Canadian
Mathematics
Competition

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2003 Solutions
Euclid Contest

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Awards

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1. (a) *Solution 1*

Since the x-intercepts of the parabola are 2 and 4, then the axis of symmetry of the parabola is $x = 3$.
Since the point $(0,8)$ is on the parabola, its image after a reflection across the vertical line $x = 3$ is the point $(6,8)$. Thus, $a = 6$.

Solution 2

Since the x-intercepts of the parabola are 2 and 4, then the equation of the parabola is of the form $y = A(x-2)(x-4)$.
Since $(0,8)$ lies on the parabola, then $8 = A(-2)(-4)$ or $A = 1$.
Therefore, the parabola has equation $y = (x-2)(x-4) = x^2 - 6x + 8$.
But the point $(a,8)$ lies on the parabola, so

$$8 = a^2 - 6a + 8$$
$$0 = a^2 - 6a$$
$$0 = a(a-6)$$

Since $a \neq 0$, then $a = 6$. Answer: $a = 6$

(b) *Solution 1*

Since the quadratic equation has two equal roots, then the expression on the left must be a perfect square. Since the leading coefficient is 1 and the coefficient of the x-term is 6, then the expression must be $(x+3)^2 = x^2 + 6x + 9$. By comparing expressions, $k = 9$.

Solution 2

Since the quadratic equation has two equal roots, the discriminant is 0, ie.
$6^2 - 4(1)(k) = 0$ or $4k = 36$ or $k = 9$. Answer: $k = 9$

(c) From the given information, the point $(1,4)$ lies on the parabola, so $4 = 1^2 - 3(1) + c$ or $c = 6$.
We now find the points of intersection of the parabola and the line by equating:

$$2x + 2 = x^2 - 3x + 6$$
$$0 = x^2 - 5x + 4$$
$$0 = (x-1)(x-4)$$

Thus the points of intersection have x-coordinates $x = 1$ and $x = 4$.
Substituting $x = 4$ into the line $y = 2x + 2$, we get the point $(4,10)$.
Therefore, the second point of intersection is $(4,10)$.

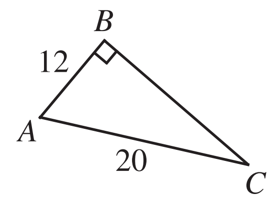
2. (a) Rearranging the equation,

$$3\sin(x) = \cos(15^\circ)$$
$$\sin(x) = \tfrac{1}{3}\cos(15^\circ)$$
$$\sin(x) \approx 0.3220$$

Using a calculator, $x \approx 18.78^\circ$. To the nearest tenth of a degree, $x = 18.8^\circ$.

Answer: $x = 18.8^\circ$

(b) *Solution 1*



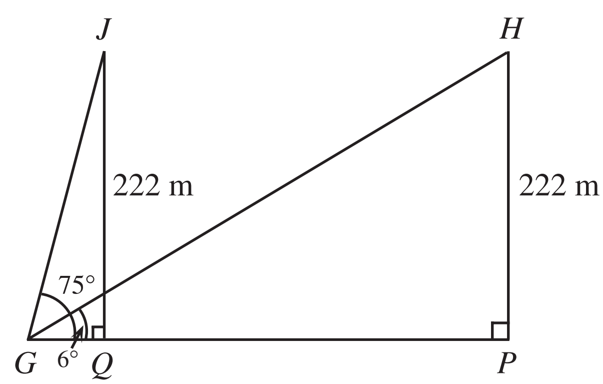
Since $\sin C = \dfrac{AB}{AC}$, then $AB = AC\sin C = 20\left(\tfrac{3}{5}\right) = 12$.
By Pythagoras, $BC^2 = AC^2 - AB^2 = 20^2 - 12^2 = 256$ or $BC = 16$.

Solution 2

Using the standard trigonometric ratios, $BC = AC\cos C$.
Since $\sin C = \tfrac{3}{5}$, then $\cos^2 C = 1 - \sin^2 C = 1 - \tfrac{9}{25} = \tfrac{16}{25}$ or $\cos C = \tfrac{4}{5}$. (Notice that $\cos C$ is positive since angle C is acute in triangle ABC.)
Therefore, $BC = 20\left(\tfrac{4}{5}\right) = 16$.

Answer: $BC = 16$

(c) Let G be the point where the goat is standing, H the position of the helicopter when the goat first measures the angle, P the point directly below the helicopter at this time, J the position of the helicopter one minute later, and Q the point directly below the helicopter at this time.



Using the initial position of the helicopter, $\tan(6^\circ) = \dfrac{HP}{PG}$ or $PG = \dfrac{222}{\tan(6^\circ)} \approx 2112.19$ m.

Using the second position of the helicopter, $\tan(75^\circ) = \dfrac{JQ}{QG}$ or $QG = \dfrac{222}{\tan(75^\circ)} \approx 59.48$ m.

