Canadian Mathematics Competition

An activity of The Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario

Euclid Contest (Grade 12)

for the

NATIONAL BANK OF CANADA

Awards

Tuesday, April 20, 1999

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Time: $2\frac{1}{2}$ hours

Calculators are permitted, provided they are non-programmable and without graphic displays.

Do not open this booklet until instructed to do so. The paper consists of 10 questions, each worth 10 marks. Parts of each question can be of two types. **SHORT ANSWER** parts are worth 2 marks each (questions 1-2) or 3 marks each (questions 3-7). **FULL SOLUTION** parts are worth the remainder of the 10 marks for the question.

Instructions for SHORT ANSWER parts

1. **SHORT ANSWER** parts are indicated like this: 
2. **Enter the answer in the appropriate box in the answer booklet.** For these questions, full marks will be given for a correct answer which is placed in the box. Part marks will be awarded **only if relevant work** is shown in the space provided in the answer booklet.

Instructions for FULL SOLUTION parts

1. **FULL SOLUTION** parts are indicated like this: 
2. **Finished solutions must be written in the appropriate location in the answer booklet.** Rough work should be done separately. If you require extra pages for your finished solutions, foolscap will be supplied by your supervising teacher. Insert these pages into your answer booklet.
3. Marks are awarded for completeness, clarity, and style of presentation. A correct solution poorly presented will not earn full marks.

NOTE: At the completion of the contest, insert the information sheet inside the answer booklet.

NOTE:
1. Please read the instructions on the front cover of this booklet.
2. Place all answers in the answer booklet provided.
3. For questions marked "💡", full marks will be given for a correct answer which is to be placed in the appropriate box in the answer booklet. **Part marks will be given for work shown**. Students are strongly encouraged to show their work.
4. It is expected that all calculations and answers will be expressed as exact numbers such as 4π, $2+\sqrt{7}$, etc., except where otherwise indicated.

1. (a) If $x^{-1} = 3^{-1} + 4^{-1}$, what is the value of x?

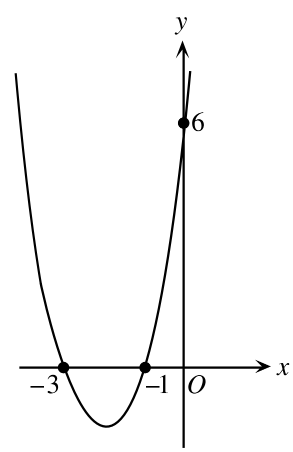
   (b) If the point $P(-3, 2)$ is on the line $3x + 7ky = 5$, what is the value of k?

   (c) If $x^2 - x - 2 = 0$, determine all possible values of $1 - \frac{1}{x} - \frac{6}{x^2}$.

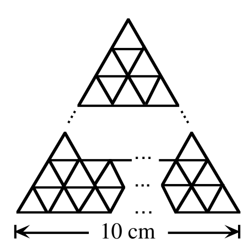
2. (a) The circle defined by the equation $(x+4)^2 + (y-3)^2 = 9$ is moved horizontally until its centre is on the line $x = 6$. How far does the centre of the circle move?

   (b) The parabola defined by the equation $y = (x-1)^2 - 4$ intersects the x-axis at the points P and Q. If (a, b) is the mid-point of the line segment PQ, what is the value of a?

   (c) Determine an equation of the quadratic function shown in the diagram.

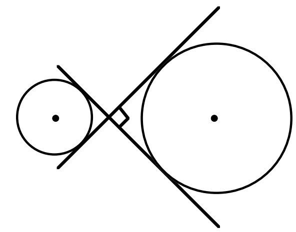


3. (a) How many equilateral triangles of side 1 cm, placed as shown in the diagram, are needed to completely cover the interior of an equilateral triangle of side 10 cm?



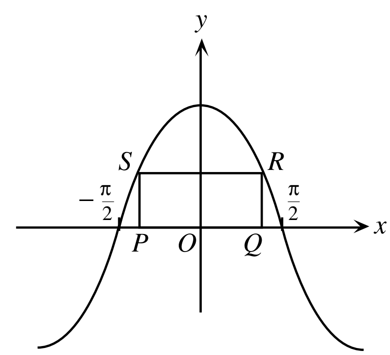
   (b) The populations of Alphaville and Betaville were equal at the end of 1995. The population of Alphaville decreased by 2.9% during 1996, then increased by 8.9% during 1997, and then increased by 6.9% during 1998. The population of Betaville increased by r% in each of the three years. If the populations of the towns are equal at the end of 1998, determine the value of r correct to one decimal place.

4. (a) In the diagram, the tangents to the two circles intersect at 90° as shown. If the radius of the smaller circle is 2, and the radius of the larger circle is 5, what is the distance between the centres of the two circles?

