Canadian
Mathematics
Competition

An activity of The Centre for Education
in Mathematics and Computing,
University of Waterloo, Waterloo, Ontario

1999 Solutions
Cayley Contest (Grade 10)

for the
NATIONAL BANK OF CANADA
Awards

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Part A

1. The value of $3^2+7^2-5^2$ is

(**A**) 75 (**B**) 83 (**C**) 33 (**D**) 25 (**E**) 10

Solution
$3^2+7^2-5^2=9+49-25=33$ ANSWER: (C)

2. If 8 is added to the square of 5 the result is divisible by

(**A**) 5 (**B**) 2 (**C**) 8 (**D**) 23 (**E**) 11

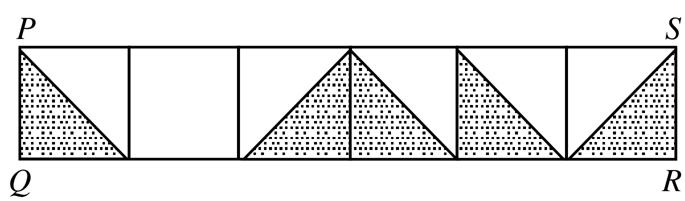
Solution
$8+5^2=33$
Of the given numbers, 11 is the only possible divisor of 33. ANSWER: (E)

3. Today is Wednesday. What day of the week will it be 100 days from now?

(**A**) Monday (**B**) Tuesday (**C**) Thursday (**D**) Friday (**E**) Saturday

Solution
Since there are 7 days in a week it will be Wednesday in 98 days.
In 100 days it will be Friday. ANSWER: (D)

4. The rectangle $PQRS$ is divided into six equal squares and shaded as shown. What fraction of $PQRS$ is shaded?



(**A**) $\frac{1}{2}$ (**B**) $\frac{7}{12}$ (**C**) $\frac{5}{11}$ (**D**) $\frac{6}{11}$ (**E**) $\frac{5}{12}$

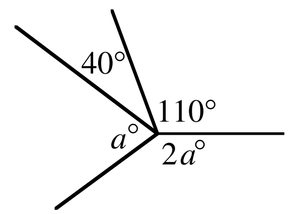
Solution
There are 5 half-squares shaded out of a total possible of 12 half-squares, hence $\frac{5}{12}$ of the area is shaded. ANSWER: (E)

5. If $x=4$ and $y=3x$ and $z=2y$, then the value of $y+z$ is

(**A**) 12 (**B**) 20 (**C**) 40 (**D**) 24 (**E**) 36

Solution
If $x = 4$ this makes $y = 12$ and $z = 24$.
Thus $y + z = 36$. ANSWER: (E)

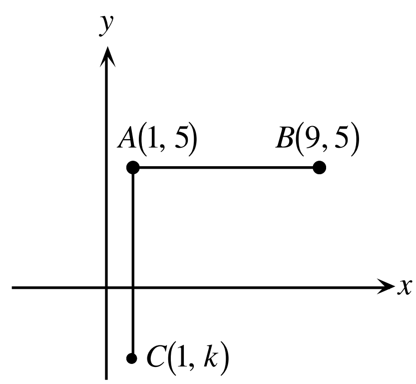
6. In the diagram, the value of a is



(**A**) 50 (**B**) 65 (**C**) 70
(**D**) 105 (**E**) 110

Solution
Since $3a° + 150° = 360°$
$3a° = 210°$
Therefore $a = 70$. ANSWER: (C)

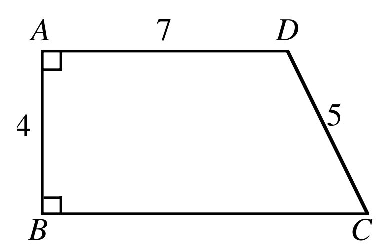
7. In the diagram, AB and AC have equal lengths. What is the value of k?



(**A**) -3 (**B**) -4 (**C**) -5
(**D**) -7 (**E**) -8

Solution
Since $AB = AC = 8$, $5 - k = 8$
$k = -3$. ANSWER: (A)

8. In the diagram, $AD < BC$. What is the perimeter of $ABCD$?

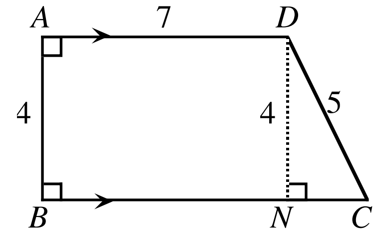


(**A**) 23 (**B**) 26 (**C**) 27
(**D**) 28 (**E**) 30

Solution

From D we draw a line perpendicular to BC that meets BC at N. Since $ADNB$ is a rectangle and $AD \| BC$, $DN = 4$. We use Pythagoras to find $NC = 3$. We now know that $BC = BN + NC = 7 + 3 = 10$.