So in the one minute that has elapsed, the helicopter has travelled
$2112.19 \text{ m} - 59.48 \text{ m} = 2052.71 \text{ m}$ or 2.0527 km.

Therefore, in one hour, the helicopter will travel $60(2.0527) = 123.162$ km.
Thus, the helicopter is travelling 123 km/h.

3. (a) Since we are looking for the value of $f(9)$, then it makes sense to use the given equation and to set $x = 3$in order to obtain $f(9) = 2f(3) + 3$.
So we need to determine the value of $f(3)$. We use the equation again and set $x = 0$ since we will then get $f(3)$ on the left side and $f(0)$ (whose value we already know) on the right side, ie.
$$f(3) = 2f(0) + 3 = 2(6) + 3 = 15$$
Thus, $f(9) = 2(15) + 3 = 33$. Answer: $f(9) = 33$

(b) *Solution 1*
We solve the system of equations for $f(x)$ and $g(x)$.
Dividing out the common factor of 2 from the second equation, we get
$f(x) + 2g(x) = x^2 + 2$.
Subtracting from the first equation, we get $g(x) = x + 4$.
Thus, $f(x) = x^2 + 2 - 2g(x) = x^2 + 2 - 2(x + 4) = x^2 - 2x - 6$.
Equating $f(x)$ and $g(x)$, we obtain
$$x^2 - 2x - 6 = x + 4$$
$$x^2 - 3x - 10 = 0$$
$$(x - 5)(x + 2) = 0$$
Therefore, $x = 5$ or $x = -2$.

Solution 2
Instead of considering the equation $f(x) = g(x)$, we consider the equation $f(x) - g(x) = 0$, and we try to obtain an expression for $f(x) - g(x)$ by manipulating the two given equations.
In fact, after some experimentation, we can see that
$$\begin{aligned} f(x) - g(x) &= 2\big(2f(x) + 4g(x)\big) - 3\big(f(x) + 3g(x)\big) \\ &= 2\left(2x^2 + 4\right) - 3\left(x^2 + x + 6\right) \\ &= x^2 - 3x - 10 \end{aligned}$$
So to solve $f(x) - g(x) = 0$, we solve $x^2 - 3x - 10 = 0$ or $(x - 5)(x + 2) = 0$. Therefore, $x = 5$ or $x = -2$.

4. (a) *Solution 1*
We label the 5 skaters A, B, C, D, and E, where D and E are the two Canadians.
There are then $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$ ways of arranging these skaters in their order of finish (for example, ADBCE indicates that A finished first, D second, etc.), because there are 5 choices for the winner, 4 choices for the second place finisher, 3 choices for the third place finisher, etc.

If the two Canadians finish without winning medals, then they must finish fourth and fifth. So the D and E are in the final two positions, and A, B and C in the first three. There are $3! = 6$ ways of arranging the A, B and C, and $2! = 2$ ways to arrange the D and E. Thus, there are $6 \times 2 = 12$ ways or arranging the skaters so that neither Canadian wins a medal.
Therefore, the probability that neither Canadian wins a medal is

$$\frac{\text{\# of ways where Canadians don't win medals}}{\text{Total \# of arrangements}} = \frac{12}{120} = \frac{1}{10}$$

Solution 2
We label the 5 skaters as A, B, C, D, and E, where D and E are the two Canadians. In any race, two of the skaters finish fourth and fifth. Also, any pair of skaters are equally as likely to finish fourth and fifth, since the probability of every skater is equally likely to finish in a given position.
How many pairs of 2 skaters can we form from the 5 skaters? There are ten such pairs:
{A,B}, {A,C}, {A,D}, {A,E}, {B,C}, {B,D}, {B,E}, {C,D}, {C,E}, {D,E}
Only one of these ten pairs is made up of the two Canadians. Therefore, the probability is $\frac{1}{10}$, since one out of ten choices gives the desired result.

Answer: $\frac{1}{10}$

(b) *Solution 1*
Since the least common multiple of 3, 5, 10 and 15 is 30, then we can count the number of positive integers less than or equal to 30 satisfying these conditions, and multiply the total by 10 to obtain the number less than 300. (This is because each group of 30 consecutive integers starting with 1 more than a multiple of 30 will have the same number of integers having these properties, because we can subtract 30 from each one and not change these properties.)
So from 1 to 30, we have:

3, 5, 6, 9, 12, 18, 21, 24, 25, 27

Thus there are 10 less than or equal to 30, and so 100 such positive integers less than or equal to 300.

