Canadian
Mathematics
Competition

An activity of The Centre for Education
in Mathematics and Computing,
University of Waterloo, Waterloo, Ontario

2000 Solutions

Euclid Contest

(Grade12)

for

The CENTRE for EDUCATION in MATHEMATICS and COMPUTING

Awards

1. (a) If $x+27^{\frac{1}{3}}=125^{\frac{1}{3}}$, what is the value of x?

Solution

$125^{\frac{1}{3}}=5,\ 27^{\frac{1}{3}}=3$

Therefore, $x=5-3=2$.

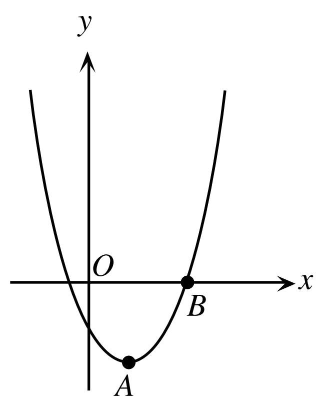
(b) The line $y=ax+c$ is parallel to the line $y=2x$ and passes through the point $(1,5)$. What is the value of c?

Solution

Since the two given lines are parallel, the line $y=ax+c$ has slope 2 and is of the form, $y=2x+c$. Since $(1,5)$ is on the line, $5=2(1)+c$

$$c=3.$$

(c) The parabola with equation $y=(x-2)^2-16$ has its vertex at A and intersects the x-axis at B, as shown. Determine the equation for the line passing through A and B.



Solution

For $y=0$, $(x-2)^2-16=0$

$$[(x-2)-4][(x-2)+4]=0$$

Therefore $x=6$ or $x=-2$.

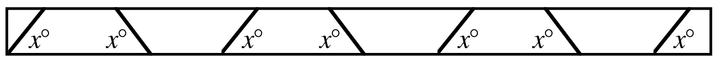
Thus, the x-intercepts of the parabola are -2 and 6, and B has coordinates $(6,0)$.

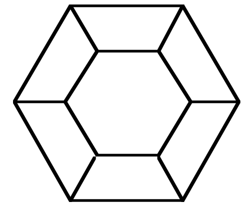
The vertex of the parabola is at $A(2,-16)$.

Equation of line containing $(6,0)$ and $(2,-16)$ has slope $\dfrac{-16}{2-6}=4$.

Thus the line has equation, $\dfrac{y-0}{x-6}=4 \Leftrightarrow y=4x-24$.

2. (a) Six identical pieces are cut from a board, as shown in the diagram. The angle of each cut is $x°$. The pieces are assembled to form a hexagonal picture frame as shown. What is the value of x?





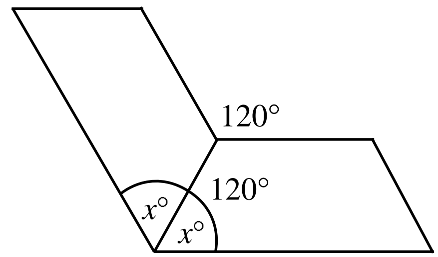
Solution

Each interior angle of a regular hexagon is $120°$.

Putting the frame together we would have the following

$2x = 120$ (in degrees)

$x = 60°$



(b) If $\log_{10} x = 3 + \log_{10} y$, what is the value of $\frac{x}{y}$?

Solution

$\log_{10} x - \log_{10} y = 3$

$\Leftrightarrow \log_{10}\left(\frac{x}{y}\right) = 3$

$\Leftrightarrow \frac{x}{y} = 10^3 = 1000$

(c) If $x + \frac{1}{x} = \frac{13}{6}$, determine all values of $x^2 + \frac{1}{x^2}$.

