



# Canadian Mathematics Competition

An activity of The Centre for Education  
in Mathematics and Computing,  
University of Waterloo, Waterloo, Ontario

## *2002 Solutions* *Cayley Contest* (Grade 10)

for

**The CENTRE for EDUCATION in MATHEMATICS and  
COMPUTING**

Awards

1. Expanding and simplifying,  

$$5x + 2(4 + x) = 5x + (8 + 2x)$$

$$= 7x + 8$$

ANSWER: (C)

2. Evaluating,  

$$(2 + 3)^2 - (2^2 + 3^2) = 5^2 - (4 + 9)$$

$$= 25 - 13$$

$$= 12$$

ANSWER: (A)

3. If  $x = -3$ ,  

$$x^2 - 4(x - 5) = (-3)^2 - 4(-3 - 5)$$

$$= 9 - 4(-8)$$

$$= 41$$

ANSWER: (D)

4. Since  $n = \frac{5}{6}(240)$ , then  $\frac{2}{5}n = \frac{2}{5}\left(\frac{5}{6}\right)(240) = \frac{1}{3}(240) = 80$ .

ANSWER: (B)

5. Using exponent laws,  

$$2^{-2} \times 2^{-1} \times 2^0 \times 2^1 \times 2^2 = 2^{-2-1+0+1+2} = 2^0 = 1.$$

Alternatively,

$$2^{-2} \times 2^{-1} \times 2^0 \times 2^1 \times 2^2 = \frac{1}{4} \times \frac{1}{2} \times 1 \times 2 \times 4 = 1.$$

ANSWER: (B)

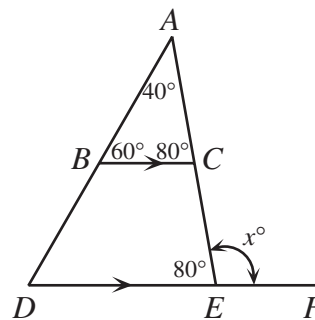
6. In  $\triangle ABC$ ,  

$$\angle ACB + 40^\circ + 60^\circ = 180^\circ$$

$$\angle ACB = 80^\circ$$
 Since  $BC$  is parallel to  $DE$ ,  $\angle AED = \angle ACB = 80^\circ$ .  
 So  

$$x^\circ = 180^\circ - \angle AED = 100^\circ$$

$$x = 100$$



ANSWER: (D)

7. Since the line has slope  $\frac{1}{2}$ , then for every 2 units we move to the right, the line rises by 1 unit. Therefore,  $(-2 + 2, 4 + 1) = (0, 5)$  lies on the line. Thus the y-intercept is 5.

ANSWER: (A)

8. Since Megan's scoring average was 18 points per game after 3 games, she must have scored  $3 \times 18 = 54$  points over her first three games. Similarly, she must have scored  $4 \times 17 = 68$  points over her first 4 games. So in the fourth game, she scored  $68 - 54 = 14$  points.

ANSWER: (E)

9. Since squares  $ABCD$  and  $DEFG$  have equal side lengths, then  $DC = DE$ , ie.  $\triangle CDE$  is isosceles. Therefore,

$$\angle DEC = \angle DCE = 70^\circ \text{ and so}$$

$$\angle CDE = 180^\circ - 70^\circ - 70^\circ = 40^\circ$$

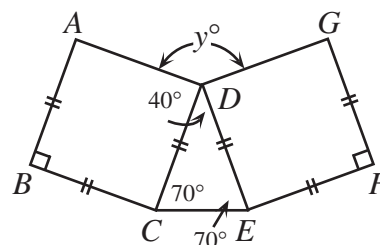
and

$$y^\circ = 360^\circ - \angle ADC - \angle CDE - \angle GDE$$

$$y^\circ = 360^\circ - 90^\circ - 40^\circ - 90^\circ$$

$$y^\circ = 140^\circ$$

$$y = 140$$



ANSWER: (E)

10. Let the original number be  $x$ . Then

$$\frac{x-5}{4} = \frac{x-4}{5}$$

$$5(x-5) = 4(x-4)$$

$$5x - 25 = 4x - 16$$

$$x = 9$$

ANSWER: (C)

