



Canadian Mathematics Competition

An activity of The Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario

Euclid Contest (Grade 12)

for

The CENTRE for EDUCATION in MATHEMATICS and COMPUTING Awards

Tuesday, April 15, 2003

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Time: $2\frac{1}{2}$ hours

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Calculators are permitted, provided they are non-programmable and without graphic displays.

Do not open this booklet until instructed to do so. The paper consists of 10 questions, each worth 10 marks. Parts of each question can be of two types. **SHORT ANSWER** parts are worth 2 marks each (questions 1-2) or 3 marks each (questions 3-7). **FULL SOLUTION** parts are worth the remainder of the 10 marks for the question.

Instructions for SHORT ANSWER parts:

1. **SHORT ANSWER** parts are indicated like this: .
2. **Enter the answer in the appropriate box in the answer booklet.** For these questions, full marks will be given for a correct answer which is placed in the box. Part marks will be awarded **only if relevant work** is shown in the space provided in the answer booklet.

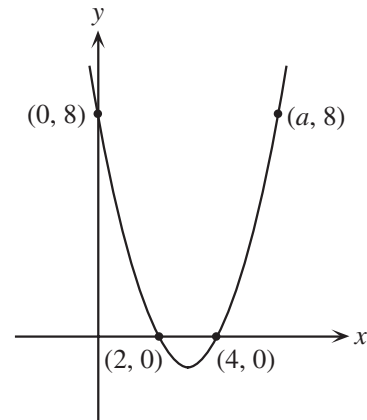
Instructions for FULL SOLUTION parts:

1. **FULL SOLUTION** parts are indicated like this: .
2. **Finished solutions must be written in the appropriate location in the answer booklet.** Rough work should be done separately. If you require extra pages for your finished solutions, foolscap will be supplied by your supervising teacher. Insert these pages into your answer booklet.
3. Marks are awarded for completeness, clarity, and style of presentation. A correct solution poorly presented will not earn full marks.

NOTE: At the completion of the contest, insert the information sheet inside the answer booklet.

- NOTE:
- Please read the instructions on the front cover of this booklet.
 - Place all answers in the answer booklet provided.
 - For questions marked “💡”, full marks will be given for a correct answer placed in the appropriate box in the answer booklet. **Marks may be given for work shown.** Students are strongly encouraged to show their work.
 - It is expected that all calculations and answers will be expressed as exact numbers such as 4π , $2 + \sqrt{7}$, etc., except where otherwise indicated.

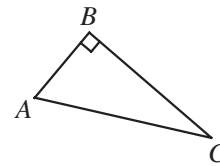
1. 💡 (a) In the diagram, the parabola cuts the y -axis at the point $(0, 8)$, cuts the x -axis at the points $(2, 0)$ and $(4, 0)$, and passes through the point $(a, 8)$. What is the value of a ?



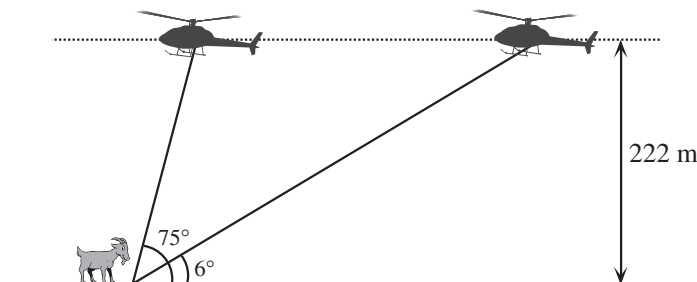
- 💡 (b) The quadratic equation $x^2 + 6x + k = 0$ has two equal roots. What is the value of k ?
- 🔍 (c) The line $y = 2x + 2$ intersects the parabola $y = x^2 - 3x + c$ at two points. One of these points is $(1, 4)$. Determine the coordinates of the second point of intersection.


2. 💡 (a) If $0^\circ < x < 90^\circ$ and $3\sin(x) - \cos(15^\circ) = 0$, what is the value of x to the nearest tenth of a degree?

- 💡 (b) In the diagram, $\triangle ABC$ is right-angled at B and $AC = 20$. If $\sin C = \frac{3}{5}$, what is the length of side BC ?



