Canadian
Mathematics
Competition

An activity of The Centre for Education
in Mathematics and Computing,
University of Waterloo, Waterloo, Ontario

2001 Solutions
Euclid Contest (Grade 12)

for

The CENTRE for EDUCATION in MATHEMATICS and COMPUTING
Awards

1. (a) What are the values of x such that $(2x-3)^2=9$?

Solution 1

$$(2x-3)^2=9$$
$$(2x-3)^2-9=0$$
$$(2x-3-3)(2x-3+3)=0 \qquad \text{[by difference of squares]}$$
$$(2x-6)(2x)=0$$
so $x=3$ or $x=0$.

Solution 2

$(2x-3)^2=9$

$2x-3=3$ or $2x-3=-3$

Therefore $x=3$ or $x=0$.

Solution 3

$$4x^2-12x+9=9$$
$$4x^2-12x=0$$
$$4x(x-3)=0$$
Therefore $x=0$ or $x=3$.

(b) If $f(x)=x^2-3x-5$, what are the values of k such that $f(k)=k$?

Solution

If $f(k)=k$, then $k^2-3k-5=k$
$$k^2-4k-5=0$$
$$(k-5)(k+1)=0$$
so $k=5$ or $k=-1$.

(c) Determine all (x, y) such that $x^2+y^2=25$ and $x-y=1$.

Solution 1 (Algebraic)

Since $x-y=1$, then $x=y+1$ (or $y=x-1$).

So since $x^2+y^2=25$, then

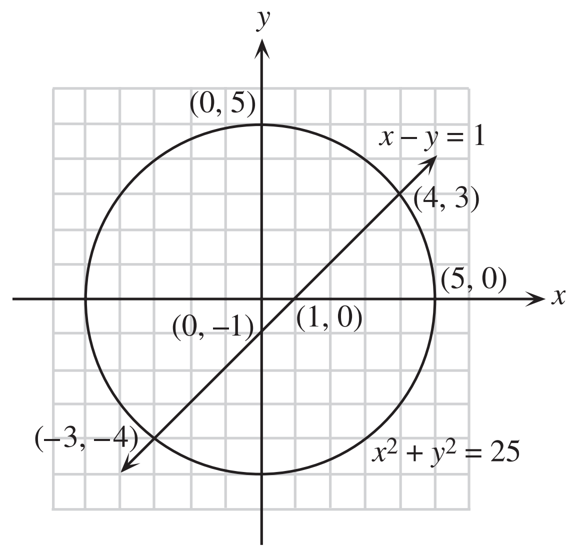
$$(y+1)^2+y^2=25 \qquad \text{or} \qquad x^2+(x-1)^2=25$$

$y^2+2y+1+y^2=25$	$x^2+x^2-2x+1=25$
$2y^2+2y-24=0$	$2x^2-2x-24=0$
$y^2+y-12=0$	$x^2-x-12=0$
$(y+4)(y-3)=0$	$(x-4)(x+3)=0$
$y=-4, 3$	$x=4, -3$
and using $x=y+1$,	and using $y=x-1$,
we get $x=-3, 4$.	we get $y=3, -4$.

So the solutions are $(x, y)=(-3,-4), (4, 3)$.

Solution 2 (Graphical)

Placing each of $x^2+y^2=25$ and $x-y=1$ on a grid we have the diagram at the right.



Therefore, the solutions are $(x, y)=(-3,-4)$, (4, 3).

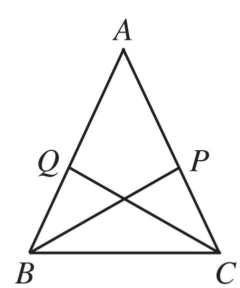
2. (a) The vertex of the parabola $y=(x-b)^2+b+h$ has coordinates $(2, 5)$. What is the value of h?

Solution

Since the x-coordinate of the vertex is 2, then $b=2$.

Since the y-coordinate of the vertex is 5, then $b+h=5$. Since $b=2$, then $h=3$.

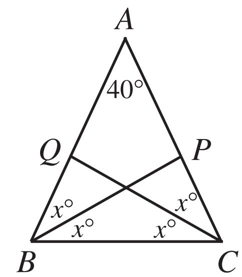
(b) In the isosceles triangle ABC, $AB=AC$ and $\angle BAC=40°$. Point P is on AC such that BP is the bisector of $\angle ABC$. Similarly, Q is on AB such that CQ bisects $\angle ACB$. What is the size of $\angle APB$, in degrees?



Solution

Let $\angle ABC = 2x°$. Since ΔABC is isosceles, then $\angle ACB = 2x°$.
Since BP bisects $\angle ABC$, $\angle ABP = \angle CBP = x°$.
Similarly, $\angle ACQ = \angle BCQ = x°$.
The angles in ΔABC add to 180°, so

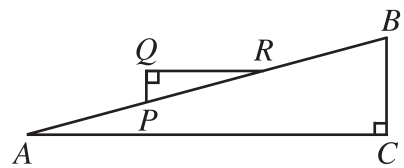
$$40° + 2x° + 2x° = 180°$$
$$x = 35.$$



In ΔAPB, the angles add to 180°, so

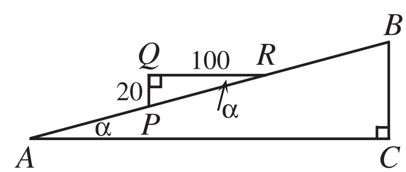
$$40° + 35° + \angle APB = 180°$$
$$\angle APB = 105°.$$

(c) In the diagram, $AB = 300$, $PQ = 20$, and $QR = 100$. Also, QR is parallel to AC. Determine the length of BC, to the nearest integer.



Solution 1

Since $QR \| AC$, $\angle QRP = \angle BAC = \alpha$ (alternating angles).
From ΔRPQ, $\tan\alpha = \frac{1}{5}$.
In ΔACB, since $\tan\alpha = \dfrac{1}{5} = \dfrac{BC}{AC}$, let $BC = x$ and $AC = 5x$. (This argument could also be made by just using the fact that ΔRQP and ΔACB are similar.)

