35th Anniversary 1963 – 1998

Canadian Mathematics Competition

An activity of The Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario

1998 Solutions Pascal Contest (Grade 9)

for the NATIONAL BANK OF CANADA Awards

PART A:

1. The value of $\frac{1+3+5}{10+6+2}$ is

(**A**) $\frac{1}{6}$ (**B**) 2 (**C**) $\frac{1}{2}$ (**D**) $1\frac{1}{2}$ (**E**) $3\frac{1}{10}$

Solution 1

$$\frac{1+3+5}{10+6+2} = \frac{9}{18} = \frac{1}{2}$$

Solution 2

$$\frac{1+3+5}{10+6+2} = \frac{(1+3+5)}{2(5+3+1)} = \frac{1}{2}$$

ANSWER: (C)

2. If $3(x-5) = 3(18-5)$, then x is

(**A**) $\frac{44}{3}$ (**B**) $\frac{32}{3}$ (**C**) 9 (**D**) 18 (**E**) 81

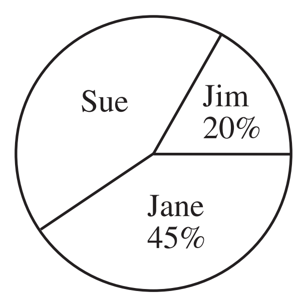
Solution

Since $3(x-5) = 3(18-5)$, divide both sides by 3 to get $(x-5) = (18-5)$.

Therefore, $x = 18$. ANSWER: (D)

3. The pie chart shows a percentage breakdown of 1000 votes in a student election. How many votes did Sue receive?

(**A**) 550 (**B**) 350 (**C**) 330
(**D**) 450 (**E**) 935



Solution

Sue received $100-(20+45) = 35$ percent of the total number of votes. Since there was a total of 1000 votes, Sue received $0.35(1000) = 350$ votes. ANSWER: (B)

4. The value of $\left(\sqrt{169}-\sqrt{25}\right)^2$ is

(**A**) 64 (**B**) 8 (**C**) 16 (**D**) 144 (**E**) 12

Solution

$$\left(\sqrt{169}-\sqrt{25}\right)^2 = (13-5)^2$$
$$= 8^2$$
$$= 64$$

ANSWER: (A)

5. The value of $\dfrac{5^6 \times 5^9 \times 5}{5^3}$ is

(**A**) 5^{18} (**B**) 25^{18} (**C**) 5^{13} (**D**) 25^{13} (**E**) 5^{51}

Solution

$$\frac{5^6 \times 5^9 \times 5}{5^3} = \frac{5^{16}}{5^3}$$
$$= 5^{13}$$

ANSWER: (C)

6. If $x = 3$, which of the following expressions is even?

(**A**) $9x$ (**B**) x^3 (**C**) $2\left(x^2+9\right)$ (**D**) $2x^2+9$ (**E**) $3x^2$

Solution 1

Since the expression $2\left(x^2+9\right)$ contains a factor of 2, it must be even regardless of the value of *x*.

Solution 2

If $x = 3$, then $9x = 27$, $x^3 = 27$, $2\left(x^2+9\right) = 36$, $2x^2+9 = 27$, and $3x^2 = 27$.

Therefore, $2\left(x^2+9\right)$ is the only expression that is an even number. ANSWER: (C)

7. The value of $490-491+492-493+494-495+\ldots-509+510$ is

(**A**) 500 (**B**) -10 (**C**) -11 (**D**) 499 (**E**) 510

Solution

The value of $490-491+492-493+494-495+\ldots-509+510$ is

$$(490-491)+(492-493)+(494-495)+\ldots+(508-509)+510 = \underbrace{(-1)+(-1)+(-1)+\ldots+(-1)}_{\text{10 pairs, each add to } -1}+510$$
$$= -10+510$$
$$= 500$$

ANSWER: (A)

8. The average (mean) of a list of 10 numbers is 0. If 72 and -12 are added to the list, the new average will be

(**A**) 30 (**B**) 6 (**C**) 0 (**D**) 60 (**E**) 5

Solution

If the average (mean) of a list of 10 numbers is 0, then the sum of the numbers is 10(0) = 0. When 72 and –12 are added to the list, the sum of these 12 numbers is $0+72-12=60$.