- 🔍 (c) A helicopter is flying due west over level ground at a constant altitude of 222 m and at a constant speed. A lazy, stationary goat, which is due west of the helicopter, takes two measurements of the angle between the ground and the helicopter. The first measurement the goat makes is 6° and the second measurement, which he makes 1 minute later, is 75° . If the helicopter has not yet passed over the goat, as shown, how fast is the helicopter travelling to the nearest kilometre per hour?



3.  (a) The function $f(x)$ has the property that $f(2x+3) = 2f(x) + 3$ for all x . If $f(0) = 6$, what is the value of $f(9)$?




- (b) Suppose that the functions $f(x)$ and $g(x)$ satisfy the system of equations

$$f(x) + 3g(x) = x^2 + x + 6$$


$$2f(x) + 4g(x) = 2x^2 + 4$$

for all x . Determine the values of x for which $f(x) = g(x)$.

4.  (a) In a short-track speed skating event, there are five finalists including two Canadians. The first three skaters to finish the race win a medal. If all finalists have the same chance of finishing in any position, what is the probability that neither Canadian wins a medal?




- (b) Determine the number of positive integers less than or equal to 300 that are multiples of 3 or 5, but are not multiples of 10 or 15.

5.  (a) In the series of odd numbers $1 + 3 + 5 - 7 - 9 - 11 + 13 + 15 + 17 - 19 - 21 - 23 \dots$ the signs alternate every three terms, as shown. What is the sum of the first 300 terms of the series?



- (b) A two-digit number has the property that the square of its tens digit plus ten times its units digit equals the square of its units digit plus ten times its tens digit. Determine all two-digit numbers which have this property, and are prime numbers.


6.  (a) A lead box contains samples of two radioactive isotopes of iron. Isotope A decays so that after every 6 minutes, the number of atoms remaining is halved. Initially, there are twice as many atoms of isotope A as of isotope B, and after 24 minutes there are the same number of atoms of each isotope. How long does it take the number of atoms of isotope B to halve?

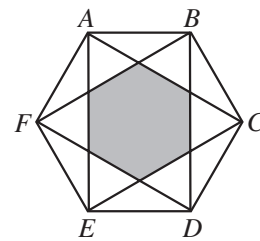


- (b) Solve the system of equations:

$$\log_{10}(x^3) + \log_{10}(y^2) = 11$$

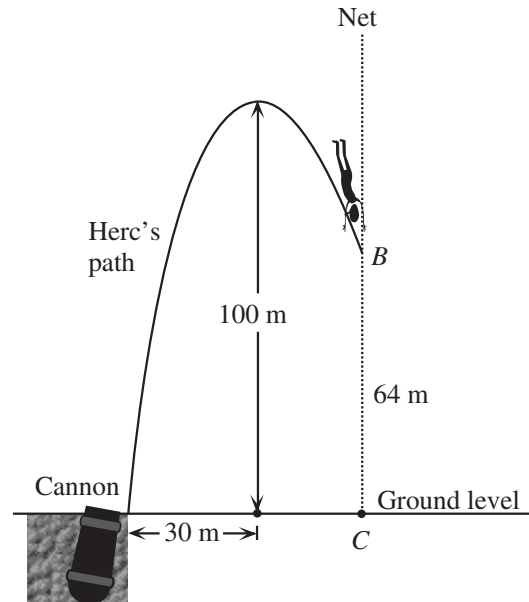
$$\log_{10}(x^2) - \log_{10}(y^3) = 3$$

7.  (a) A regular hexagon is a six-sided figure which has all of its angles equal and all of its side lengths equal. In the diagram, $ABCDEF$ is a regular hexagon with an area of 36. The region common to the equilateral triangles ACE and BDF is a hexagon, which is shaded as shown. What is the area of the shaded hexagon?