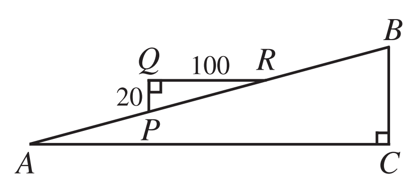


By Pythagoras, $x^2 + 25x^2 = 300^2$, $x = \sqrt{\frac{90\,000}{25}} \doteq 58.83$.
Therefore $BC = 59$ m to the nearest metre.

Solution 2

Since $QR \| AC$, $\angle QRP = \angle BAC$ (alternating angles).
This means $\Delta ABC \sim \Delta RPQ$ (two equal angles).
By Pythagoras,

$$PR^2 = QP^2 + QR^2$$
$$PR = \sqrt{100^2 + 20^2} = \sqrt{10\,400}.$$



Since $\Delta ABC \sim \Delta RPQ$,

$$\frac{BC}{AB}=\frac{PQ}{RP}$$
$$BC=\frac{AB\cdot PQ}{RP}$$
$$=\frac{300\cdot 20}{\sqrt{10\,400}}$$
$$\doteq 58.83$$

BC is 59 m (to the nearest metre).

3. (a) In an increasing sequence of numbers with an odd number of terms, the difference between any two consecutive terms is a constant *d*, and the middle term is 302. When the last 4 terms are removed from the sequence, the middle term of the resulting sequence is 296. What is the value of *d*?

Solution 1

Let the number of terms in the sequence be $2k+1$.
We label the terms $a_1, a_2, ..., a_{2k+1}$.
The middle term here is $a_{k+1}=302$.
Since the difference between any two consecutive terms in this increasing sequence is *d*, $a_{m+1}-a_m=d$ for $m=1, 2, ..., 2k$.
When the last 4 terms are removed, the last term is now a_{2k-3} so the middle term is then $a_{k-1}=296$. (When four terms are removed from the end, the middle term shifts two terms to the left.)
Now $6=a_{k+1}-a_{k-1}=(a_{k+1}-a_k)+(a_k-a_{k-1})=d+d=2d$.
Therefore $d=3$.

Solution 2

If the last four terms are removed from the sequence this results in 302 shifting 2 terms to the left in the new sequence meaning that $302-296=2d$, $d=3$.

(b) There are two increasing sequences of five consecutive integers, each of which have the property that the sum of the squares of the first three integers in the sequence equals the sum of the squares of the last two. Determine these two sequences.

Solution

Let *n* be the smallest integer in one of these sequences.
So we want to solve the equation $n^2+(n+1)^2+(n+2)^2=(n+3)^2+(n+4)^2$ (translating the given problem into an equation).
Thus $n^2+n^2+2n+1+n^2+4n+4=n^2+6n+9+n^2+8n+16$

$$n^2 - 8n - 20 = 0$$
$$(n-10)(n+2) = 0.$$

So $n = 10$ or $n = -2$.
Therefore, the sequences are 10, 11, 12, 13, 14 and -2, -1, 0, 1, 2.

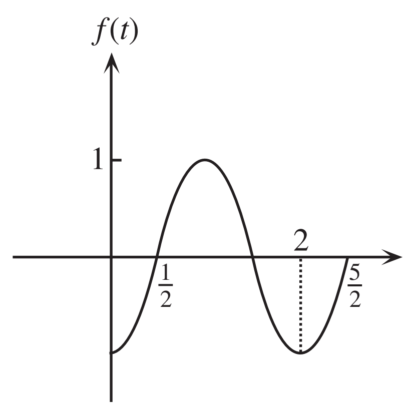
Verification
$(-2)^2 + (-1)^2 + 0^2 = 1^2 + 2^2 = 5$ and $10^2 + 11^2 + 12^2 = 13^2 + 14^2 = 365$

4. (a) If $f(t) = \sin\left(\pi t - \frac{\pi}{2}\right)$, what is the smallest positive value of t at which $f(t)$ attains its minimum value?

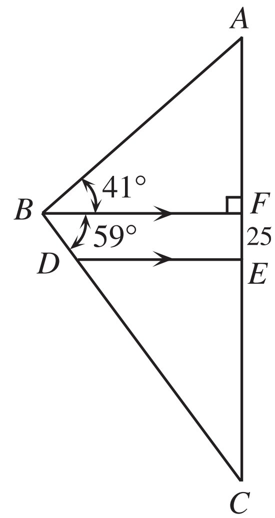
Solution 1
Since $t > 0$, $\pi t - \frac{\pi}{2} > -\frac{\pi}{2}$. So $\sin\left(\pi t - \frac{\pi}{2}\right)$ first attains its minimum value when

$$\pi t - \frac{\pi}{2} = \frac{3\pi}{2}$$
$$t = 2.$$

Solution 2
Rewriting $f(t)$ as, $f(t) = \sin\left[\pi\left(t - \frac{1}{2}\right)\right]$.
Thus $f(t)$ has a period $\frac{2\pi}{\pi} = 2$ and appears in the diagram at the right.
Thus $f(t)$ attains its minimum at $t = 2$. Note that $f(t)$ attains a minimum value at $t = 0$ but since $t > 0$, the required answer is $t = 2$.



(b) In the diagram, $\angle ABF = 41°$, $\angle CBF = 59°$, DE is parallel to BF, and $EF = 25$. If $AE = EC$, determine the length of AE, to 2 decimal places.



Solution

Let the length of $AE = EC$ be x.
Then $AF = x - 25$.

In, ΔBCF, $\dfrac{x+25}{BF} = \tan(59°)$.