Solution 2
We proceed by doing a (careful!) count.
The number of positive multiples of 3 less than or equal to 300 is 100.
The number of positive multiples of 5 less than or equal to 300 is 60.
Thus, we have 160 candidates, but have included multiples of 15 twice (since 15 is a multiple of each of 3 and 5), and have also included multiples of 10.
The number of multiples of 15 less than or equal to 300 is 20, so to remove the multiples of 15, we must remove 40 from 160 to get 120 positive integers less than or equal to 300 which are multiples of 3 or 5 but not of 15.

This total still included some multiples of 10 that are less or equal to 300 (but not all, since we have already removed 30, for instance).
In fact, there are 30 multiples of 10 less than or equal 300, 10 of which are multiples of 15 as well (that is, the multiples of 30). So we must remove 20 from the total of 120.
We then obtain that there are 100 positive integers less than or equal to 300 which are multiples of 3 or 5, but not of 10 or 15.

5. (a) Since the signs alternate every three terms, it makes sense to look at the terms in groups of 6.
The sum of the first 6 terms is $1+3+5-7-9-11=-18$.
The sum of the next 6 terms is $13+15+17-19-21-23=-18$.
In fact, the sum of each group of 6 terms will be the same, since in each group, 12 has been added to the numerical value of each term when compared to the previous group of 6, so overall 12 has been added three times and subtracted three times.
Since we are looking for the sum of the first 300 terms, then we are looking at 50 groups of 6 terms, so the sum must be $50(-18)=-900$.

Answer: -900

(b) Let the two digit integer have tens digit *a* and units digit *b*. Then the given information tells us

$$a^2+10b=b^2+10a$$
$$a^2-b^2-10a+10b=0$$
$$(a+b)(a-b)-10(a-b)=0$$
$$(a-b)(a+b-10)=0$$

and so $a=b$ or $a+b=10$.
So the possibilities for the integer are 11, 22, 33, 44, 55, 66, 77, 88, 99, 19, 28, 37, 46, 55, 64, 73, 82, 91. We now must determine which integers in this list are prime.
We can quickly reject all multiples of 11 bigger than 11 and all of the even integers, to reduce the list to 11, 19, 37, 73, 91.
All of these are prime, except for $91=13\times 7$.
Therefore, the required integers are 11, 19, 37, and 73.

6. (a) *Solution 1*
In 24 minutes, the number of atoms of isotope A has halved 4 times, so the initial number of atoms is $2^4=16$ times the number of atoms of isotope A at time 24 minutes.
But there were initially half as many atoms of isotope B as of isotope B, so there was 8 times the final number of atoms. Therefore, the number of atoms of isotope B halves 3 times in the 24 minutes, so it takes 8 minutes for the number of atoms of isotope B to halve.

Solution 2

Initially, there is twice as many atoms of isotope A as of isotope B, so let the original numbers of atoms of each be $2x$ and x, respectively.

Considering isotope A, after 24 minutes, if it loses half of its atoms every 6 minutes, there will be $2x\left(\frac{1}{2}\right)^{\frac{24}{6}}$ atoms remaining.

Similarly for isotope B, after 24 minutes, there will be $x\left(\frac{1}{2}\right)^{\frac{24}{T}}$ atoms remaining, where T is the length of time (in minutes) that it takes for the number of atoms to halve.

From the given information,

$$\begin{aligned} 2x\left(\tfrac{1}{2}\right)^{\frac{24}{6}} &= x\left(\tfrac{1}{2}\right)^{\frac{24}{T}} \\ 2\left(\tfrac{1}{2}\right)^{4} &= \left(\tfrac{1}{2}\right)^{\frac{24}{T}} \\ \left(\tfrac{1}{2}\right)^{3} &= \left(\tfrac{1}{2}\right)^{\frac{24}{T}} \\ \frac{24}{T} &= 3 \\ T &= 8 \end{aligned}$$

Therefore, it takes 8 minutes for the number of atoms of isotope B to halve.

Answer: 8 minutes

(b) *Solution 1*

Using the facts that $\log_{10} A + \log_{10} B = \log_{10} AB$ and that $\log_{10} A - \log_{10} B = \log_{10} \frac{A}{B}$, then we can convert the two equations to

$$\log_{10}\left(x^3y^2\right) = 11$$

$$\log_{10}\left(\frac{x^2}{y^3}\right) = 3$$

Raising both sides to the power of 10, we obtain

$$x^3y^2 = 10^{11}$$

$$\frac{x^2}{y^3} = 10^3$$

To eliminate the y's, we raise the first equation to the power 3 and the second to the power 2 to obtain

$$x^9y^6 = 10^{33}$$

$$\frac{x^4}{y^6} = 10^6$$

and multiply to obtain $x^9x^4 = x^{13} = 10^{39} = 10^{33}10^6$.

Therefore, since $x^{13} = 10^{39}$, then $x = 10^3$.

Substituting back into $x^3y^2 = 10^{11}$, we get $y^2 = 10^2$, and so $y = \pm 10$. However, substituting into $\frac{x^2}{y^3} = 10^3$ we see that y must be positive, so $y = 10$.

