \frac{1}{2}(8)(4) = 16$.



(b) The total area of the two shaded regions is equal to the difference between the area of the semi-circle and the area of the triangle.
We know that the area of the triangle is 16, and the semi-circle has radius 4, so its area is $\frac{1}{2}\pi r^2 = \frac{1}{2}\pi(4)^2 = 8\pi$.
Therefore, the total area of the two shaded regions is $8\pi - 16$.

(c) In (b), we saw that the area of the semi-circle drawn on *AB* is 8π.
Since $AC = CB$, then the areas of the semi-circles drawn on *AC* and *CB* are equal.
Since $AC = 4\sqrt{2}$, then the diameter of the semi-circle on *AC* is $4\sqrt{2}$, and so the radius of the semi-circle is $2\sqrt{2}$. Therefore, the area of the semi-circle on *AC* is $\frac{1}{2}\pi\left(2\sqrt{2}\right)^2 = \frac{1}{2}\pi(8) = 4\pi$.
So the sum of the areas of the semi-circles on *AC* and *CB* is $4\pi + 4\pi = 8\pi$, which is the area of the semi-circle on *AB*, as required.

3. (a) If Bob places a 3, then the total of the two numbers so far is 8, so Avril should place a 7 to bring the total up to 15.
Since Bob can place a 3 in any the eight empty circles, Avril should place a 7 in the circle directly opposite the one in which Bob places the 3. This allows Avril to win on her next turn.

(b) As in (a), Bob can place any of the numbers 1, 2, 3, 4, 6, 7, 8, 9 in any of the eight empty circles. On her next turn, Avril should place a disc in the circle directly opposite the one in which Bob put his number. What number should Avril use? Avril should place the number that brings the total up to 15, as shown below:

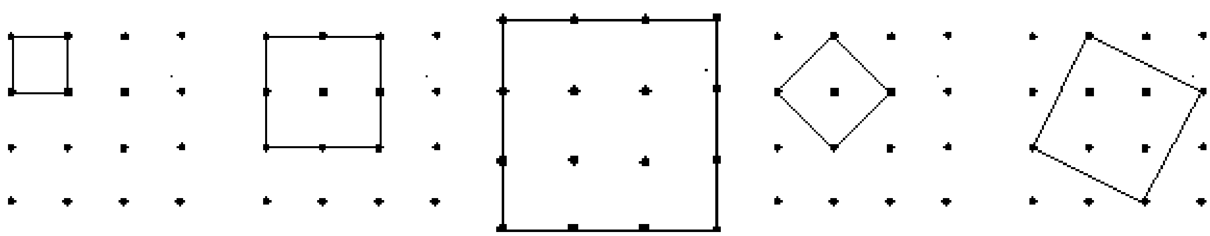
Bob's First Turn	Total so far	Avril's Second Turn
1	6	9
2	7	8
3	8	7
4	9	6
6	11	4
7	12	3
8	13	2
9	14	1

Since each of these possibilities is available to Avril on her second turn (since 5 is not in the list and none is equal to Bob's number), then she can always win on her second turn.

(c) Bob can place any of the numbers 4, 5, 6, 7, 8, 9 in any of the six empty circles. Which of these numbers can we pair up so that the sum of the pair is 14 (so that placing these numbers in the two circles at opposite ends of a line gives 15)? We can pair 5 and 9, and