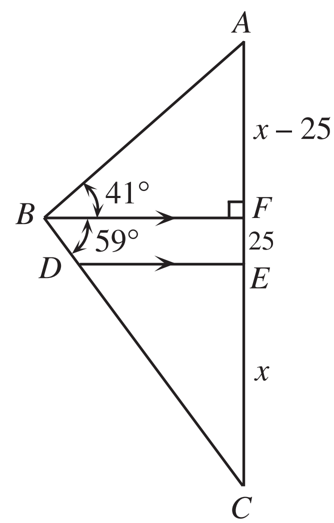
In ΔABF, $\dfrac{x-25}{BF} = \tan(41°)$.

Solving for BF in these two equations and equating,

$$BF = \frac{x+25}{\tan 59°} = \frac{x-25}{\tan 41°}$$

so
$$(\tan 41°)(x+25) = (\tan 59°)(x-25)$$
$$25(\tan 59° + \tan 41°) = x(\tan 59° - \tan 41°)$$
$$x = \frac{25(\tan 59° + \tan 41°)}{\tan 59° - \tan 41°}$$
$$x \doteq 79.67.$$

Therefore the length of AE is 79.67.



5. (a) Determine all integer values of x such that $(x^2-3)(x^2+5) < 0$.

Solution

Since $x^2 \ge 0$ for all x, $x^2 + 5 > 0$. Since $(x^2-3)(x^2+5) < 0$, $x^2 - 3 < 0$, so $x^2 < 3$ or $-\sqrt{3} < x < \sqrt{3}$. Thus $x = -1, 0, 1$.

(b) At present, the sum of the ages of a husband and wife, P, is six times the sum of the ages of their children, C. Two years ago, the sum of the ages of the husband and wife was ten times the sum of the ages of the same children. Six years from now, it will be three times the sum of the ages of the same children. Determine the number of children.

Solution

Let n be the number of children.

At the present, $P = 6C$, where P and C are as given. (1)

Two years ago, the sum of the ages of the husband and wife was $P - 4$, since they were each two years younger.

Similarly, the sum of the ages of the children was $C - n(2)$ (n is the number of children).

So two years ago, $P - 4 = 10(C - 2n)$ (2), from the given condition.

Similarly, six years from now, $P + 12 = 3(C + 6n)$ (3), from the given condition.

We want to solve for n.

Substituting (1) into each of (2) and (3),

$6C - 4 = 10(C - 2n)$ or $20n - 4C = 4$ or $5n - C = 1$

$6C + 12 = 3(C + 6n)$ or $-18n + 3C = -12$ or $-6n + C = -4$

Adding these two equations, $-n = -3$, so $n = 3$.

Therefore, there were three children.

6. (a) Four teams, A, B, C, and D, competed in a field hockey tournament. Three coaches predicted who would win the Gold, Silver and Bronze medals:

Medal	Gold	Silver	Bronze
Team			

- Coach 1 predicted Gold for A, Silver for B, and Bronze for C,
- Coach 2 predicted Gold for B, Silver for C, and Bronze for D,
- Coach 3 predicted Gold for C, Silver for A, and Bronze for D.

Each coach predicted exactly one medal winner correctly. Complete the table **in the answer booklet** to show which team won which medal.

Solution

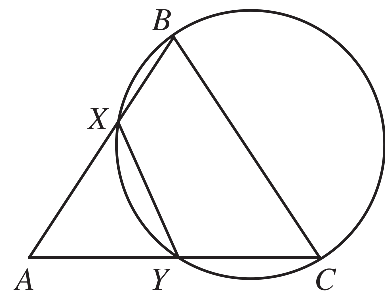
If A wins gold, then Coach 1 has one right. For Coach 3 to get one right, D must win bronze, since A cannot win silver. Since D wins bronze, Coach 2 gets one right. So C can't win silver, so B does which means Coach 1 has two right, which can't happen. So A doesn't win gold.

If B wins gold, then Coach 2 has one right. For Coach 1 to get one right, C wins bronze, as B can't win silver.

For Coach 3 to get one right, A wins silver.

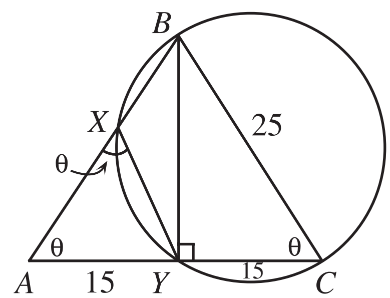
So Gold to B, Silver to A and Bronze to C satisfies the conditions.

(b) In triangle ABC, $AB = BC = 25$ and $AC = 30$. The circle with diameter BC intersects AB at X and AC at Y. Determine the length of XY.



Solution 1

Join BY. Since BC is a diameter, then $\angle BYC = 90°$.
Since $AB = BC$, ΔABC is isosceles and BY is an altitude in ΔABC, then $AY = YC = 15$.
Let $\angle BAC = \theta$.
Since ΔABC is isosceles, $\angle BCA = \theta$.
Since $BCYX$ is cyclic, $\angle BXY = 180 - \theta$ and so $\angle AXY = \theta$.
Thus ΔAXY is isosceles and so $XY = AY = 15$.
Therefore $XY = 15$.



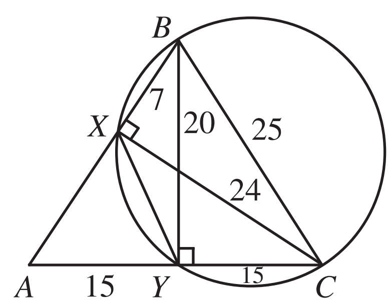
Solution 2

Join BY. $\angle BYC = 90°$ since it is inscribed in a semicircle.
ΔBAC is isosceles, so altitude BY bisects the base.
Therefore $BY = \sqrt{25^2 - 15^2} = 20$.
Join CX. $\angle CXB = 90°$ since it is also inscribed in a semicircle.
The area of ΔABC is

$$\tfrac{1}{2}(AC)(BY) = \tfrac{1}{2}(AB)(CX)$$

$$\cancel{\tfrac{1}{2}}(30)(20) = \cancel{\tfrac{1}{2}}(25)(CX)$$

$$CX = \tfrac{600}{25} = 24.$$



From ΔABY we conclude that $\cos \angle ABY = \dfrac{BY}{AB} = \dfrac{20}{25} = \dfrac{4}{5}$.