Therefore, the solution to the system of equation is $x = 10^3$ and $y = 10$.

Solution 2

Since the domain of the logarithm is the positive real numbers, then the quantities $\log_{10}\left(x^3\right)$ and $\log_{10}\left(y^3\right)$ tell us that x and y are positive.

Using the fact that $\log_{10}\left(a^b\right) = b\log_{10}(a)$, we rewrite the equations as

$$3\log_{10} x + 2\log_{10} y = 11$$
$$2\log_{10} x - 3\log_{10} y = 3$$

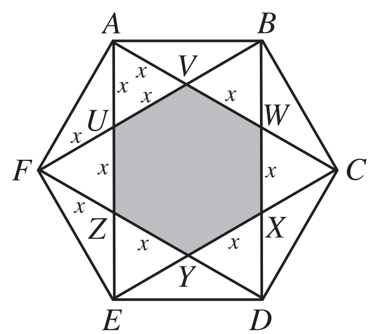
We solve the system of equations for $\log_{10} x$ and $\log_{10} y$ by multiplying the first equation by 3 and adding two times the second equation in order to eliminate $\log_{10} y$.

Thus we obtain $13\log_{10} x = 39$ or $\log_{10} x = 3$.

Substituting back into the first equation, we obtain $\log_{10} y = 1$.

Therefore, $x = 10^3$ and $y = 10$.

7. (a) *Solution 1*



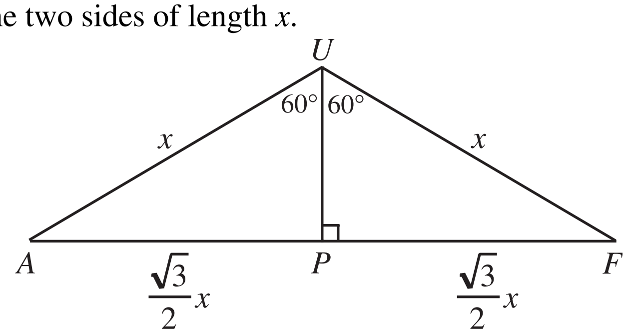
We label the vertices of the shaded hexagon U, V, W, X, Y, and Z.

By symmetry, all of the six triangles with two vertices on the inner hexagon and one on the outer hexagon (eg. triangle UVA) are congruent equilateral triangles.

In order to determine the area of the inner hexagon, we determine the ratio of the side lengths of the two hexagons.

Let the side length of the inner hexagon be x. Then $AU = UF = x$.

Then triangle AUF has a 120° between the two sides of length x.



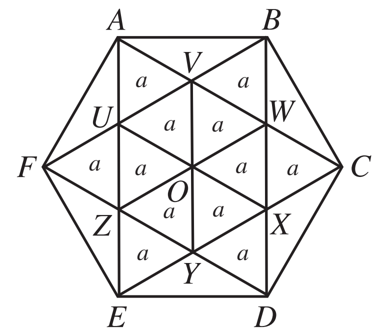
If we draw a perpendicular from U to point P on side AF, then UP divides ΔAUF into two $30^\circ - 60^\circ - 90^\circ$ triangles. Thus, $FP = PA = \frac{\sqrt{3}}{2}x$ and so $AF = \sqrt{3}x$.

So the ratio of the side lengths of the hexagons is $\sqrt{3}$: 1, and so the ratio of their areas is $\left(\sqrt{3}\right)^2$: 1 = 3 : 1.

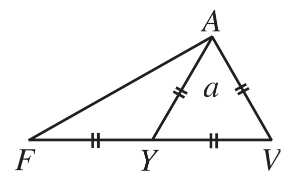
Since the area of the larger hexagon is 36, then the area of the inner hexagon is 12.

Solution 2

We label the vertices of the hexagon U, V, W, X, Y, and Z.
By symmetry, all of the six triangles with two vertices on the inner hexagon and one on the outer hexagon (eg. triangle UVA) are congruent equilateral triangles.
We also join the opposite vertices of the inner hexagon, ie. we join U to X, V to Y, and W to Z. (These 3 line segments all meet at a single point, say O.) This divides the inner hexagon into 6 small equilateral triangles identical to the six earlier mentioned equilateral triangles.