The required perimeter is $7 + 5 + 10 + 4 = 26$.



ANSWER: (B)

9. Three CD's are bought at an average cost of \$15 each. If a fourth CD is purchased, the average cost becomes \$16. What is the cost of the fourth CD?

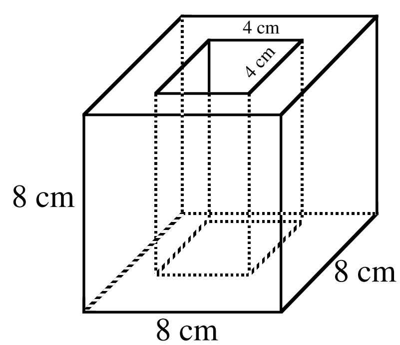
(**A**) \$16 (**B**) \$17 (**C**) \$18 (**D**) \$19 (**E**) \$20

Solution

If four C.D.'s have an average cost of \$16 this implies that \$64 was spent in purchasing the four of them. Using the same reasoning, \$45 was spent buying the first three. Thus, the fourth C.D. must have cost $\$64 - \$45 = \$19$. ANSWER: (D)

10. An 8 cm cube has a 4 cm square hole cut through its centre, as shown. What is the remaining volume, in cm^3?

(**A**) 64 (**B**) 128 (**C**) 256
(**D**) 384 (**E**) 448



Solution

$$\begin{aligned}\text{Remaining volume } &= 8 \times 8 \times 8 - 8 \times 4 \times 4 \text{ (in cm}^3\text{)}\\ &= 8(64 - 16)\\ &= 8 \times 48\\ &= 384\end{aligned}$$

ANSWER: (D)

Part B

11. The time on a digital clock is 5:55. How many minutes will pass before the clock next shows a time with all digits identical?

(**A**) 71 (**B**) 72 (**C**) 255 (**D**) 316 (**E**) 436

Solution

The digits on the clock will next be identical at 11:11. This represents a time difference of 316 minutes. (Notice that times like 6:66, 7:77 etc. are not possible.) ANSWER: (D)

12. The numbers 49, 29, 9, 40, 22, 15, 53, 33, 13, 47 are grouped in pairs so that the sum of each pair is the same. Which number is paired with 15?

(**A**) 33 (**B**) 40 (**C**) 47 (**D**) 49 (**E**) 53

Solution

If we arrange the numbers in ascending order we would have: 9, 13, 15, 22, 29, 33, 40, 47, 49, 53. If the sum of each pair is equal they would be paired as: $9 \leftrightarrow 53$, $13 \leftrightarrow 49$, $15 \leftrightarrow 47$, $22 \leftrightarrow 40$, $29 \leftrightarrow 33$.

ANSWER: (C)

13. The units digit in the product $\left(5^2+1\right)\left(5^3+1\right)\left(5^{23}+1\right)$ is

(**A**) 0 (**B**) 1 (**C**) 2 (**D**) 5 (**E**) 6

Solution

Since 5^2, 5^3 and 5^{23} all end in 5, then 5^2+1, 5^3+1 and $5^{23}+1$ all end in 6. When we multiply these three numbers together their product must also end in a 6.

ANSWER: (E)

14. In an election for class president, 61 votes are cast by students who are voting to choose one of four candidates. Each student must vote for only one candidate. The candidate with the highest number of votes is the winner. The smallest number of votes the winner can receive is

(**A**) 15 (**B**) 16 (**C**) 21 (**D**) 30 (**E**) 31

Solution

After 60 votes are cast, theoretically it is possible for each candidate to have 15 votes.

The final vote, the 61st, would mean that the winning candidate would need just 16 votes to have the minimum number possible.

ANSWER: (B)

15. A chocolate drink is 6% pure chocolate, by volume. If 10 litres of pure milk are added to 50 litres of this drink, the percent of chocolate in the new drink is

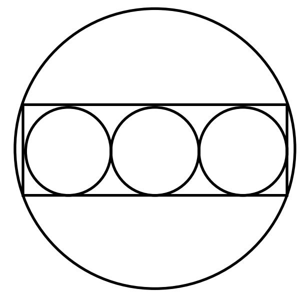
(**A**) 5 (**B**) 16 (**C**) 10 (**D**) 3 (**E**) 26

Solution

If 6% of the 50 litres is pure chocolate, this means that there will be three litres of pure chocolate in the final mixture. If the final mixture contains sixty litres of which three litres are pure chocolate this represents $\frac{3}{60}$ or 5% of the total.