In ΔBXY, applying the Law of Cosines we get $(XY)^2 = (BX)^2 + (BY)^2 - 2(BX)(BY)\cos \angle XBY$.

Now (by Pythagoras ΔBXC),

$$\begin{aligned} BX^2 &= BC^2 - CX^2 \\ &= 25^2 - 24^2 \\ &= 49 \\ BX &= 7. \end{aligned}$$

Therefore $XY^2 = 7^2 + 20^2 - 2(7)(20)\frac{4}{5}$

$$\begin{aligned} &= 49 + 400 - 224 \\ &= 225. \end{aligned}$$

Therefore $XY = 15$.

7. (a) What is the value of x such that $\log_2(\log_2(2x-2)) = 2$?

Solution

$$\begin{aligned} \log_2(\log_2(2x-2)) &= 2 \\ \log_2(2x-2) &= 2^2 \\ 2x-2 &= 2^{\left(2^2\right)} \\ 2x-2 &= 2^4 \\ 2x-2 &= 16 \\ 2x &= 18 \\ x &= 9 \end{aligned}$$

(b) Let $f(x) = 2^{kx} + 9$, where k is a real number. If $f(3) : f(6) = 1 : 3$, determine the value of $f(9) - f(3)$.

Solution

From the given condition,

$$\frac{f(3)}{f(6)} = \frac{2^{3k}+9}{2^{6k}+9} = \frac{1}{3}$$

$$3\left(2^{3k}+9\right) = 2^{6k}+9$$

$$0 = 2^{6k} - 3\left(2^{3k}\right) - 18.$$

We treat this as a quadratic equation in the variable $x = 2^{3k}$, so

$$0 = x^2 - 3x - 18$$

$$0 = (x-6)(x+3).$$

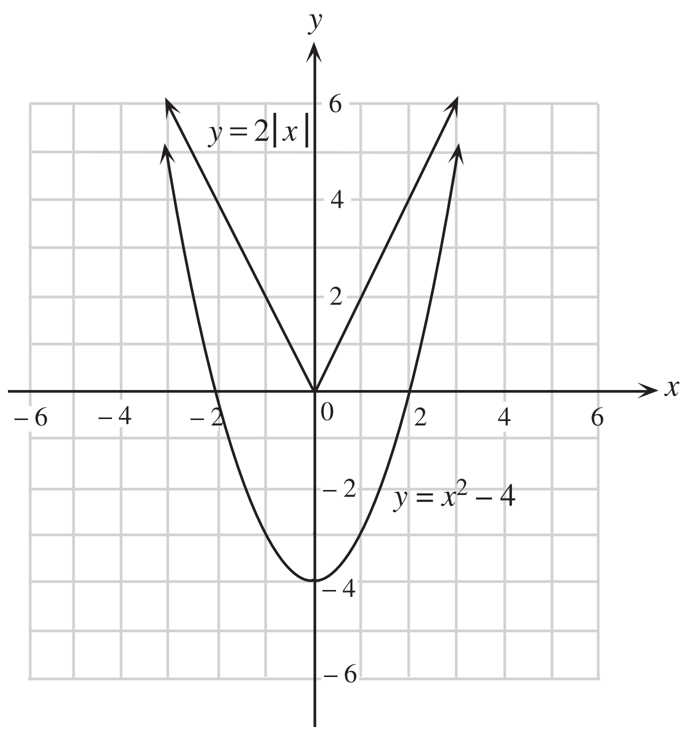
Therefore, $2^{3k} = 6$ or $2^{3k} = -3$. Since $2^a > 0$ for any a, then $2^{3k} \neq -3$.

So $2^{3k} = 6$. We could solve for k here, but this is unnecessary.

We calculate
$$\begin{aligned} f(9)-f(3) &= \left(2^{9k}+9\right)-\left(2^{3k}+9\right) \\ &= 2^{9k}-2^{3k} \\ &= \left(2^{3k}\right)^3-2^{3k} \\ &= 6^3-6 \\ &= 210. \end{aligned}$$
Therefore $f(9)-f(3)=210$.

8. (a) On the grid provided in the answer booklet, sketch $y=x^2-4$ and $y=2|x|$.

Solution



(b) Determine, with justification, all values of k for which $y=x^2-4$ and $y=2|x|+k$ do **not** intersect.

Solution

Since each of these two graphs is symmetric about the y-axis (i.e. both are even functions), then we only need to find k so that there are no points of intersection with $x \geq 0$.
So let $x \geq 0$ and consider the intersection between $y=2x+k$ and $y=x^2-4$.
Equating, we have, $2x+k=x^2-4$.
Rearranging, we want $x^2-2x-(k+4)=0$ to have no solutions.

For no solutions, the discriminant is negative, i.e.