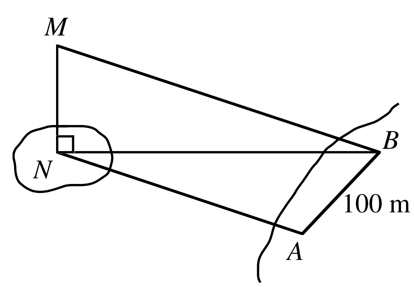


(b) A circular ferris wheel has a radius of 8 m and rotates at a rate of 12° per second. At $t = 0$, a seat is at its lowest point which is 2 m above the ground. Determine how high the seat is above the ground at $t = 40$ seconds.

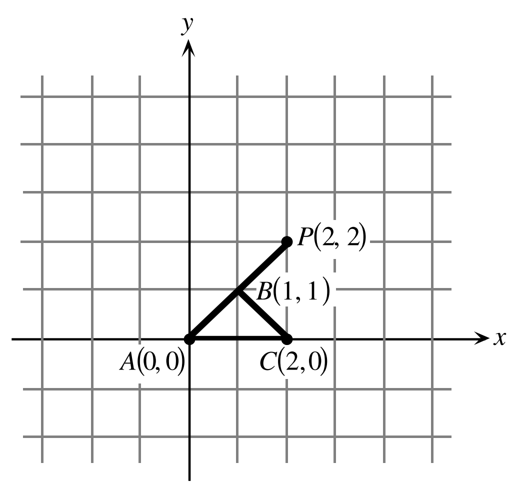
5. (a) A rectangle $PQRS$ has side PQ on the x-axis and touches the graph of $y = k \cos x$ at the points S and R as shown. If the length of PQ is $\frac{\pi}{3}$ and the area of the rectangle is $\frac{5\pi}{3}$, what is the value of k?



(b) In determining the height, MN, of a tower on an island, two points A and B, 100 m apart, are chosen on the same horizontal plane as N. If $\angle NAB = 108°$, $\angle ABN = 47°$ and $\angle MBN = 32°$, determine the height of the tower to the nearest metre.



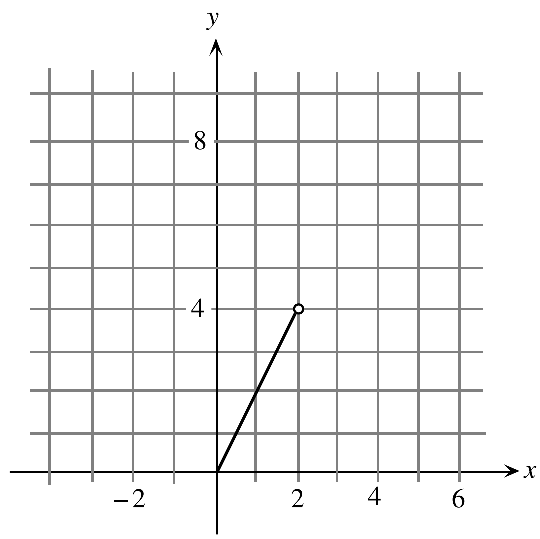
6. (a) The points A, P and a third point Q (not shown) are the vertices of a triangle which is similar to triangle ABC. What are the coordinates of all possible positions for Q?



(b) Determine the coordinates of the points of intersection of the graphs of $y = \log_{10}(x - 2)$ and $y = 1 - \log_{10}(x + 1)$.

7\. (a) On the grid provided in the answer booklet, draw the graphs of the functions $y = -2\sqrt{x+1}$ and $y = \sqrt{x-2}$. For what value(s) of k will the graphs of the functions $y = -2\sqrt{x+1}$ and $y = \sqrt{x-2} + k$ intersect? (Assume x and k are real numbers.)

(b) Part of the graph for $y = f(x)$ is shown, $0 \le x < 2$.
If $f(x+2) = \frac{1}{2} f(x)$ for all real values of x, draw the graph for the intervals, $-2 \le x < 0$ and $2 \le x < 6$.



8\. (a) The equation $y = x^2 + 2ax + a$ represents a parabola for all real values of a. Prove that each of these parabolas pass through a common point and determine the coordinates of this point.

(b) The vertices of the parabolas in part (a) lie on a curve. Prove that this curve is itself a parabola whose vertex is the common point found in part (a).

9\. A ‘millennium’ series is any series of consecutive integers with a sum of 2000. Let m represent the first term of a ‘millennium’ series.

(a) Determine the minimum value of m.

(b) Determine the smallest possible positive value of m.

10\. $ABCD$ is a cyclic quadrilateral, as shown, with side $AD = d$, where d is the diameter of the circle. $AB = a$, $BC = a$ and $CD = b$. If a, b and d are integers $a \ne b$,

(a) prove that d cannot be a prime number.

(b) determine the *minimum* value of d.