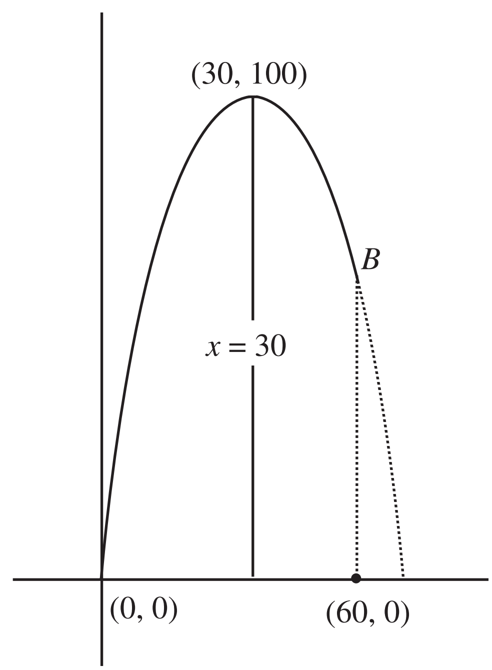
Let the area of one of these triangles be a. Then we can label the 12 small equilateral triangles as all having area a.
But triangle AUF also has area a, because if we consider triangle AFV, then AU is a median (since $FU = AU = UV$ by symmetry) and so divides triangle AFV into two triangles of equal area. Since the area of triangle AUV is a, then the area of triangle AUF is also a.



Therefore, hexagon $ABCDEF$ is divided into 18 equal areas. Thus, $a = 2$ since the area of the large hexagon is 36.
Since the area of $UVWXYZ$ is $6a$, then its area is 12.

Answer: 12

(b) We assign coordinates to the diagram, with the mouth of the cannon at the point $(0,0)$, with the positive x-axis in the horizontal direction towards the safety net from the cannon, and the positive y-axis upwards from $(0,0)$.
Since Herc reaches his maximum height when his horizontal distance is 30 m, then the axis of symmetry of the parabola is the line $x = 30$.
Since the parabola has a root at $x = 0$, then the other root must be at $x = 60$.
Therefore, the parabola has the form $y = ax(x-60)$.
In order to determine the value of a, we note that Herc passes through the point $(30,100)$, and so



$$100 = 30a(-30)$$
$$a = -\tfrac{1}{9}$$

Thus, the equation of the parabola is $y = -\frac{1}{9}x(x-60)$.

(Alternatively, we could say that since the parabola has its maximum point at $(30,100)$, then it must be of the form $y = a(x-30)^2 + 100$.
Since the parabola passes through $(0,0)$, then we have

$$0 = a(0-30)^2 + 100$$
$$0 = 900a + 100$$
$$a = -\tfrac{1}{9}$$

Thus, the parabola has the equation $y = -\frac{1}{9}(x-30)^2 + 100$.)

We would like to find the points on the parabola which have y-coordinate 64, so we solve

$$64 = -\tfrac{1}{9}x(x-60)$$
$$0 = x^2 - 60x + 576$$
$$0 = (x-12)(x-48)$$

Since we want a point after Herc has passed his highest point, then $x = 48$, ie. the horizontal distance from the cannon to the safety net is 48 m.

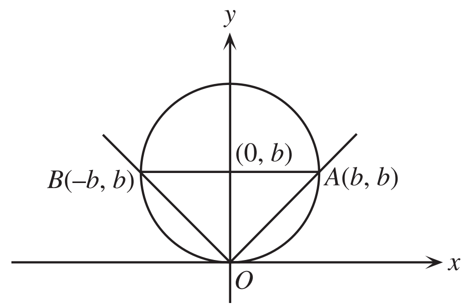
8. (a) Since both the circle with its centre on the y-axis and the graph of $y = |x|$ are symmetric about the y-axis, then for each point of intersection between these two graphs, there should be a corresponding point of intersection symmetrically located across the y-axis. Thus, since there are exactly three points of intersection, then one of these points must be on the y-axis, ie. has x-coordinate 0. Since this point is on the graph of $y = |x|$, then this point must be $(0,0)$.
Since the circle has centre on the y-axis (say, has coordinates $(0,b)$), then its radius is equal to b (and b must be positive for there to be three points of intersection).
So the circle has equation $x^2 + (y-b)^2 = b^2$. Where are the other two points of intersection? We consider the points with x positive and use symmetry to get the other point of intersection.

When $x \geq 0$, then $y = |x|$ has equation $y = x$. Substituting into the equation of the circle,

$$x^2 + (x-b)^2 = b^2$$
$$2x^2 - 2bx = 0$$
$$2x(x-b) = 0$$

Therefore, the points of intersection are $(0,0)$ and (b,b) on the positive side of the y-axis, and so at the point $(-b,b)$ on the negative side of the y-axis.

Thus the points O, A and B are the points $(0,0)$, (b,b) and $(-b,b)$.



Since the radius of the circle is b, then the area of the circle is πb^2.

Triangle OAB has a base from $(-b,b)$ to (b,b) of length $2b$, and a height from the line $y = b$ to the point $(0,0)$ of length b, and so an area of $\frac{1}{2}b(2b) = b^2$.