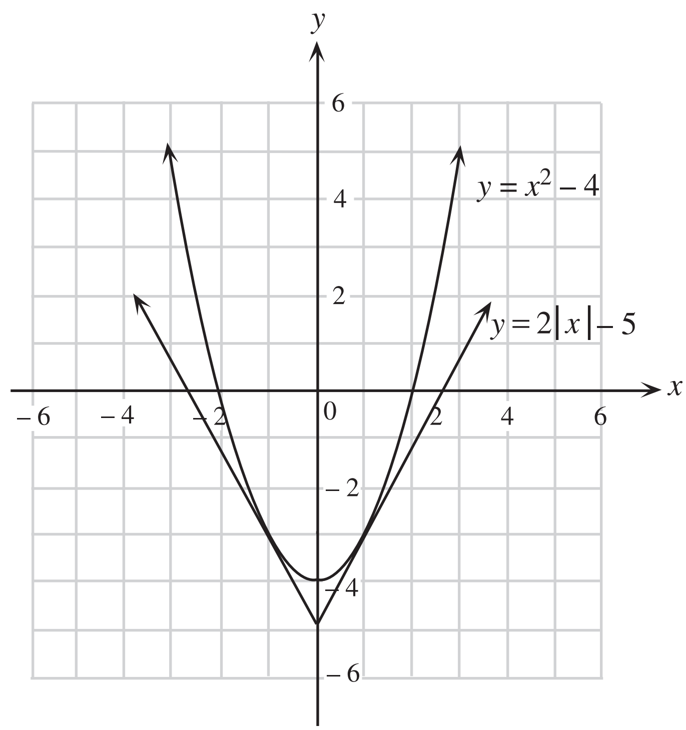
$$\begin{aligned} 20+4k &< 0 \\ 4k &< -20 \\ k &< -5. \end{aligned}$$

So $y = x^2 - 4$ and $y = 2|x| + k$ have no intersection points when $k < -5$.

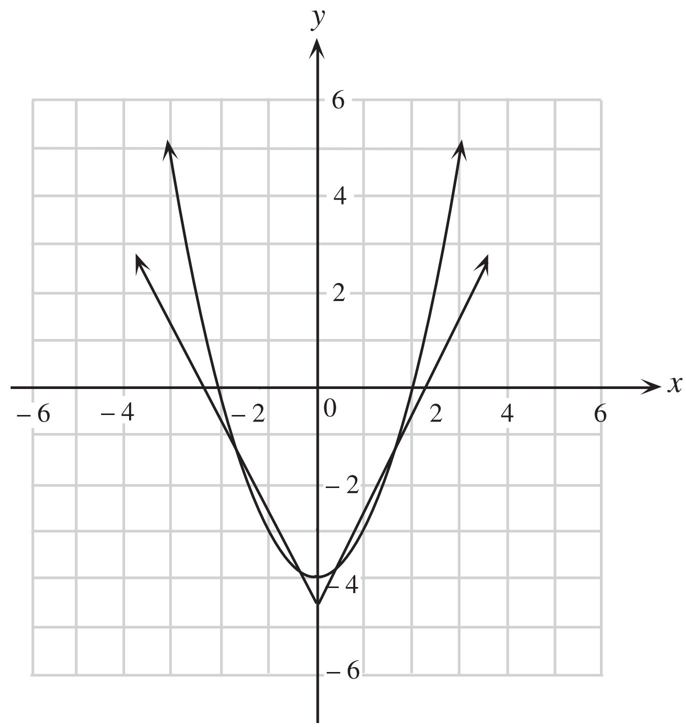
(c) State the values of k for which $y = x^2 - 4$ and $y = 2|x| + k$ intersect in exactly two points. (Justification is not required.)

Solution Analysing Graphs

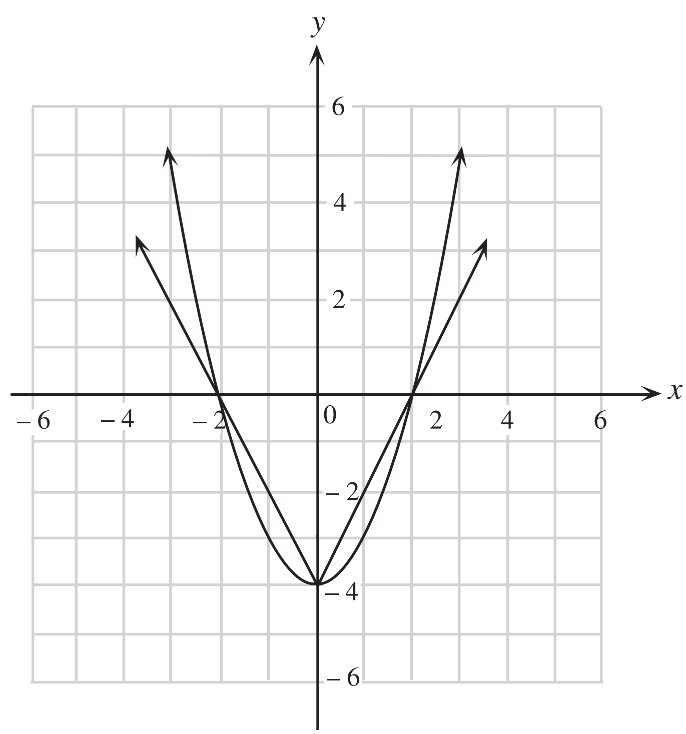
For $k < -5$, there are no points of intersection. When $k = -5$, the graph with equation $y = 2|x| + k$ is tangent to the graph with equation $y = x^2 - 4$ for both $x \geq 0$ and $x \leq 0$. So $k = -5$ is one possibility for two intersection points.



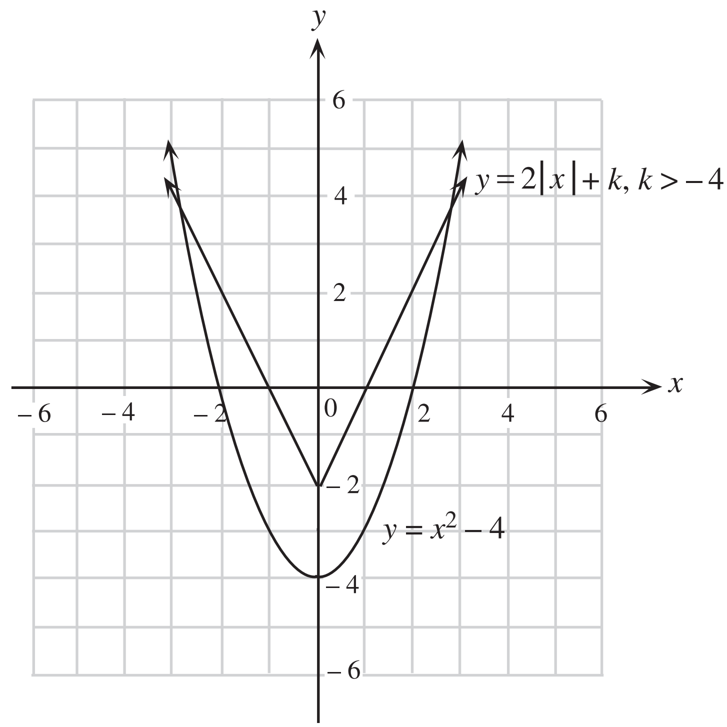
For $-5 < k < -4$ a typical graph appears on the right.
i.e. for $-5 < k < -4$, there will be 4 points of intersection.