Thus, the average of the 12 numbers is $60 \div 12 = 5$. ANSWER: (E)

9. What is one-half of 1.2×10^{30}?

(**A**) 6.0×10^{30} (**B**) 6.0×10^{29} (**C**) 0.6×5^{30} (**D**) 1.2×10^{15} (**E**) 1.2×5^{30}

Solution

$$\begin{aligned}\frac{1}{2}\left(1.2 \times 10^{30}\right) &= 0.6 \times 10^{30}\\ &= 0.6 \times 10 \times 10^{29}\\ &= 6.0 \times 10^{29}\end{aligned}$$

ANSWER: (B)

10. If $x+y+z=25$ and $y+z=14$, then x is

(**A**) 8 (**B**) 11 (**C**) 6 (**D**) -6 (**E**) 31

Solution

We are given that $x+y+z=25$ (1)

and $y+z=14$ (2).

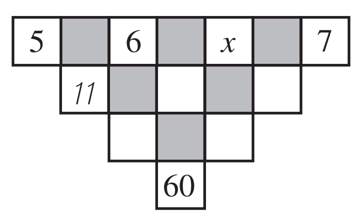
Subtract equation (2) from equation (1) to get $x = 11$. ANSWER: (B)

PART B:

11. The number in an unshaded square is obtained by adding the numbers connected to it from the row above. (The '*11*' is one such number.) The value of x is

(**A**) 4 (**B**) 6 (**C**) 9

(**D**) 15 (**E**) 10



Solution

The three entries in row two, from left to right, are 11, $6+x$, and $x+7$. The two entries in row three, from left to right, are $11+(6+x)=17+x$ and $(6+x)+(x+7)=2x+13$. The single entry in row four is $(17+x)+(2x+13)=3x+30$.

Thus, $3x + 30 = 60$

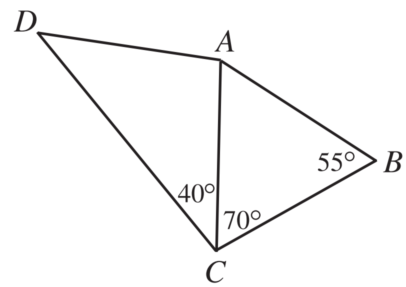
$3x = 30$

$x = 10$

ANSWER: (E)

12. In the diagram, $DA = CB$. What is the measure of $\angle DAC$?

(**A**) 70° (**B**) 100° (**C**) 95°
(**D**) 125° (**E**) 110°



Solution

In ΔABC, $\angle BAC = 180^\circ - (70^\circ + 55^\circ)$
$= 55^\circ$

Since $\angle BAC = \angle ABC$, then ΔABC is isosceles with $AC = CB$.

We are given that $DA = CB$, so $DA = AC$ and ΔADC is also isosceles.

Thus, $\angle ADC = \angle ACD = 40^\circ$

and $\angle DAC = 180^\circ - (40^\circ + 40^\circ)$
$= 100^\circ$

ANSWER: (B)

13. A three-wheeled vehicle travels 100 km. Two spare wheels are available. Each of the five wheels is used for the same distance during the trip. For how many kilometres is each wheel used?