Solution 1 'Squaring both sides'

$\left(x + \frac{1}{x}\right)^2 = \left(\frac{13}{6}\right)^2$; squaring

$x^2 + 2 + \frac{1}{x^2} = \frac{169}{36}$

$x^2 + \frac{1}{x^2} = \frac{169}{32} - 2$

$x^2 + \frac{1}{x^2} = \frac{169}{36} - \frac{72}{36} = \frac{97}{36}$

Solution 2 'Creating a quadratic equation and solving'

$6x\left(x + \frac{1}{x}\right) = 6x\left(\frac{13}{6}\right)$

$6x^2 + 6 = 13x$

$6x^2 - 13x + 6 = 0$

$(3x - 2)(2x - 3) = 0$

$x=\frac{2}{3}$ ***or*** $x=\frac{3}{2}$

For $x=\frac{2}{3}$, $x^2+\frac{1}{x^2}$

$=\left(\frac{2}{3}\right)^2+\frac{1}{\left(\frac{2}{3}\right)^2}$

$=\frac{4}{9}+\frac{9}{4}$

$=\frac{81+16}{36}$

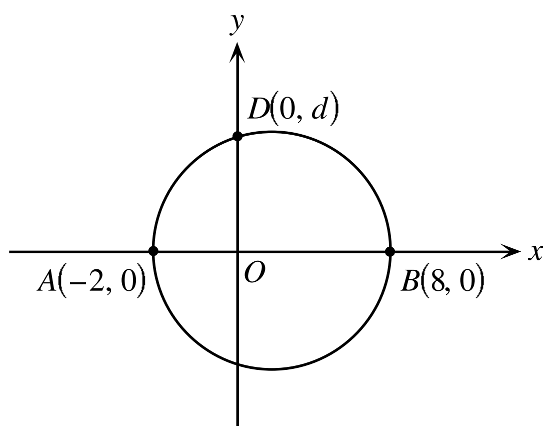
$=\frac{97}{36}$

For $x=\frac{3}{2}$, $\left(\frac{3}{2}\right)^2+\frac{1}{\left(\frac{3}{2}\right)^2}$

$=\frac{9}{4}+\frac{4}{9}$

$=\frac{97}{36}$

3. (a) A circle, with diameter AB as shown, intersects the positive y-axis at point $D(0, d)$. Determine d.



Solution 1

The centre of the circle is $(3, 0)$ and the circle has a radius of 5.

Thus $\sqrt{d^2+3^2}=5$

$d^2=5^2-3^2$

$d^2=16$

Therefore $d=4$, since $d>0$.

Solution 2

Since AB is a diameter of the circle, $\angle ADB=90°$ and $\angle AOD=90°$.

$\Delta ADO \sim \Delta DBO$

Therefore, $\frac{OD}{AO}=\frac{BO}{OD}$

and $d^2=2(8)$

$d^2=16$

$d=4$, since $d>0$.

Solution 3

$\angle ADB = \angle AOD = \angle BOD = 90°$

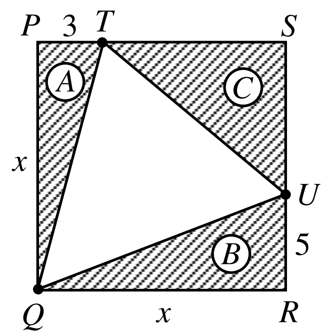
In ΔAOD, $AD^2 = 4 + d^2$.

In ΔBOD, $DB^2 = 64 + d^2$.

In ΔADB, $\left(4 + d^2\right) + \left(64 + d^2\right) = 100$

$$2d^2 = 32$$
$$d = 4,\ d > 0.$$

(b) A square $PQRS$ with side of length x is subdivided into four triangular regions as shown so that area Ⓐ + area Ⓑ = area Ⓒ. If $PT = 3$ and $RU = 5$, determine the value of x.



Solution

Since the side length of the square is x, $TS = x - 3$ and $VS = x - 5$

Area of triangle $A = \frac{1}{2}(3)(x)$.

Area of triangle $B = \frac{1}{2}(5)(x)$

Area of triangle $C = \frac{1}{2}(x - 5)(x - 3)$.