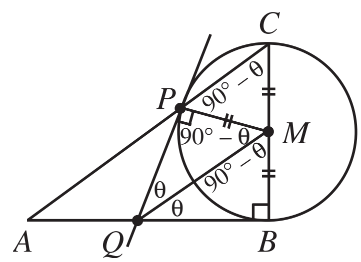
Therefore, the ratio of the area of the triangle to the area of the circle is $b^2 : \pi b^2 = 1 : \pi$.

(b) *Solution 1*

Since M is the midpoint of a diameter of the circle, M is the centre of the circle.

Join P to M. Since QP is tangent to the circle, PM is perpendicular to QP.

Since PM and BM are both radii of the circle, then $PM = MB$.



Therefore, ΔQPM and ΔQBM are congruent (Hypotenuse - Side).

Thus, let $\angle MQB = \angle MQP = \theta$. So $\angle QMB = \angle QMP = 90^\circ - \theta$

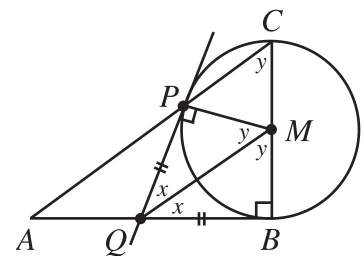
Then $\angle PMC = 180^\circ - \angle PMQ - \angle BMQ = 180^\circ - \left(90^\circ - \theta\right) - \left(90^\circ - \theta\right) = 2\theta$.

But ΔPMC is isosceles with $PM = MC$ since PM and MC are both radii.

Therefore, $\angle CPM = \frac{1}{2}\left(180^\circ - \angle PMC\right) = 90^\circ - \theta$.

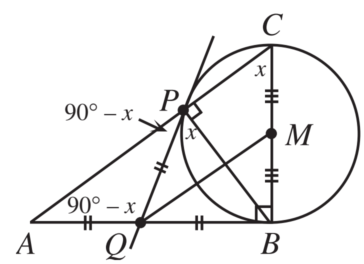
But then $\angle CPM = \angle PMQ$, and since PM is a transversal between AC and QM, then QM is parallel to AC because of equal alternating angles.

Solution 2



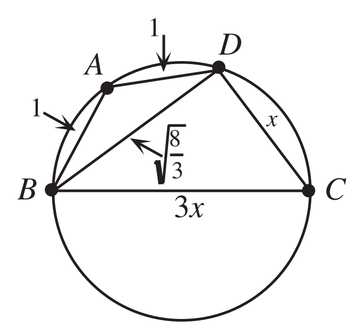
Join M to P and B to P.
Since QP and QB are tangents to the circle coming from the same point, they have the same length.
Since QM joins the point of intersection of the tangents to the centre of the circle, then by symmetry, $\angle PQM = \angle BQM$ and $\angle PMQ = \angle BMQ$. So let $\angle PQM = \angle BQM = x$ and $\angle PMQ = \angle BMQ = y$.
Looking at ΔQMB, we see that $x + y = 90^\circ$, since ΔQMB is right-angled.
Now if we consider the chord PB, we see that its central angle is $2y$, so any angle that it subtends on the circle (eg. $\angle PCB$) is equal to y.
Thus, $\angle ACB = \angle QMB$, so QM is parallel to AC.

Solution 3



Join PB.
Since QP is tangent to the circle, then by the Tangent-Chord Theorem, $\angle QPB = \angle PCB = x$ (ie. the inscribed angle of a chord is equal to the angle between the tangent and chord.
Since BC is a diameter of the circle, then $\angle CPB = 90^\circ$ and so $\angle APB = 90^\circ$, whence $\angle APQ = 90^\circ - \angle QPB = 90^\circ - x$.
Looking at ΔABC, we see that $\angle PAQ = 90^\circ - x$, so $\angle PAQ = \angle APQ$, and so $AQ = QP$.
But QP and QB are both tangents to the circle (QB is tangent since it is perpendicular to a radius), so $QP = QB$.
But then $AQ = QB$ and $BM = MC$, so Q is the midpoint of AB and M is the midpoint of BC. Thus we can conclude that QM is parallel to AC.
(To justify this last statement, we can show very easily that ΔQBM is similar to ΔABC, and so show that $\angle CAB = \angle MQB$.)

9. *Solution 1*



Consider ΔBAD. Since we know the lengths of sides BA and AD and the cosine of the angle between them, we can calculate the length of BD using the cosine law:

$$\begin{aligned} BD &= \sqrt{BA^2 + AD^2 - 2(BA)(AD)\cos\angle BAD} \\ &= \sqrt{2 - 2\left(-\tfrac{1}{3}\right)} \\ &= \sqrt{\tfrac{8}{3}} \end{aligned}$$

Next, let $x = \cos\angle ABC$. Note that $DC = x$.