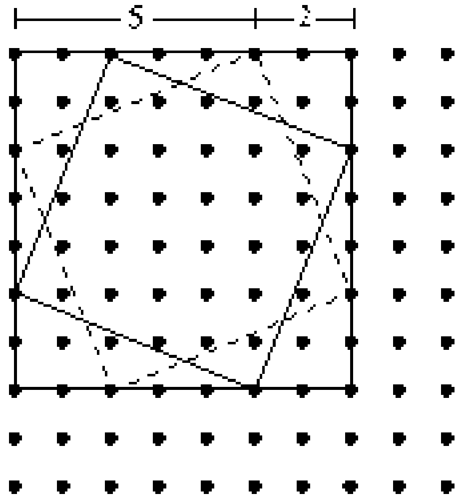
6 and 8 in this way. However, the 4 and 7 do not pair up with any other number to give a sum of 14.
So if Bob on his next turn places 5, 6, 8, or 9, Avril should place the second number from the pair (that is, 9, 8, 6, or 5, respectively) opposite the number Bob places and she will win.
If Bob places the 4 or the 7, Avril should then place the 7 or the 4, respectively, in the opposite circle. She will not win on this turn, but this forces Bob to then place one of the remaining 4 paired numbers on his next turn, so Avril can win for sure on her following turn.

4. (a) There are three rows of three 1 by 1 squares, or 9 in total.
There are two rows of two 2 by 2 squares, or 4 in total.
There is one 3 by 3 square.



There are squares which have side length $\sqrt{2}$, formed by cutting off the four corners of a 2 by 2 square, as in the third example in the 3 by 3 grid. Each 2 by 2 square can be cut to give one $\sqrt{2}$ by $\sqrt{2}$ square, so there are 4 of these.
We can also cut the 3 by 3 square. We can do by cutting off right-angled triangles which have legs 1 and 2 (and hypotenuse $\sqrt{5}$) to give a square of side-length $\sqrt{5}$. We can do this cutting in two different ways – either by starting the long side of the triangle horizontal, or with the long side of the triangle vertical. So there are two squares which are $\sqrt{5}$ by $\sqrt{5}$.
This gives $9+4+1+4+2=20$ squares in total.

(b) We start by noting that there are nine squares that measure 7 by 7. These squares have sides that are parallel to the sides of the 10 by 10 grid. For each of these squares, there are two squares that have a side length of $\sqrt{29}$. We have drawn one such example to illustrate this.



(c) First we count the number of squares with sides parallel to the sides of the 10 by 10 grid. There are nine rows of nine 1 by 1 square, or 9^2 squares in total.
There are eight rows of eight 2 by 2 squares, or 8^2 squares in total.
This pattern continues, until we find that there are two rows of two 8 by 8 squares, or 2^2 in total, and there is only $1 = 1^2$ 9 by 9 square.
As in (b), each of these squares starting with the 2 by 2 can be cut to give squares whose sides are not parallel to the side of the grid.
Each 2 by 2 square can have its sides divided 1 and 1 to give a square of side length $\sqrt{2}$. So the 2 by 2 squares account for $2(8^2)$ squares in total (the original 2 by 2 squares, plus the squares of side length $\sqrt{2}$).
Each 3 by 3 square can have its sides divided 1 and 2 (to give a square of side length $\sqrt{5}$) or 2 and 1 (to give a square of side length $\sqrt{5}$). So the 3 by 3 squares account for $3(7^2)$ squares in total.
Each 4 by 4 square can have its sides divided 1 and 3, 2 and 2, or 3 and 1 to give squares of side length $\sqrt{10}$, $\sqrt{8}$ or $\sqrt{10}$, respectively. So the 4 by 4 squares account for $4(6^2)$ squares in total.
We continue this pattern up to the 9 by 9 squares. Each 9 by 9 square can have its sides divided 1 and 8, 2 and 7, 3 and 6, 4 and 5, 5 and 4, 6 and 3, 7 and 2, or 8 and 1, to give squares of side length $\sqrt{65}$, $\sqrt{53}$, $\sqrt{45}$, $\sqrt{41}$, $\sqrt{41}$, $\sqrt{45}$, $\sqrt{53}$ or $\sqrt{65}$. So the 9 by 9 squares account for $9(1^2)$ squares in total, since there is only one 9 by 9 square, and it gives rise to 9 different squares.
Therefore, the total number of squares that can be formed is

$$1(9^2) + 2(8^2) + 3(7^2) + 4(6^2) + 5(5^2) + 6(4^2) + 7(3^2) + 8(2^2) + 9(1^2).$$