From the given information, $\frac{1}{2}(3x) + \frac{1}{2}(5x) = \frac{1}{2}(x - 5)(x - 3)$.

$3x + 5x = x^2 - 8x + 15$

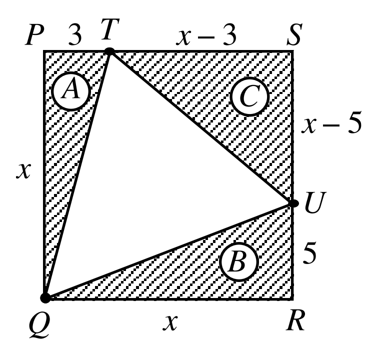
$x^2 - 16x + 15 = 0$

$(x - 15)(x - 1) = 0$

Thus $x = 15$ ***or*** $x = 1$.

Therefore $x = 15$ since $x = 1$ is inadmissible.

Labelled diagram



4. (a) A die, with the numbers 1, 2, 3, 4, 6, and 8 on its six faces, is rolled. After this roll, if an odd number appears on the top face, all odd numbers on the die are doubled. If an even number appears on the top face, all the even numbers are halved. If the given die changes in this way, what is the probability that a 2 will appear on the second roll of the die?

Solution

There are only two possibilities on the first roll - it can either be even or odd.

Possibility 1 'The first roll is odd'

The probability of an odd outcome on the first roll is $\frac{1}{3}$.

After doubling all the numbers, the possible outcomes on the second roll would now be 2, 2, 6, 4, 6, 8 with the probability of a 2 being $\frac{1}{3}$.

Thus the probability of a 2 on the second roll would be $\frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$.

Possibility 2 'The first is even'

The probability of an even outcome on the first roll is $\frac{2}{3}$.

After halving all the numbers, the possible outcomes on the second roll would be 1, 1, 3, 2, 3, 8.

The probability of a 2 on the second die would now be $\frac{1}{6}$.

Thus the probability of a 2 on the second roll is $\frac{2}{3} \times \frac{1}{6} = \frac{1}{9}$.

The probability of a 2 appear on the top face is $\frac{1}{9} + \frac{1}{9} = \frac{2}{9}$.

b) The table below gives the final standings for seven of the teams in the English Cricket League in 1998. At the end of the year, each team had played 17 matches and had obtained the total number of points shown in the last column. Each win *W*, each draw *D*, each bonus bowling point *A*, and each bonus batting point *B* received *w*, *d*, *a* and *b* points respectively, where *w*, *d*, *a* and *b* are positive integers. *No points* are given for a loss. Determine the values of *w*, *d*, *a* and *b* if total points awarded are given by the formula: Points $= w \times W + d \times D + a \times A + b \times B$.

Final Standings

	W	*Losses*	*D*	*A*	*B*	*Points*
Sussex	6	7	4	30	63	201
Warks	6	8	3	35	60	200
Som	6	7	4	30	54	192
Derbys	6	7	4	28	55	191
Kent	5	5	7	18	59	178
Worcs	4	6	7	32	59	176
Glam	4	6	7	36	55	176

Solution

There are a variety of ways to find the unknowns.
The most efficient way is to choose equations that have like coefficients. Here is one way to solve the problem using this method.

For Sussex: $6w+4d+30a+63b=201$
For Som: $6w+4d+30a+54b=192$
Subtracting, $9b=9$ $b=1$