11. Point  $B$  is the  $y$ -intercept of  $y = 2x - 8$ , and so has coordinates  $(0, -8)$  from the form of the line. Point  $A$  is the  $x$ -intercept of  $y = 2x - 8$ , so we set  $y = 0$  and obtain

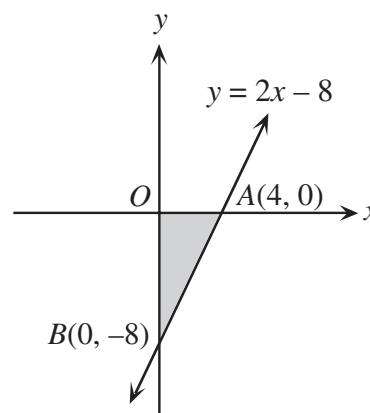
$$0 = 2x - 8$$

$$2x = 8$$

$$x = 4$$

and so  $A$  has coordinates  $(4, 0)$ . This tells us that

$$\text{Area of } \triangle AOB = \frac{1}{2}(OA)(OB) = \frac{1}{2}(4)(8) = 16.$$



ANSWER: (B)

12. After the price is increased by 40%, the new price is 140% of \$10.00, or \$14.00.

Then after this price is reduced by 30%, the final price is 70% of \$14.00, which is

$$0.7(\$14.00) = \$9.80.$$

ANSWER: (A)

13. By Pythagoras,

$$BC^2 = 120^2 + 160^2 = 40\,000$$

so  $BC = 200$  m.

Let  $FB = x$ . Then  $CF = 200 - x$ . From the given information,

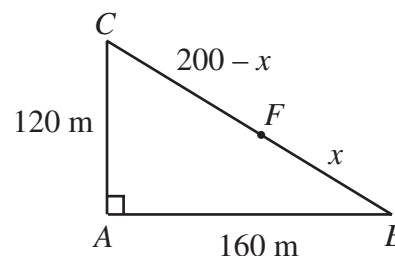
$$AC + CF = AB + BF$$

$$120 + 200 - x = 160 + x$$

$$160 = 2x$$

$$x = 80$$

Thus, the distance from  $F$  to  $B$  is 80 m.



ANSWER: (D)

14. *Solution 1*

We factor out a common factor of  $a$  from the first two terms and a common factor of  $b$  from the last two terms, and so

$$ac + ad + bc + bd = 68$$

$$a(c + d) + b(c + d) = 68$$

$$a(4) + b(4) = 68$$

$$4(a + b) = 68$$

$$a + b = 17$$

using the fact that  $c + d = 4$ .

Then  $a + b + c + d = (a + b) + (c + d) = 17 + 4 = 21$ .

*Solution 2*

We let  $c = d = 2$ , since  $c + d = 4$ , and then substitute these values to find that

$$ac + ad + bc + bd = 68$$

$$2a + 2a + 2b + 2b = 68$$

$$a + b = 17$$

Thus  $(a + b) + (c + d) = 17 + 4 = 21$ .

ANSWER: (D)

15. From  $A$ , we can travel to  $D$ ,  $E$  or  $B$ .

If we travel  $A \rightarrow D$ , we must then go  $D \rightarrow E \rightarrow F$ , following the arrows.

If we travel  $A \rightarrow E$ , we must then go  $E \rightarrow F$ .

If we travel  $A \rightarrow B$ , we can then travel from  $B$  to  $E$ ,  $C$  or  $F$ .

From  $E$  or  $C$ , we must travel directly to  $F$ .

Thus, there are 5 different paths from  $A$  to  $F$ .

ANSWER: (B)

16. Let the four consecutive positive integers be  $n$ ,  $n + 1$ ,  $n + 2$ ,  $n + 3$ .

So we need to solve the equation  $n(n+1)(n+2)(n+3) = 358\,800$ .

Now  $n(n+1)(n+2)(n+3)$  is approximately equal to  $n^4$ , so  $n^4$  is close to 358 800, and so  $n$  is close to  $\sqrt[4]{358\,800} \approx 24.5$ .