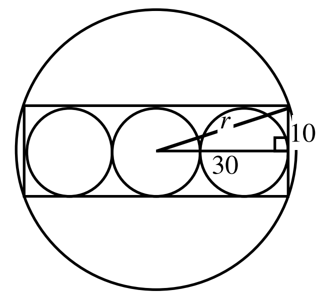
ANSWER: (A)

16. Three circles, each with a radius of 10 cm, are drawn tangent to each other so that their centres are all in a straight line. These circles are inscribed in a rectangle which is inscribed in another circle. The area of the largest circle is



(**A**) 1000π (**B**) 1700π (**C**) 900π
(**D**) 1600π (**E**) 1300π

Solution
By symmetry, the centre of the large circle is the centre of the smaller middle circle. If the constructions are made as shown and with the appropriate representation of the lengths we find, $r^2 = 30^2 + 10^2 = 1000$.
Thus, $A = \pi r^2 = \pi(1000) = 1000\pi$.



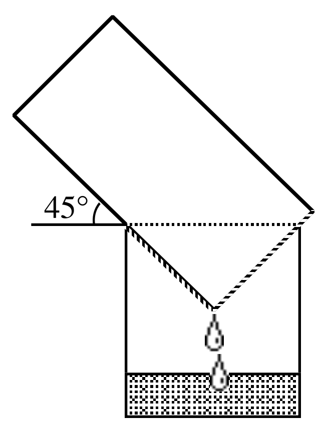
ANSWER: (A)

17. Let N be the smallest positive integer whose digits have a product of 2000. The sum of the digits of N is

(**A**) 21 (**B**) 23 (**C**) 25 (**D**) 27 (**E**) 29

Solution
Since $2000 = 2^4 \cdot 5^3$, the smallest possible positive integer satisfying the required conditions is $25\,558$ which gives the sum $2+5+5+5+8 = 25$. A natural answer might be 23 since $44\,555$ satisfies the given conditions. However, since $25\,558 < 44\,555$ and the question requires the smallest number then the answer must be 25 and not 23. ANSWER: (C)

18. A cylindrical pail containing water drains into a cylindrical tub 40 cm across and 50 cm deep, while resting at an angle of 45° to the horizontal, as shown. How deep is the water in the tub when its level reaches the pail?



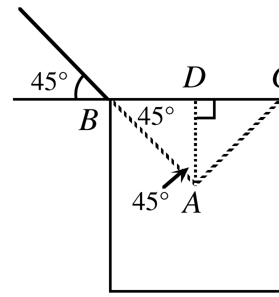
(**A**) 10 cm (**B**) 20 cm (**C**) 30 cm
(**D**) 35 cm (**E**) 40 cm

Solution

We label the points A, B and C as shown in the diagram. From symmetry, we note that ΔBAC is an isosceles right-angled triangle.

From A we draw a line perpendicular to BC meeting the line at point D. This construction allows us to conclude that ΔABD is also a right-angled isosceles triangle and specifically that $BD = DA$. Since $BD = DA$ and $BD = DC = 20$, we find $DA = 20$.

This makes the depth of the water $50 - 20$ or 30.



ANSWER: (C)

19. A number is *Beprisque* if it is the only natural number between a prime number and a perfect square (e.g. 10 is Beprisque but 12 is not). The number of *two-digit* Beprisque numbers (including 10) is

(**A**) 1 (**B**) 2 (**C**) 3 (**D**) 4 (**E**) 5

Solution

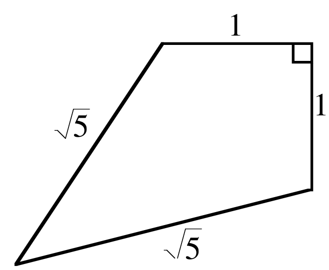
We start with the observation that it is necessary to consider only the odd perfect squares and the integers adjacent to them. It is not necessary to consider the even perfect squares because if we add 2 or subtract 2 from an even number the result is even and it is required by the conditions set out in the question that this number be prime. Considering then the odd perfect squares we have: $\{9, 10, 11\}$, $\{23, 24, 25, 26, 27\}$, $\{47, 48, 49, 50, 51\}$, $\{79, 80, 81, 82, 83\}$.