If $b=1$
For Derbys: $6w+4d+28a+55=191$

$$6w+4d+28a=136 \qquad (1)$$

For Sussex: $6w+4d+30a+63=201$

$$6w+4d+30a=138 \qquad (2)$$

Subtracting, (2) – (1) $2a=2$

$$a=1.$$

We can now calculate d and w by substituting $a=1$, $b=1$ into a pair of equations.
An efficient way of doing this is by substituting $a=1$, $b=1$ into Som and Worcs.
For Som: $6w+4d+84=192$

$$6w+4d=108 \qquad (3)$$

For Worcs: $6w+3d+85=200$

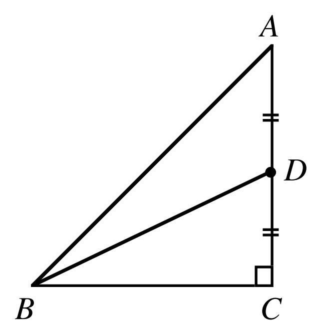
$$6w+3d=105 \qquad (4)$$

Subtracting, (3) – (4) $d=3$.
Substituting $d=3$ in either (3) or (4), $6w+4(3)=108$ (substituting in (3))

$$6w=96$$
$$w=16.$$

Therefore $w=16$, $d=3$, $a=b=1$.

5. (a) In the diagram, $AD=DC$, $\sin\angle DBC=0.6$ and $\angle ACB=90°$. What is the value of $\tan\angle ABC$?

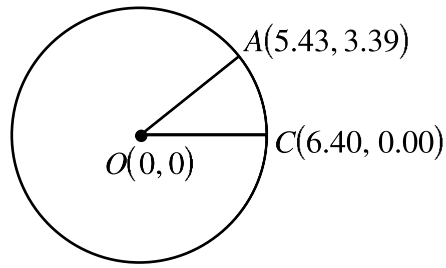


Solution

Let $DB=10$.
Therefore, $DC=AD=6$.
By the theorem of Pythagoras, $BC^2=10^2-6^2=64$.
Therefore, $BC=8$.

Thus, $\tan \angle ABC = \frac{12}{8} = \frac{3}{2}$.

(b) On a cross-sectional diagram of the Earth, the x and y-axes are placed so that $O(0, 0)$ is the centre of the Earth and $C(6.40, 0.00)$ is the location of Cape Canaveral. A space shuttle is forced to land on an island at $A(5.43, 3.39)$, as shown. Each unit represents 1 000 km. Determine the distance from Cape Canaveral to the island, measured on the surface of the earth, to the nearest 10 km.



Solution

Calculating $\angle AOC$

Calculating arc length

Distance

Solution

$$\tan \angle AOC = \frac{3.39}{5.43}$$

$$\angle AOC = \tan^{-1}\left(\frac{3.39}{5.43}\right) = 31.97°$$

The arc length $\overset{\frown}{AC} = \frac{31.97}{360°}[(2\pi)(6.40)] = 3.57$ units

The distance is approximately 3570 km.

6. (a) Let $\lfloor x \rfloor$ represent the greatest integer which is less than or equal to x. For example, $\lfloor 3 \rfloor = 3$, $\lfloor 2.6 \rfloor = 2$. If x is positive and $x\lfloor x \rfloor = 17$, what is the value of x?

Solution

We deduce that $4 < x < 5$.

Otherwise, if $x \le 4$, $x\lfloor x \rfloor \le 16$, and if $x \ge 5$, $x\lfloor x \rfloor \ge 25$.

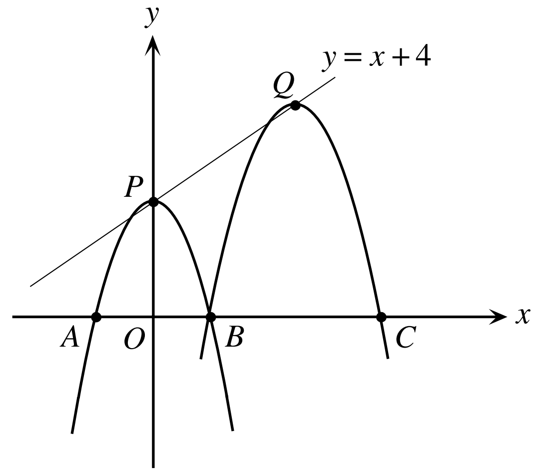
Therefore $\lfloor x \rfloor = 4$

Since $x\lfloor x \rfloor = 17$

$$4x = 17$$

$$x = 4.25$$

(b) The parabola $y=-x^2+4$ has vertex P and intersects the x-axis at A and B. The parabola is translated from its original position so that its vertex moves along the line $y=x+4$ to the point Q. In this position, the parabola intersects the x-axis at B and C. Determine the coordinates of C.