So we try  $24(25)(26)(27) = 421\,200$  which is too big, and so we try

$23(24)(25)(26) = 358\,800$ , which is the value we want.

So the sum is  $23 + 24 + 25 + 26 = 98$ .

ANSWER: (B)

17. A “double-single” number is of the form  $aab$ , where  $a$  and  $b$  are different digits. There are 9 possibilities for  $a$  (since  $a$  cannot be 0). For each of these possibilities, there are 9 possibilities for  $b$  (since  $b$  can be any digit from 0 to 9, provided that it doesn't equal  $a$ ). So there are  $9 \times 9 = 81$  possibilities in total.

ANSWER: (A)

18. *Solution 1*

Since  $\triangle ADF$  and  $\triangle ABC$  share  $\angle DAF$  and

$$\frac{AD}{AB} = \frac{AF}{AC} = \frac{1}{2}$$

then  $\triangle ADF$  is similar to  $\triangle ABC$ . So the ratio of their areas is the square of the ratio of their side lengths, that is

$$\text{Area of } \triangle ADF = \left(\frac{1}{2}\right)^2 \times \text{Area of } \triangle ABC.$$

Also from the similar triangles,  $\angle ADF = \angle ABC$ , so  $DF$  is parallel to  $BC$ . Therefore, since  $AG$  is perpendicular to  $BC$ , then  $AG$  is also perpendicular to  $DE$ , so  $AE$  is an altitude in  $\triangle ADF$ . Since  $\triangle ADF$  is isosceles,  $DE = EF$ .

Therefore,

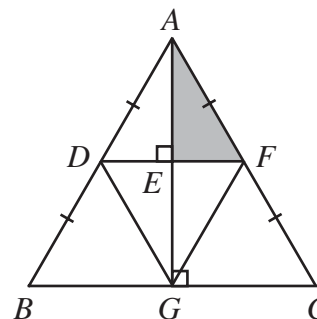
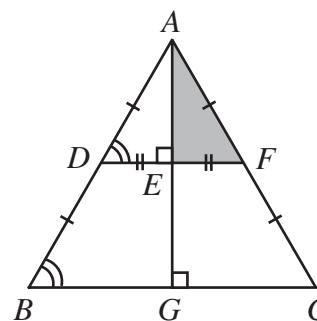
$$\begin{aligned} \text{Area of } \triangle ADF &= \frac{1}{2} \times \text{Area of } \triangle ADF \\ &= \frac{1}{8} \times \text{Area of } \triangle ABC \end{aligned}$$

*Solution 2*

Join  $G$  to  $D$  and  $G$  to  $F$ . This implies that we have four identical isosceles triangles:  $\triangle ADF$ ,  $\triangle GDF$ ,  $\triangle BDG$ , and  $\triangle CFG$ . Each of these identical triangles has an equal area.

Thus,

$$\begin{aligned} \text{Area of } \triangle AEF &= \frac{1}{2} \left( \frac{1}{4} \times \text{Area of } \triangle ABC \right) \\ &= \frac{1}{8} \times \text{Area of } \triangle ABC \end{aligned}$$



ANSWER: (E)

19. Let  $x = 777\,777\,777\,777\,777$  and  $y = 222\,222\,222\,222\,223$ .

So the required quantity is  $x^2 - y^2 = (x + y)(x - y)$ .

Now  $x + y = 1\,000\,000\,000\,000\,000$  and  $x - y = 555\,555\,555\,555\,554$ .

Thus,  $x^2 - y^2 = 555\,555\,555\,555\,554\,000\,000\,000\,000\,000$ , ie. the sum of the digits is  $14 \times 5 + 4 = 74$ .