When $k = -4$, a typical graph appears on the right.

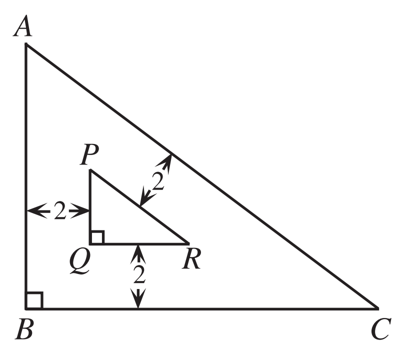


So when $k > -4$, there will only be two points of intersection, as the contact point at the cusp of $y = 2|x| - 4$ will be eliminated. An example where $k = -2$ is shown.



So the possibility for exactly two distinct points of intersection are $k = -5$, $k > -4$.

9. Triangle ABC is right-angled at B and has side lengths which are integers. A second triangle, PQR, is located inside ΔABC as shown, such that its sides are parallel to the sides of ΔABC and the distance between parallel lines is 2. Determine the side lengths of all possible triangles ABC, such that the area of ΔABC is 9 times that of ΔPQR.



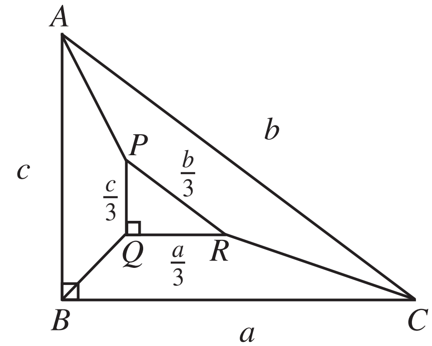
Solution 1

Let the sides of ΔABC be $AB = c$, $BC = a$, $AC = b$, a, b, c are all integers.

Since the sides of ΔPQR are all parallel to the sides of ΔABC, then ΔABC is similar to ΔPQR.

Now the ratio of areas of ΔABC to ΔPQR is $9 = 3^2$ to 1, so the ratio of side lengths will be 3 to 1.

So the sides of ΔPQR are $PQ = \frac{c}{3}$, $QR = \frac{a}{3}$, $PR = \frac{b}{3}$.



So we can label the diagram as indicated.
We join the corresponding vertices of the two triangles as

Area of trapezoid $BQRC$
Area of trapezoid $CRPA$
Area of trapezoid $APQB$
+ Area of ΔPQR

Area of ΔABC.

Doing so gives,

$$2\left(\frac{2}{3}a\right)+2\left(\frac{2}{3}b\right)+2\left(\frac{2}{3}c\right)+\frac{ac}{18}=\frac{ac}{72}$$

Or upon simplifying $ac = 3a + 3b + 3c$ (Note that this relationship can be derived in a variety of ways.)

$$\begin{aligned}
ac &= 3c + 3b + 3a \\
ac - 3c - 3a &= 3b \\
ac - 3c - 3a &= 3\sqrt{a^2 + c^2} && \text{(since } b = \sqrt{a^2+c^2}\text{)} \\
a^2c^2 + 9c^2 + 9a^2 - 6ac^2 - 6a^2c + 18ac &= 9\left(a^2 + c^2\right) && \text{(squaring both sides)} \\
ac(ac - 6c - 6a + 18) &= 0 \\
ac - 6c - 6a + 18 &= 0 && \text{(as } ac \neq 0\text{)} \\
c(a-6) &= 6a - 18 \\
c &= \frac{6a-18}{a-6} \\
c &= 6 + \frac{18}{a-6}.
\end{aligned}$$

Since a is a side of a triangle, $a > 0$. We are now looking for positive integer values such that $\frac{18}{a-6}$ is also an integer.

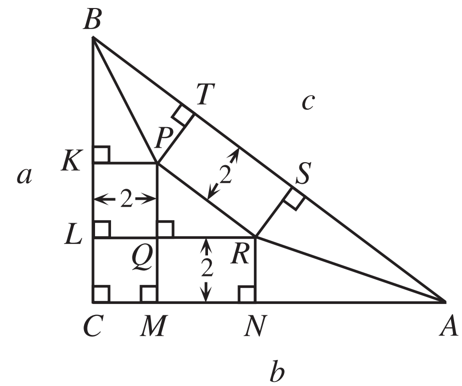
The only possible values for a are 3, 7, 8, 9, 12, 15 and 24.
Tabulating the possibilities and calculating values for b and c gives,

a	3	7	8	9	12	15	24
c	0	24	15	12	9	8	7
b	–	25	17	15	15	17	25

Thus the only possibilities for the triangle are $(7, 24, 25)$, $(8, 15, 7)$ and $(9, 12, 15)$.

Solution 2

The two triangles are similar with areas in the ratio 1:9.
Therefore the sides are in the ratio 1:3.
Let $a = BC$, $b = CA$, $c = BA$.



Then $\frac{a}{3} = PQ$, $\frac{b}{3} = QR$, $\frac{c}{3} = PR$.

Locate points K, L on BC; M, N on CA; and T, S on AB as shown.