Solution 1

The parabola $y=-x^2+4$ has vertex $P(0,4)$ and intersects the x-axis at $A(-2,0)$ and $B(2,0)$.
The intercept $B(2,0)$ has its pre-image, B' on the parabola $y=-x^2+4$. To find B', we find the point of intersection of the line passing through $B(2,0)$, with slope 1, and the parabola $y=-x^2+4$.

The equation of the line is $y=x-2$.

Intersection points, $x-2=-x^2+4$

$$x^2+x-6=0$$
$$(x+3)(x-2)=0.$$

Therefore, $x=-3$ or $x=2$.

For $x=-3$, $y=-3-2=-5$. Thus B' has coordinates $(-3,-5)$.

If $(-3,-5)\to(2,0)$ then the required general translation mapping $y=-x^2+4$ onto the parabola with vertex Q is $(x,y)\to(x+5,y+5)$.

Possibility 1

Using the general translation, we find the coordinates of Q to be,
$P(0,4)\to Q(0+5,4+5)=Q(5,9)$.

If C is the reflection of B in the axis of symmetry of the parabola, i.e. $x=5$, C has coordinates $(8,0)$.

Possibility 2

If B' has coordinates $(-3,-5)$ then C' is the reflection of B' in the y-axis. Thus C' has coordinates $(3,-5)$.

If we apply the general translation then C has coordinates $(3+5,-5+5)$ or $(8,0)$.

Thus C has coordinates $(8,0)$.

Possibility 3

Using the general translation, we find the coordinates of Q to be,
$P(0,4)\to Q(0+5,4+5)=Q(5,9)$.

The equation of the image parabola is $y=-(x-5)^2+9$.

To find its intercepts, $-(x-5)^2+9=0$

$$(x-5)^2=9$$

$$x-5=\pm 3.$$

Therefore $x=8$ or $x=2$.
Thus C has coordinates $(8,0)$.

Solution 2

The translation moving the parabola with equation $y=-x^2+4$ onto the parabola with vertex Q is $T(t,t)$ because the slope of the line $y=x+4$ is 1.
The pre-image of B' is $(2-t,-t)$.
Since B' is on the parabola with vertex P, we have

$$-t=-(2-t)^2+4$$

$$-t=-4+4t-t^2+4$$

$$t^2-5t=0$$

$$t(t-5)=0.$$

Therefore, $t=0$ or $t=5$.
Thus B' is $(-3,-5)$.
Let C have coordinates $(c,0)$.
The pre-image of C is $(c-5,-5)$.
Therefore, $-5=-(c-5)^2+4$.
Or, $(c-5)^2=9$.
Therefore $c-5=3$ or $c-5=-3$.

$$c=8 \text{ or } c=2$$

Thus C has coordinates $(8,0)$.

Solution 3

The translation moving the parabola with equation $y=-x^2+4$ onto the parabola with vertex Q is $T(p,p)$ because the slope of the line $y=x+4$ is 1.
Q will have coordinates $(p,p+4)$.
Thus the equation of the image parabola is $y=-(x-p)^2+p+4$.
Since $(2,0)$ is on the parabola,

$$0=-(2-p)^2+p+4$$

$$p^2-5p=0$$

$$p(p-5)=0.$$

Therefore $p=0$ or $p=5$.
The coordinates of Q are $(5,9)$.

As in solution 1, we can use either reflection properties or the equation of the parabola to find that C has coordinates $(8,0)$.