ANSWER: (C)

20. Let the final depth of the water be  $h$ .

The total initial volume of water is

$$\pi(4\text{ m})^2(10\text{ m}) = 160\pi\text{ m}^3.$$

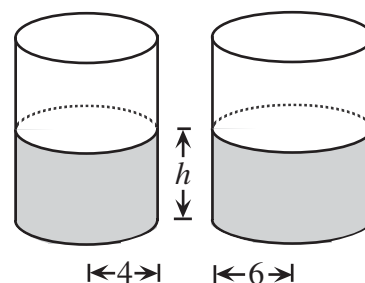
When the depths of water are equal,

$$\pi(4\text{ m})^2 h + \pi(6\text{ m})^2 h = 160\pi\text{ m}^3$$

$$(52\pi\text{ m}^2)h = 160\pi\text{ m}^3$$

$$h = \frac{160}{52}\text{ m}$$

$$h = \frac{40}{13}\text{ m}$$



ANSWER: (E)

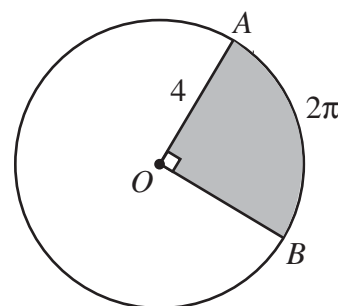
21. Let  $r$  be the radius of the circle.

Since  $\angle AOB = 90^\circ$ , then this sector is one quarter of the whole circle, so the circumference of the circle is

$$4 \times 2\pi = 8\pi.$$

So  $2\pi r = 8\pi$ , or  $r = 4$ .

Thus the area of sector  $AOB$  is  $\frac{1}{4}\pi r^2 = \frac{1}{4}\pi(16) = 4\pi$ .



ANSWER: (A)

22. When we add up a number of consecutive integers, the sum of these integers will be equal to the number of integers being added times the average of the integers being added.

Let  $N$  be the number of consecutive integers being added, and let  $A$  be the average of the consecutive integers being added. Thus,  $NA = 75$ .

Before we determine the possibilities for  $N$ , we make the observation that  $N$  must be less than 12, since the sum of the first 12 positive integers is 78, and thus the sum of any 12 or more consecutive positive integers is at least 78.

Case 1:  $N$  is odd.

In this case,  $A$  (the average) is an integer (there is a “middle number” among the integers being added). Therefore, since  $NA = 75$ ,  $N$  must be an odd positive factor of 75 which is bigger than 1 and less than 12, ie.  $N$  is one of 3 or 5. So there are two possibilities when  $N$  is odd, namely  $24 + 25 + 26 = 75$  and  $13 + 14 + 15 + 16 + 17 = 75$ .

Case 2:  $N$  is even.

In this case  $A$  will be half-way between two integers.

Set  $N = 2k$  and  $A = \frac{2l+1}{2}$ , where  $k$  and  $l$  are integers.

Then

$$2k \left( \frac{2l+1}{2} \right) = 75$$

$$k(2l+1) = 75$$

Thus  $k$  is a factor of 75, and so  $N = 2k$  is 2 times a factor of 75, ie.  $N$  could be 2, 6 or 10. So the possibilities here are  $37 + 38 = 75$ ,  $10 + 11 + 12 + 13 + 14 + 15 = 75$ , and  $3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12 = 75$ .

So there are 5 ways in total.

ANSWER: (E)

23. By Pythagoras in  $\triangle AFB$ ,

$$AF^2 + 9^2 = 41^2$$

$$AF = 40$$

By Pythagoras in  $\triangle AFD$ ,

$$FD^2 + 40^2 = 50^2$$

$$FD = 30$$

Since  $AD$  is parallel to  $BC$ ,  $\angle FAD = \angle FEB$ . Therefore,  $\triangle AFD$  is similar to  $\triangle FEB$ , so

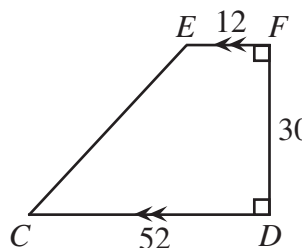
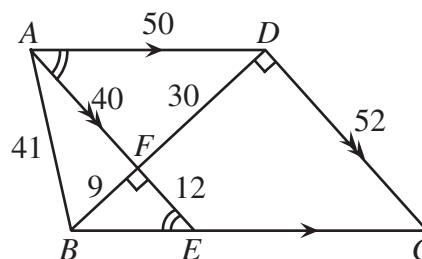
$$\frac{AF}{FD} = \frac{EF}{FB} \text{ or } \frac{40}{30} = \frac{EF}{9} \text{ or } EF = 12.$$

Since  $AE$  is perpendicular to  $BD$  and  $DC$  is perpendicular to  $BD$ , then  $AE$  is parallel to  $DC$ , so  $AECD$  is a parallelogram. Thus,  $DC = AE = 52$ .