The Beprisque numbers are those that are circled. ANSWER: (E)

20. The area of the given quadrilateral is

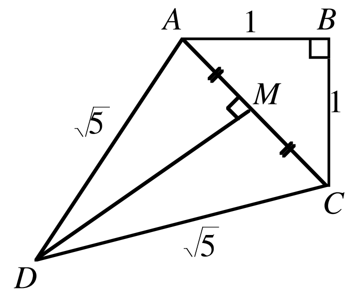
(**A**) $\frac{3}{2}$ (**B**) $\sqrt{5}$ (**C**) $\frac{1+\sqrt{10}}{2}$

(**D**) 2 (**E**) 3



Solution

Label the quadrilateral as shown. We join A to C and using Pythagoras in ΔABC we calculate BC to be $\sqrt{2}$. Since ΔACD is isosceles, we can draw a line perpendicular to AC which passes through M, the mid-point of AC. Since $AC = \sqrt{2}$, $AM = \frac{\sqrt{2}}{2}$. Since $AM = \frac{\sqrt{2}}{2}$, $DA = \sqrt{5}$ and ΔADM is right-angled, we can once again use Pythagoras to find that $DM = \frac{3}{\sqrt{2}}$. The line segment DM is also the height of ΔDAC. The area of ΔADC is $\frac{1}{2}\left(\frac{3}{\sqrt{2}}\right)\left(\sqrt{2}\right) = \frac{3}{2}$ and because the area of $\Delta ABC = \frac{1}{2}(1)(1) = \frac{1}{2}$, the total area of the quadrilateral is $\frac{1}{2} + \frac{3}{2} = 2$ square units.



ANSWER: (D)

Part C

21. A number is formed using the digits 1, 2, ..., 9. Any digit can be used more than once, but adjacent digits cannot be the same. Once a pair of adjacent digits has occurred, that pair, in that order, cannot be used again. How many digits are in the largest such number?

(**A**) 72 (**B**) 73 (**C**) 144 (**D**) 145 (**E**) 91

Solution

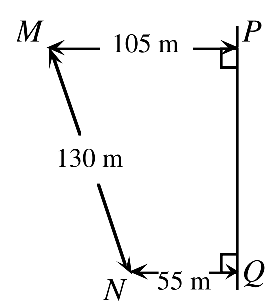
Since there are $9(8) = 72$ ordered pairs of consecutive digits, and since the final digit has no successor, we can construct a 73 digit number by adding a 9. The question is, of course, can we actually construct this number? The answer is 'yes' and the largest such number is,

98 97 96 95 94 93 92 91 87 86 85 84 83 82 81 76 75 74 73 72 71 65 64 63 62 61
54 53 52 51 43 42 41 32 31 21 9.

If we count the numbers in the string we can see that there are actually 73 numbers contained within it.

ANSWER: (B)

22. A main gas line runs through P and Q. From some point T on PQ, a supply line runs to a house at point M. A second supply line from T runs to a house at point N. What is the minimum total length of pipe required for the two supply lines?



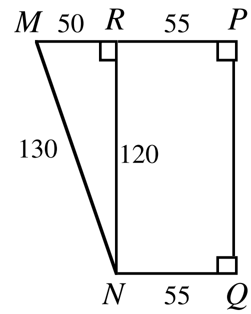
(**A**) 200 (**B**) 202 (**C**) 198
(**D**) 210 (**E**) 214

Solution

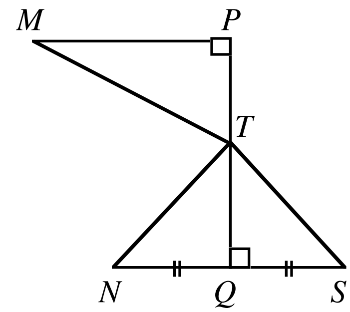
We start by choosing point R so that $RPQN$ is a rectangle.

Thus, $MR = 105 - 55$
$= 50$.

Using Pythagoras, $RN = \sqrt{130^2 - 50^2} = 120$.

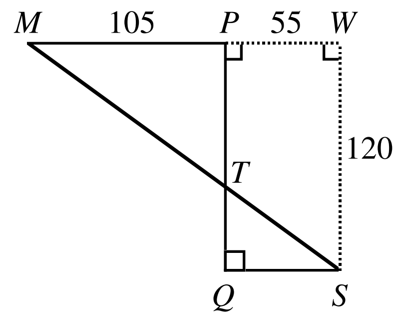


Let S be the image of N reflected in PQ.
Join M to T, T to S and T to N.
Since $\Delta TNQ \equiv \Delta TSQ$, it follows that $TN = TS$.
The length of the supply line is $MT + TN = MT + TS$.