7. (a) A cube has edges of length n, where n is an integer. Three faces, meeting at a corner, are painted red. The cube is then cut into n^3 smaller cubes of unit length. If exactly 125 of these cubes have no faces painted red, determine the value of n.

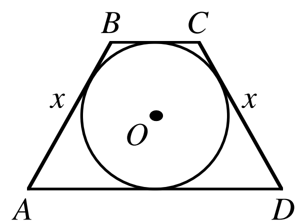
Solution

If we remove the cubes which have red paint, we are left with a smaller cube with measurements, $(n-1)\times(n-1)\times(n-1)$

Thus, $(n-1)^3 = 125$

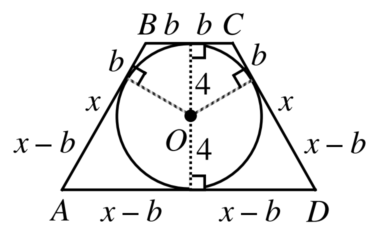
$n = 6$.

(b) In the isosceles trapezoid $ABCD$, $AB = CD = x$. The area of the trapezoid is 80 and the circle with centre O and radius 4 is tangent to the four sides of the trapezoid. Determine the value of x.



Solution

Using the tangent properties of a circle, the lengths of line segments are as shown on the diagram.



Area of trapezoid $ABCD = \frac{1}{2}(8)(BC + AD)$

$= 4(2b + 2x - 2b)$

$= 8x$.

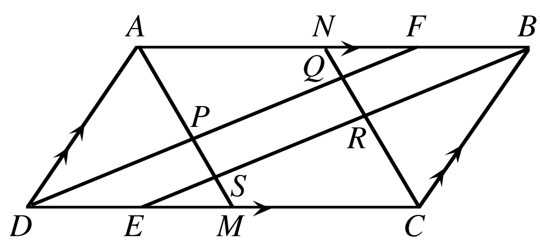
Thus, $8x = 80$.

Therefore, $x = 10$.

8. In parallelogram $ABCD$, $AB = a$ and $BC = b$, where $a > b$. The points of intersection of the angle bisectors are the vertices of quadrilateral $PQRS$.

(a) Prove that $PQRS$ is a rectangle.

(b) Prove that $PR = a - b$.



Solution

(a) In a parallelogram opposite angles are equal.

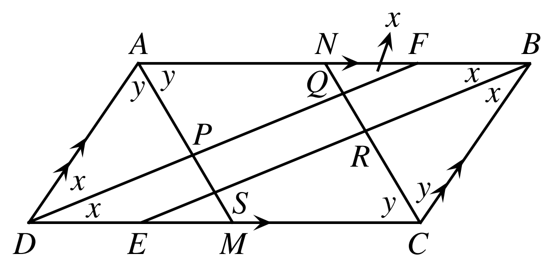
Since DF and BE bisect the two angles, let $\angle ADF = \angle CDF = \angle ABE = \angle CBE$

$= x$ (in degrees)

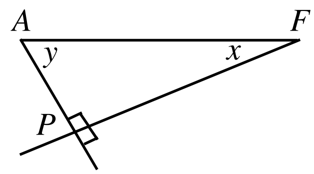
Also $\angle CDF = \angle AFD = x$ (alternate angles)

Let $\angle DAM = \angle BAM = \angle DCN = \angle BCN = y$ (in degrees)

For any parallelogram, any two consecutive angles add to $180°$, $\therefore 2x+2y=180$
or, $x+y=90$.



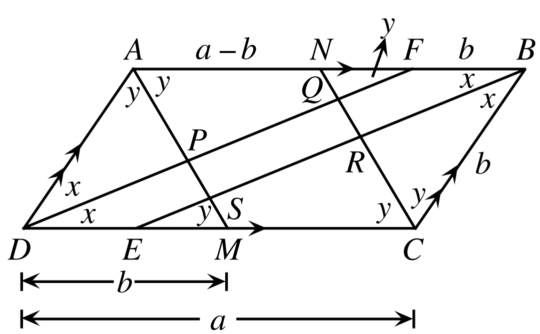
Therefore in ΔPAF, $\angle APF = 90°$.