$BC = BK + KL + LC$

$a = BK + \frac{a}{3} + 2$

Therefore $BK = \frac{2}{3}a - 2$.

In a similar way, $AN = \frac{2}{3}b - 2$.

Now $\Delta BKP \cong \Delta BTP$ and $\Delta ANR \cong \Delta ASR$, both by HL.

Therefore $BT = BK = \frac{2}{3}a - 2$ and $AS = AN = \frac{2}{3}b - 2$.

Now, $AB = AS + ST + BT$

$$c = \frac{2}{3}b - 2 + \frac{c}{3} + \frac{2}{3}a - 2$$

$$\frac{2}{3}c = \frac{2}{3}b + \frac{2}{3}a - 4$$

$$c = b + a - 6$$

$$b = c + (6 - a).$$

By Pythagoras, $a^2 + b^2 = c^2$

$$a^2 + [c + (6 - a)]2 = c^2$$
$$a^2 + \cancel{c^2} + 2c(6 - a) + (6 - a)2 = \cancel{c^2}$$
$$a^2 + (6 - a)^2 = -2c(6 - \mathrm{a})$$
$$2a^2 - 12a + 36 = 2c(a - 6)$$
$$a^2 - 6a + 18 = c(a - 6)$$

$$c = \frac{a^2 - 6a + 18}{a - 6}$$

$$c = \frac{a(a - 6) + 18}{a - 6}$$

$$c = a + \frac{18}{a - 6}.$$

Since a and c are integers, $a - 6$ is a divisor of 18.

Also since $b < c$ and $b = c + (6 - a)$, we conclude that $6 - a < 0$ so $a - 6 > 0$.

Thus $a - 6$ can be 1, 2, 3, 6, 9, 18.

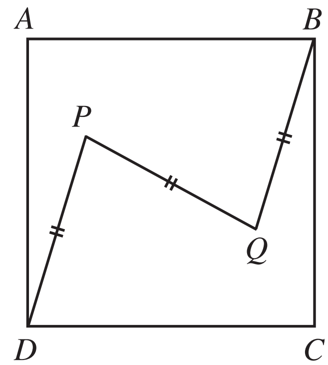
The values of a are: 7, 8, 9, 12, 15, 24.

Matching values for c: 25, 17, 15, 15, 17, 25

Matching values for b: 24, 15, 12, 9, 8, 7

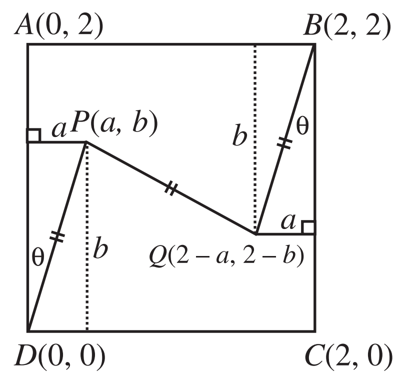
The distinct triangles are (7, 24, 25), (8, 15, 17) and (9, 12, 15).

10. Points P and Q are located inside the square $ABCD$ such that DP is parallel to QB and $DP = QB = PQ$. Determine the minimum possible value of $\angle ADP$.



Solution 1

Placing the information on the coordinate axes, the diagram is indicated to the right.



We note that P has coordinates (a, b).

By symmetry (or congruency) we can label lengths a and b as shown. Thus Q has coordinates $(2-a, 2-b)$.

Since $PD = PQ$, $a^2 + b^2 = (2-2a)^2 + (2-2b)^2$

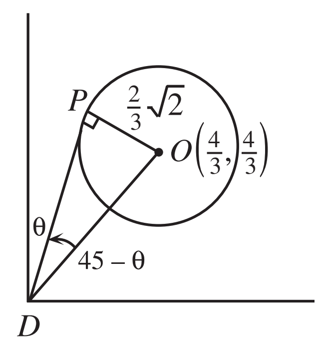
or, $$3a^2 + 3b^2 - 8a - 8b + 8 = 0$$

$$\left(a - \tfrac{4}{3}\right)^2 + \left(b - \tfrac{4}{3}\right)^2 = \tfrac{8}{9}$$

P is on a circle with centre $O\left(\frac{4}{3}, \frac{4}{3}\right)$ with $r = \frac{2}{3}\sqrt{2}$.

The minimum angle for θ occurs when DP is tangent to the circle.

So we have the diagram noted to the right.



Since OD makes an angle of $45°$ with the x-axis then $\angle PDO = 45 - \theta$ and $OD = \frac{4}{3}\sqrt{2}$.

Therefore $\sin(45 - \theta) = \dfrac{\frac{2}{3}\sqrt{2}}{\frac{4}{3}\sqrt{2}} = \dfrac{1}{2}$ which means $45° - \theta = 30°$ or $\theta = 15°$.

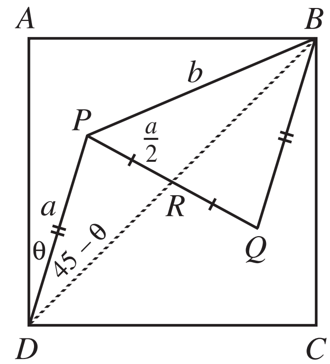
Thus the minimum value for θ is $15°$.

Solution 2

Let $AB = BC = CD = DA = 1$.

Join D to B. Let $\angle ADP = \theta$. Therefore, $\angle PDB = 45 - \theta$.

Let $PD = a$ and $PB = b$ and $PQ = \dfrac{a}{2}$.