Now consider quadrilateral  $FECD$ .

This quadrilateral is a trapezoid, so

$$\begin{aligned} \text{Area of } FECD &= \frac{1}{2}(EF + CD)(FD) \\ &= \frac{1}{2}(12 + 52)(30) \\ &= 960 \end{aligned}$$



ANSWER: (C)

24. We examine a vertical cross-section of the cylinder and the spheres that passes through the vertical axis of the cylinder and the centres of the spheres. (There is such a cross-section since the spheres will be pulled into this position by gravity.)

Let the centres of the spheres be  $O_1$  and  $O_2$ , as shown.

Join the centres of the spheres to each other and to the respective points of tangency of the spheres to the walls of the cylinder.

Then  $O_1O_2 = 6 + 9 = 15$  ( $O_1O_2$  passes through the point of tangency of the two spheres),  $O_1P = 6$  and  $O_2Q = 9$ , using the radii of the spheres.

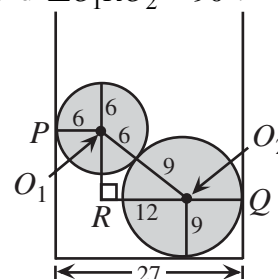
Next, draw  $\triangle O_1RO_2$  so that  $RO_1$  is perpendicular to  $O_1P$  and  $\angle O_1RO_2 = 90^\circ$ .

Then looking at the width of the cylinder, since  $PO_1 = 6$

and  $O_2Q = 9$ , we have

$$PO_1 + RO_2 + O_2Q = 27$$

$$RO_2 = 12$$



By Pythagoras in  $\triangle O_1RO_2$ , we see that  $O_1R = 9$ .

Then the depth of the water will be

$$\text{Radius of lower sphere} + RO_1 + \text{Radius of higher sphere} = 9 + 9 + 6 = 24.$$

So

$$\text{Volume of water} = (\text{Volume of cylinder to height of 24}) - (\text{Volume of spheres})$$

$$= \pi \left(\frac{27}{2}\right)^2 (24) - \frac{4}{3}\pi(6)^3 - \frac{4}{3}\pi(9)^3$$

$$= 4374\pi - 288\pi - 972\pi$$

$$= 3114\pi$$

Therefore, the volume of water required is  $3114\pi$  cubic units.

ANSWER: (D)

25. Subtracting the first equation from the second,

$$k^2x - kx - 6 = 0$$

$$(k^2 - k)x = 6$$

$$[k(k-1)]x = 6$$

Since we want both  $k$  and  $x$  to be integers, then  $k(k-1)$  is a factor of 6, ie. is equal to one of  $\pm 1, \pm 2, \pm 3, \pm 6$ . Now  $k(k-1)$  is the product of two consecutive integers, so on this basis we can eliminate six of these eight possibilities to obtain

$$k(k-1) = 2 \quad \text{or} \quad k(k-1) = 6$$

which yields

$$k^2 - k - 2 = 0 \quad \text{or} \quad k^2 - k - 6 = 0$$

and so  $k = 2, -1, 3, -2$ .



We now make a table of values of  $k$ ,  $x$  and  $y$  to check when  $y$  is also an integer. (We note that from the first equation,  $y = \frac{1}{5}(kx + 7)$ .)

$k$	$x$	$y$
2	3	$\frac{13}{5}$
-1	3	$\frac{4}{5}$
3	1	2
-2	1	1

So there are two values of  $k$  for which the lines intersect at a lattice point.

ANSWER: (B)