Using similar reasoning and properties of parallel lines we get right angles at Q, R and S. Thus $PQRS$ is a rectangle.

Solution

(b) Since AM is a bisector of $\angle DAB$, let $\angle DAM = \angle BAM = y$.
Also, $\angle DMA = y$ (alternate angles)
This implies that ΔADM is isosceles.
Using the same reasoning in ΔCBN, we see that it is also isosceles and so the diagram may now be labelled as:



$AN = a - b$

Thus ΔADM and ΔCBN are identical isosceles triangles.
Also, $AM \parallel NC$ (corresponding angles)
or, $AP \parallel NR$.
By using properties of isosceles triangles (or congruency), $AP = NR$ implying that $APRN$ is a parallelogram.
Thus $AN = PR$ and since $AN = a - b$, $PR = a - b$ (as required)

9. A permutation of the integers 1, 2, ..., n is a listing of these integers in some order. For example, $(3, 1, 2)$ and $(2, 1, 3)$ are two different permutations of the integers 1, 2, 3. A permutation $(a_1, a_2, ..., a_n)$ of the integers 1, 2, ..., n is said to be "fantastic" if $a_1 + a_2 + ... + a_k$ is divisible by k, for ***each*** k from 1 to n. For example, $(3, 1, 2)$ is a fantastic permutation of 1, 2, 3 because 3 is divisible by 1, $3+1$ is divisible by 2, and $3+1+2$ is divisible by 3. However, $(2, 1, 3)$ is not fantastic because $2+1$ is not divisible by 2.

(a) Show that no fantastic permutation exists for $n = 2000$.

(b) Does a fantastic permutation exist for $n = 2001$? Explain.

Solution

(a) In our consideration of whether there is a fantastic permutation for $n = 2000$, we start by looking at the 2000th position.
Using our definition of fantastic permutation, it is necessary that $2000 \mid (1+2+3+\cdots+2000)$.
Since $1+2+3+\cdots+2000 = \frac{(2000)(2001)}{2} = (1000)(2001)$, it is required that $2000 \mid 1000(2001)$.
This is not possible and so no fantastic permutation exists for $n = 2000$.

Solution

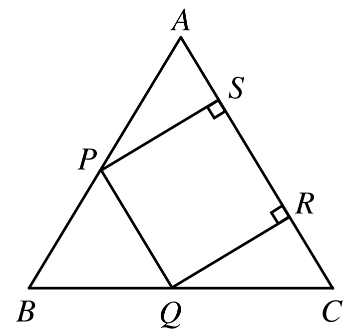
(b) The sum of the integers from 1 to 2001 is $\frac{(2001)(2002)}{2} = (2001)(1001)$ which is divisible by 2001. If $t_1, t_2, ..., t_{2001}$ is a fantastic permutation, when we remove t_{2001} from the above sum, and what remains must be divisible by 2000.
We now consider $t_1 + t_2 + \cdots + t_{2000}$ and determine what integer is not included in the permutation.

$$\begin{aligned} t_1 + t_2 + \cdots + t_{2000} &= \frac{(2001)(2002)}{2} - t_{2001} \\ &= (1001)(2001) - t_{2001} \end{aligned}$$

Since $(1001)(2001) = 2\,003\,001$, t_{2001} must be a number of the form $k001$ where k is odd.
The only integer less than or equal to 2001 with this property is 1001. Therefore $t_{2001} = 1001$.

So the sum up to t_{2000} is $2\,003\,001 - 1001 = 2\,002\,000$.
When we remove t_{2000} we must get a multiple of 1999.
The largest multiple of 1999 less than $2\,002\,000$ is $(1999)(1001) = 2\,000\,999$. This would make $t_{2000} = 2\,002\,000 - 2\,000\,999 = 1001$ which is impossible since $t_{2000} \neq t_{2001}$. If we choose lesser multiples of 1999 to subtract from $2\,002\,000$ we will get values of t_{2000} which are greater than 2001, which is also not possible.
Thus, a fantastic permutation is not possible for $n = 2001$.