Since $ABCD$ is a cyclic quadrilateral, then $\angle ADC = 180^\circ - \angle ABC$, and so $\cos\angle ADC = -\cos\angle ABC = -x$.
Similarly, $\cos\angle BCD = -\cos\angle BAD = \frac{1}{3}$ (since $ABCD$ is a cyclic quadrilateral).
So we can now use the cosine law simultaneously in ΔADC and ΔABC (since side AC is common) in order to try to solve for BC:

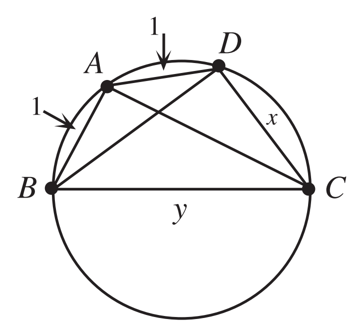
$$\begin{aligned}
1^2 + x^2 - 2(1)(x)\cos\angle ADC &= 1^2 + BC^2 - 2(1)(BC)\cos\angle ABC \\
1^2 + x^2 - 2(1)(x)(-x) &= 1^2 + BC^2 - 2(1)(BC)(x) \\
0 &= BC^2 - 2(BC)x - 3x^2 \\
0 &= (BC - 3x)(BC + x)
\end{aligned}$$

Since x is already a side length, then x must be positive (ie. $\angle ABC$ is acute), so $BC = 3x$.

Since $\cos\angle BCD = \frac{1}{3}$ and sides DC and BC are in the ratio 1 : 3, then ΔBCD must indeed be right-angled at D. (We could prove this by using the cosine law to calculate $BD^2 = 8x^2$ and then noticing that $DC^2 + BD^2 = BC^2$.)
Since ΔBCD is right-angled at D, then BC is a diameter of the circle.

Solution 2

Let $x = \cos\angle ABC = CD$, and let $BC = y$.



Since the opposite angles in a cyclic quadrilateral are supplementary, their cosines are negatives of each other. Thus, $\cos\angle ADC = -x$ and $\cos\angle BCD = \frac{1}{3}$.
Next, we use the cosine law four times: twice to calculate AC^2 in the two triangles ABC and ADC, and then twice to calculate BD^2 in the triangles ADB and CDB to obtain:

$$\begin{aligned}
1^2 + x^2 - 2(1)(x)\cos\angle ADC &= 1^2 + y^2 - 2(1)(y)\cos\angle ABC \\
1 + x^2 - 2x(-x) &= 1 + y^2 - 2y(x) \\
0 &= y^2 - 2xy - 3x^2 \\
0 &= (y - 3x)(y + x)
\end{aligned}$$

and

$$\begin{aligned}
1^2 + 1^2 - 2(1)(1)\cos\angle BAD &= x^2 + y^2 - 2xy\cos\angle BCD \\
2 - 2\left(-\tfrac{1}{3}\right) &= x^2 + y^2 - 2xy\left(\tfrac{1}{3}\right) \\
\tfrac{8}{3} &= x^2 + y^2 - \tfrac{2}{3}xy
\end{aligned}$$

From the first equation, since x is already a side length and so is positive, we must have that $y = 3x$.
Substituting into the second equation, we obtain

$$\tfrac{8}{3} = x^2 + (3x)^2 - \tfrac{2}{3}x(3x)$$
$$\tfrac{8}{3} = 8x^2$$
$$x = \tfrac{1}{\sqrt{3}}$$

since x must be positive. Thus, since $y = 3x$, then $y = \sqrt{3}$.

Looking then at ΔBDC, we have side lengths $BC = \sqrt{3}$, $CD = \frac{1}{\sqrt{3}}$ and $BD = \sqrt{\frac{8}{3}}$. (The last is from the left side of the second cosine law equation.) Thus, $BC^2 = CD^2 + BD^2$, and so ΔBDC is right-angled at D, whence BC is a diameter of the circle.

10. (a) To show that 8 is a savage integer, we must partition the set $\{1,2,3,4,5,6,7,8\}$ according to the given criteria.

Since the sum of the integers from 1 to 8 is 36, then the sum of the elements in each of the sets A, B, and C must be 12.

C must contain both 3 and 6.

A can contain only the numbers 1, 5, 7, and may not contain all of these.

B can contain only the numbers 2, 4, 8, and may not contain all of these.

So if we let $C = \{1,2,3,6\}$, $A = \{5,7\}$ and $B = \{4,8\}$, then these sets have the desired properties.

Therefore, 8 is a savage integer.

(b) We use the strategy of putting all of the multiples of 3 between 1 and n in the set C, all of the remaining even numbers in the set B, and all of the remaining numbers in the set A. The sums of these sets will not likely all be equal, but we then try to adjust the sums to by moving elements out of A and B into C, as we did in part (a), to try to make these sums equal. (Notice that we can't move elements either into A or B, or out of C.) We will use the notation $|C|$ to denote the sum of the elements of C.