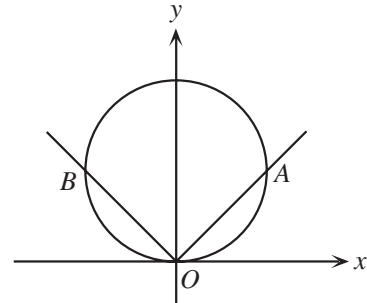




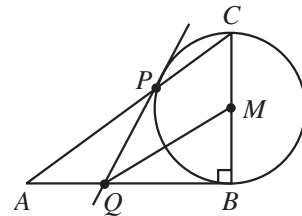
- (b) At the Big Top Circus, Herc the Human Cannonball is fired out of the cannon at ground level. (For the safety of the spectators, the cannon is partially buried in the sand floor.) Herc's trajectory is a parabola until he catches the vertical safety net, on his way down, at point B . Point B is 64 m directly above point C on the floor of the tent. If Herc reaches a maximum height of 100 m, directly above a point 30 m from the cannon, determine the horizontal distance from the cannon to the net.




8. (a) A circle with its centre on the y -axis intersects the graph of $y = |x|$ at the origin, O , and exactly two other distinct points, A and B , as shown. Prove that the ratio of the area of triangle ABO to the area of the circle is always $1 : \pi$.



- (b) In the diagram, triangle ABC has a right angle at B and M is the midpoint of BC . A circle is drawn using BC as its diameter. P is the point of intersection of the circle with AC . The tangent to the circle at P cuts AB at Q . Prove that QM is parallel to AC .



9. Cyclic quadrilateral $ABCD$ has $AB = AD = 1$, $CD = \cos \angle ABC$, and $\cos \angle BAD = -\frac{1}{3}$. Prove that BC is a diameter of the circumscribed circle.

10.  A positive integer n is called “savage” if the integers $\{1, 2, \dots, n\}$ can be partitioned into three sets A , B and C such that
- i) the sum of the elements in each of A , B , and C is the same,
 - ii) A contains only odd numbers,
 - iii) B contains only even numbers, and
 - iv) C contains every multiple of 3 (and possibly other numbers).
- (a) Show that 8 is a savage integer.
- (b) Prove that if n is an even savage integer, then $\frac{n+4}{12}$ is an integer.
- (c) Determine all even savage integers less than 100.

PUBLICATIONS

Students and parents who enjoy solving problems for fun and recreation may find the following publications of interest. They are an excellent resource for enrichment, problem solving and contest preparation.

Copies of Previous Canadian Mathematics Competitions

Copies of previous contests and solutions are available at no cost in both English and French at <http://www.cemc.uwaterloo.ca>

Problems Problems Problems Books

Each volume is a collection of problems (multiple choice and full solution), grouped into 9 or more topics. Questions are selected from previous Canadian Mathematics Competition contests, and full solutions are provided for all questions. The price is \$15. **(Available in English only.)**

Volume 1

- over 300 problems and full solutions
- 10 topics
- for students in Grades 9, 10, & 11
- French version of Volume 1 is available

Volume 3

- over 235 problems and full solutions
- 12 topics
- for senior high school students

Volume 5

- over 200 problems and full solutions
- 9 topics (different from Volume 3)
- for senior high school students

Volume 7

- over 300 problems and full solutions
- 12 topics
- for students in Grades 9 and 10



Volume 2

- over 325 problems and full solutions
- 10 topics (different from Volume 1)
- for students in Grades 9, 10, & 11

Volume 4

- over 325 problems and full solutions
- 12 topics
- for students in Grades 7, 8, & 9

Volume 6

- over 300 problems and full solutions
- 11 topics
- for students in Grades 7, 8, & 9

Volume 8

- over 200 problems and full solutions
- 10 topics
- for students in Grades 11 and 12



Problems and How To Solve Them - Volume 2

This new book continues the collection of problems available for senior level students. Included for each of the nine chapters is a discussion on solving problems, with suggested approaches. There are more than 160 new problems, almost all from Canadian Mathematics Competitions, with complete solutions. The price is \$20. **(Available in English only.)**

Orders should be addressed to: Canadian Mathematics Competition
Faculty of Mathematics, Room 5181
University of Waterloo
Waterloo, ON N2L 3G1

Include your name, address (with postal code), and telephone number.

Cheques or money orders in Canadian funds should be made payable to "Centre for Education in Mathematics and Computing". In Canada, add \$3.00 for the first item ordered for shipping and handling, plus \$1.00 for each subsequent item. No Provincial Sales Tax is required, but 7% GST must be added. Orders *outside of Canada ONLY*, add \$10.00 for the first item ordered for shipping and handling, plus \$2.00 for each subsequent item. **Prices for these publications will remain in effect until September 1, 2003.**

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