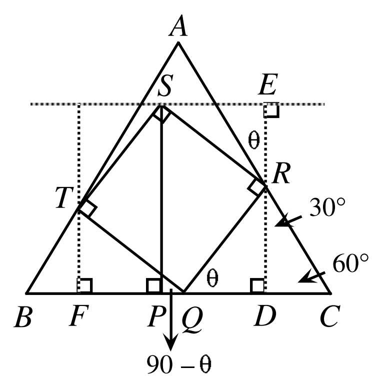
10. An equilateral triangle ABC has side length 2. A square, $PQRS$, is such that P lies on AB, Q lies on BC, and R and S lie on AC as shown. The points P, Q, R, and S move so that P, Q and R always remain on the sides of the triangle and S moves from AC to AB through the interior of the triangle. If the points P, Q, R and S always form the vertices of a square, show that the path traced out by S is a straight line parallel to BC.



In essence, this solution establishes that the perpendicular distance from S to BC is $s(\sin\theta+\cos\theta)$ and then showing that this is a constant by finding $s(\sin\theta+\cos\theta)$ as part of the base which is itself a constant length.

Solution

Let $\angle RQC=\theta$ and from S draw a line perpendicular to the base at P.
Then $\angle TQB=180-(90+\theta)=90-\theta$.
Let s be the length of the side of the square.
From R draw a line perpendicular to BC at D and then through S draw a line parallel to BC. From R draw a line perpendicular to this line at E.



From ΔRQD, $RD=s\sin\theta$.
Since $\angle QRD=90-\theta$ then $\angle SRE=\theta$.
From ΔSER, $ER=s\cos\theta$.
The perpendicular distance from S to BC is $RD+ER=s\sin\theta+s\cos\theta$ which we must now show is a constant.

We can now take each of the lengths DC, DQ, PF, FB and express them in terms of s.

From ΔRDC which is a $30°-60°-90°$ triangle, $\frac{DC}{RD}=\frac{1}{\sqrt{3}}$.

Since $RD=s\sin\theta$ (from above)

$$DC=\frac{1}{\sqrt{3}}(s\sin\theta)=\frac{\sqrt{3}}{3}s\sin\theta.$$

From ΔRDQ, $\frac{QD}{RQ} = \cos\theta$, $QD = s\cos\theta$.

From ΔTFQ, $\sin\theta = \frac{FQ}{s}$ and $\cos\theta = \frac{TF}{s}$.
or, $FQ = s\sin\theta$ and $TF = s\sin\theta$.

From ΔTFB, $\frac{BF}{TF} = \frac{1}{\sqrt{3}}$, $BF = \frac{1}{\sqrt{3}}TF = \frac{1}{\sqrt{3}}s\cos\theta = \frac{\sqrt{3}}{3}s\cos\theta$.

Since $DC + QD + FQ + BF = 2$, $\frac{\sqrt{3}}{3}s\sin\theta + s\cos\theta + s\sin\theta + \frac{\sqrt{3}}{3}s\cos\theta = 2$.

$$\frac{\sqrt{3}}{3}(s\cos\theta + s\sin\theta) + (s\cos\theta + s\sin\theta) = 2$$

$$s\cos\theta + s\sin\theta = \frac{2}{\left(\frac{\sqrt{3}}{3} + 1\right)}$$

Thus $s\cos\theta + s\sin\theta$ is a constant and the path traced out by S is a straight line parallel to BC.

Note: A number of enquiries have been made about this question. Several individuals have made the comment that it is not possible to do this under the given conditions. What is not mentioned, and what is not realized, is that the size of the square *changes*. This makes it possible for the square to exist under the given conditions.