We now establish a relationship between a and b.
In ΔPDB, $b^2 = a^2 + 2 - 2(a)\left(\sqrt{2}\right)\cos(45 - \theta)$

or, $\cos(45-\theta) = \dfrac{a^2 - b^2 + 2}{2\sqrt{2}a}$ (1)

In ΔPDR, $\left(\dfrac{a}{2}\right)^2 = a^2 + \left(\dfrac{\sqrt{2}}{2}\right)^2 - 2a\dfrac{\sqrt{2}}{2}\cos(45-\theta)$

or, $\cos(45-\theta) = \dfrac{\frac{3}{4}a^2 + \frac{1}{2}}{a\sqrt{2}}$ (2)

Comparing (1) and (2) gives, $\dfrac{a^2 - b^2 + 2}{2\sqrt{2}a} = \dfrac{\frac{3}{4}a^2 + \frac{1}{2}}{a\sqrt{2}}$.

Simplifying this, $a^2 + 2b^2 = 2$

or, $b^2 = \dfrac{2 - a^2}{2}$.

Now $\cos(45-\theta) = \dfrac{a^2 + 2 - \left(\dfrac{2-a^2}{2}\right)}{2a\sqrt{2}} = \dfrac{1}{4\sqrt{2}}\left(3a + \dfrac{2}{a}\right)$.

Now considering $3a + \dfrac{2}{a}$, we know $\left(\sqrt{3a} - \sqrt{\dfrac{2}{a}}\right)^2 \geq 0$

or, $3a + \dfrac{2}{a} \geq 2\sqrt{6}$.

Thus, $\cos(45-\theta) \geq \dfrac{1}{4\sqrt{2}}\left(2\sqrt{6}\right) = \dfrac{\sqrt{3}}{2}$

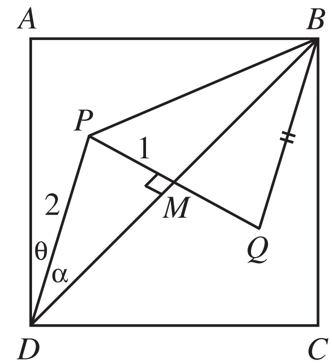
$\cos(45-\theta) \geq \dfrac{\sqrt{3}}{2}$.

$\cos(45-\theta)$ has a minimum value for $45° - \theta = 30°$ or $\theta = 15°$.

Solution 3

Join BD. Let BD meet PQ at M. Let $\angle ADP = \theta$.
By interior alternate angles, $\angle P = \angle Q$ and $\angle PDM = \angle QBM$.
Thus $\Delta PDM \cong \Delta QBM$ by A.S.A., so $PM = QM$ and $DM = BM$.
So M is the midpoint of BD and the centre of the square.

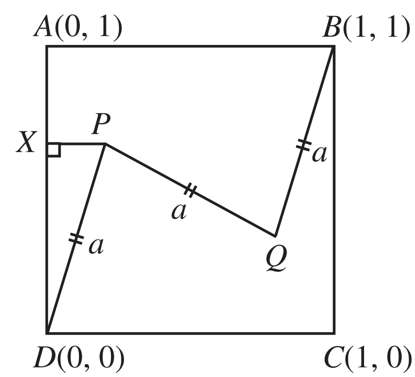
Without loss of generality, let $PM = 1$. Then $PD = 2$.
Since $\theta + \alpha = 45°$ (see diagram), θ will be minimized when α is maximized.



Consider ΔPMD.
Using the sine law, $\dfrac{\sin\alpha}{1} = \dfrac{\sin(\angle PMD)}{2}$.
To maximize α, we maximize $\sin\alpha$.
But $\sin\alpha = \dfrac{\sin(\angle PMD)}{2}$, so it is maximized when $\sin(\angle PMD) = 1$.
In this case, $\sin\alpha = \frac{1}{2}$, so $\alpha = 30°$.
Therefore, $\theta = 45° - 30° = 15°$, and so the minimum value of θ is 15°.

Solution 4

We place the diagram on a coordinate grid, with $D(0,0)$, $C(1,0)$, $B(0,1)$, $A(1,1)$.
Let $PD = PQ = QB = a$, and $\angle ADP = \theta$.
Drop a perpendicular from P to AD, meeting AD at X.
Then $PX = a\sin\theta$, $DX = a\cos\theta$.
Therefore the coordinates of P are $(a\sin\theta, a\cos\theta)$.
Since $PD \| BQ$, then $\angle QBC = \theta$.
So by a similar argument (or by using the fact that PQ are symmetric through the centre of the square), the coordinates of Q are $(1 - a\sin\theta, 1 + a\cos\theta)$.



Now $(PQ)^2 = a^2$, so $(1 - 2a\sin\theta)^2 + (1 - 2a\cos\theta)^2 = a^2$

$$2 + 4a^2\sin^2\theta + 4a^2\cos^2\theta - 4a(\sin\theta + \cos\theta) = a^2$$

$$2 + 4a^2 - a^2 = 4a(\sin\theta + \cos\theta)$$

$$\frac{2+3a^2}{4a} = \sin\theta + \cos\theta$$

$$\frac{2+3a^2}{4\sqrt{2}a} = \frac{1}{\sqrt{2}}\sin\theta + \frac{1}{\sqrt{2}}\cos\theta = \cos(45°)\sin\theta + \sin(45°)\cos\theta$$

$$\frac{2+3a^2}{4\sqrt{2}a} = \sin(\theta + 45°)$$

Now $\left(a - \sqrt{\tfrac{2}{3}}\right)^2 \geq 0$

$$a^2 - 2a\sqrt{\tfrac{2}{3}} + \tfrac{2}{3} \geq 0$$

$$3a^2 - 2a\sqrt{6} + 2 \geq 0$$

$$3a^2 + 2 \geq 2a\sqrt{6}$$

$$\frac{3a^2+2}{4\sqrt{2}a} \geq \frac{\sqrt{3}}{2}$$

and equality occurs when $a = \sqrt{\tfrac{2}{3}}$.

So $\sin(\theta + 45°) \geq \dfrac{\sqrt{3}}{2}$ and thus since $0° \leq \theta \leq 90°$, then $\theta + 45° \geq 60°$ or $\theta \geq 15°$.

Therefore the minimum possible value of $\angle ADP$ is $15°$.