Clearly the length $MT + TS$ is a minimum when M, T and S are collinear. In that case, $MT + TS = MS$.
Create ΔMSW as shown.

By Pythagoras, $MS = \sqrt{160^2 + 120^2}$
$= 200$.



ANSWER: (A)

23. How many integers can be expressed as a sum of three distinct numbers chosen from the set $\{4, 7, 10, 13, ..., 46\}$?

(**A**) 45 (**B**) 37 (**C**) 36 (**D**) 43 (**E**) 42

Solution

Since each number is of the form $1 + 3n$, $n = 1, 2, 3, ..., 15$, the sum of the three numbers will be of the form $3 + 3k + 3l + 3m$ where k, l and m are chosen from $\{1, 2, 3, ..., 15\}$. So the question is equivalent to the easier question of, 'How many distinct integers can be formed by adding three numbers from, $\{1, 2, 3, ..., 15\}$?'

The smallest is $1 + 2 + 3 = 6$ and the largest is $13 + 14 + 15 = 42$.

It is clearly possible to get every sum between 6 and 42 by:

(a) increasing the sum by one replacing a number with one that is 1 larger or,

(b) decreasing the sum by one by decreasing one of the addends by 1.

Thus all the integers from 6 to 42 inclusive can be formed.

This is the same as asking, 'How many integers are there between 1 and 37 inclusive?' The answer, of course, is 37. ANSWER: (B)

24. The sum of all values of x that satisfy the equation $\left(x^2-5x+5\right)^{x^2+4x-60}=1$ is

(**A**) -4 (**B**) 3 (**C**) 1 (**D**) 5 (**E**) 6

Solution
We consider the solution in three cases.

Case 1 It is possible for the base to be 1.
Therefore, $x^2-5x+5=1$
$x^2-5x+4=0$
$(x-1)(x-4)=0$
Therefore $x=1$ or $x=4$.
Both these values are acceptable for x.

Case 2 It is possible that the exponent be 0.
Therefore, $x^2+4x-60=0$
$(x+10)(x-6)=0$
$x=-10$ or $x=6$
Note: It is easy to verify that neither $x=-10$ nor $x=6$ is a zero of x^2-5x+5, $x^2-5x+5=0$, so that the indeterminate form $0°$ does not occur.

Case 3 It is possible that the base is –1 and the exponent is even.
Therefore, $x^2-5x+5=-1$ but $x^2+4x-60$ must also be even.
$x^2-5x+5=-1$
$x^2-5x+6=0$
$(x-2)(x-3)=0$
$x=2$ or $x=3$
If $x=2$, then $x^2-4x-60$ is even, so $x=2$ is a solution.
If $x=3$, then $x^2-4x-60$ is odd, so $x=3$ is ***not*** a solution.

Therefore the sum of the solutions is $1+4-10+6+2=3$. ANSWER: (B)

25. If $a=3^p$, $b=3^q$, $c=3^r$, and $d=3^s$ and if p, q, r, and s are positive integers, determine the smallest value of $p+q+r+s$ such that $a^2+b^3+c^5=d^7$.

(**A**) 17 (**B**) 31 (**C**) 106 (**D**) 247 (**E**) 353

Solution
If we rewrite the given expression by substituting we arrive at the new expression

$3^{2p}+3^{3q}+3^{5r}=3^{7s}$. (This is derived by replacing a with 3^p, b with 3^q and so on.)
On the left side we remove the lowest power of 3 (whatever it is), $3^{2p}\left(1+3^{3q-2p}+3^{5r-2p}\right)=3^{7s}$.

Both factors on the left side must be multiples of 3 but $1+3^{3q-2p}+3^{5r-2p}$ cannot be a multiple of 3 unless 3^{3q-2p} and 3^{5r-2p} are both exactly 1. This means that $2p=3q=5r$ or that the exponents are themselves multiples of 30, say $30m$.

We now have, $3^{30m}+3^{30m}+3^{30m}=3^{7s}$

or, $(3)\left(30^{30m}\right)=3^{7s}$

$3^{30m+1}=3^{7s}$.

We are now looking for the smallest integers, m and s, such that $30m+1=7s$.
If we try $m=1, 2, 3, 4, ...$ we find that $m=3$ and $s=13$. Thus $2p=90$, $p=45$; $3q=90$, $q=30$, $5r=90$, $r=18$ and $7s=91$, $s=13$.
From this, $p+q+r+s=45+30+18+13=106$. ANSWER: (C)