Since we are considering the case of n even and we want to examine multiples of 3 less than or equal to n, it makes sense to consider n as having one of the three forms $6k$, $6k+2$ or $6k+4$. (These forms allow us to quickly tell what the greatest multiple of 3 less than n is.)

Case 1: $n = 6k$

In this case, C contains at least the integers 3, 6, 9, …, $6k$, and so the sum of C is greater than one-third of the sum of the integers from 1 to n, since if we divide the integers from 1 to $n = 6k$ into groups of 3 consecutive integers starting with 1, 2, 3, then the set C will always contain the largest of the 3.

Case 2: $n = 6k + 4$

Here, the sum of the integers from 1 to $n = 6k + 4$ is $\frac{1}{2}(6k+4)(6k+5) = 18k^2 + 27k + 10 = 3\left(6k^2 + 9k + 3\right) + 1$, which is never divisible by 3. Therefore, *n* cannot be savage in this case because the integers from 1 to *n* cannot be partitioned into 3 sets with equal sums.

Case 3: $n = 6k + 2$

Here, the sum of the integers from 1 to $n = 6k + 2$ is $\frac{1}{2}(6k+2)(6k+3) = 18k^2 + 15k + 3$, so the sum of the elements of each of the sets *A*, *B* and *C* should be $6k^2 + 5k + 1$, so that the sums are equal.

In this case *C*, contains at least the integers 3, 6, 9, …, 6*k*, and so

$$|C| \geq 3 + 6 + 9 + \cdots 6k = 3(1 + 2 + 3 + \cdots + 2k) = 3\left(\tfrac{1}{2}(2k)(2k+1)\right) = 6k^2 + 3k$$

The set *A* contains at most the integers 1, 3, 5, 7, …, $6k + 1$, but does not contain the odd multiples of 3 less than *n*, ie. the integers 3, 9, 15, …, $6k - 3$. Therefore,

$$\begin{aligned}|A| &\leq (1 + 3 + 5 + \cdots + 6k + 1) - (3 + 9 + \cdots + 6k - 3)\\ &= \tfrac{1}{2}(3k+1)[1 + 6k + 1] - \tfrac{1}{2}(k)[3 + 6k - 3]\\ &= (3k+1)(3k+1) - k(3k)\\ &= 6k^2 + 6k + 1\end{aligned}$$

(To compute the sum of each of these arithmetic sequences, we use the fact that the sum of an arithmetic sequence is equal to half of the number of terms times the sum of the first and last terms.)

The set *B* contains at most the integers 2, 4, 6, 8, …, $6k + 2$, but does not contain the even multiples of 3 less than *n*, ie. the integers 6, 12, …, $6k$. Therefore,

$$\begin{aligned}|B| &\leq (2 + 4 + 6 + \cdots + 6k + 2) - (6 + 12 + \cdots + 6k)\\ &= \tfrac{1}{2}(3k+1)[2 + 6k + 2] - \tfrac{1}{2}(k)[6 + 6k]\\ &= (3k+1)(3k+2) - k(3k+3)\\ &= 6k^2 + 6k + 2\end{aligned}$$

Thus, the set *C* is $2k + 1$ short of the desired sum, while the set *A* has a sum that is *k* too big and the set *B* has a sum that is $k + 1$ too big.

So in order to correct this, we would like to move elements from *A* adding to *k*, and elements from *B* which add to $k + 1$ all to set *C*.

Since we are assuming that n is savage, then this is possible, which means that $k+1$ must be even since every element in B is even, so the sum of any number of elements of B is even.
Therefore, k is odd, and so $k = 2l+1$ for some integer l, and so $n = 6(2l+1)+2 = 12l+8$, ie. $\frac{n+4}{12}$ is an integer.

Having examined all cases, we see that if n is an even savage integer, then $\frac{n+4}{12}$ is an integer.

(c) From (b), the only possible even savage integers less than 100 are those satisfying the condition that $\frac{n+4}{12}$ is an integer, ie. 8, 20, 32, 44, 56, 68, 80, 92. We already know that 8 is savage, so we examine the remaining 7 possibilities.
We make a table of the possibilities, using the notation from (b):

n	k	Sum of elements to remove from A	Sum of elements to remove from B	Possible?
20	3	3	4	No – cannot remove a sum of 3 from A.
32	5	5	6	Yes – remove 5 from A, 2 and 4 from B
44	7	7	8	Yes – remove 7 from A, 8 from B
56	9	9	10	No – cannot remove a sum of 9 from A.
68	11	11	12	Yes – remove 11 from A, 4 and 8 from B
80	13	13	14	Yes – remove 13 from A, 14 from B
92	15	15	16	No – cannot remove a sum of 15 from A (since could only use 1, 5, 7, 11, 13)

Therefore, the only even savage integers less than 100 are 8, 32, 44, 68 and 80.