(**A**) 20 (**B**) 25 (**C**) $33\frac{1}{3}$ (**D**) 50 (**E**) 60

Solution

Since only three of the five wheels are in use at any time, the total distance travelled by all the wheels is 3(100) = 300 kilometres. However, each of the five wheels is used for the same distance during the trip. Thus, each wheel is used for $300 \div 5 = 60$ kilometres. ANSWER: (E)

14. The sum of the digits of a five-digit positive integer is 2. (A five-digit integer cannot start with zero.) The number of such integers is

(**A**) 1 (**B**) 2 (**C**) 3 (**D**) 4 (**E**) 5

Solution

If the sum of the digits of a five-digit positive integer is 2, then the only possible integers are 20 000, 11 000, 10 100, 10 010, and 10 001. There are 5 such integers. ANSWER: (E)

15. Four points are on a line segment, as shown.
If $AB : BC = 1 : 2$ and $BC : CD = 8 : 5$, then $AB : BD$ equals

(**A**) 4:13 (**B**) 1:13 (**C**) 1:7
(**D**) 3:13 (**E**) 4:17

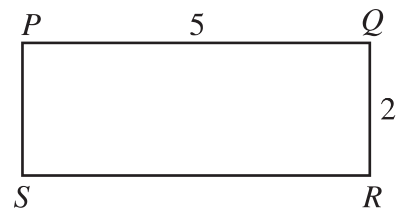
Solution

In order to compare the given ratios, we must rewrite the ratio $AB : BC = 1 : 2$ as $AB : BC = 4:8$. Now both ratios express BC as 8 units and we can write $AB : BC : CD = 4:8:5$.

Thus, $AB : BD = 4:(8+5)$
$= 4:13$

ANSWER: (A)

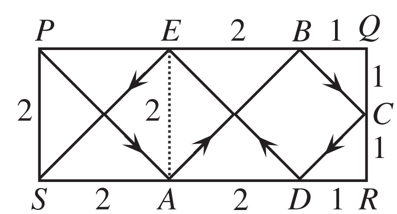
16. On a rectangular table 5 units long and 2 units wide, a ball is rolled from point P at an angle of $45°$ to PQ and bounces off SR. The ball continues to bounce off the sides at $45°$ until it reaches S. How many bounces of the ball are required?



(**A**) 9 (**B**) 8 (**C**) 7
(**D**) 5 (**E**) 4

Solution

Since the ball bounces off the sides of the rectangular table at 45°, right-angled isosceles triangles are created as shown. The ball begins at point P then bounces at points $A,B,C,D,$and E before reaching S, for a total of 5 bounces.



ANSWER: (D)

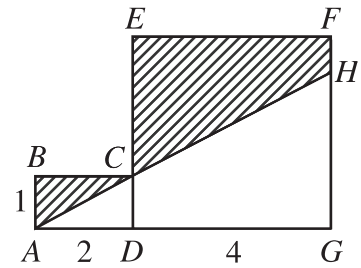
17. If $1998 = p^s q^t r^u$, where p, q and r are prime numbers, what is the value of $p + q + r$?
(**A**) 222 (**B**) 48 (**C**) 42 (**D**) 66 (**E**) 122

Solution

The prime factorization of 1998 is $2 \times 3^3 \times 37$. Thus, $p,q,$ and r have values 2, 3, and 37 (in any order), and $p + q + r = 42$.

ANSWER: (C)

18. In the diagram, $DEFG$ is a square and $ABCD$ is a rectangle. A straight line is drawn from A, passes through C and meets FG at H. The area of the shaded region is



(**A**) 8 (**B**) 8.5 (**C**) 10
(**D**) 9 (**E**) 10.5

Solution

We are given that AH is a straight line segment, and C is a point on AH. Since $AD : DC = 2{:}1$, then $AG : GH = 2{:}1$. Since the length of AG is 6, the length of GH is 3.

The area of rectangle $ABCD$ is $1 \times 2 = 2$. The area of square $DEFG$ is $4^2 = 16$. The area of ΔAHG is $\frac{1}{2}(6)(3) = 9\cdot$

Therefore, the area of the shaded region is $2 + 16 - 9 = 9$. ANSWER: (D)

19. Using only digits 1, 2, 3, 4, and 5, a sequence is created as follows: one 1, two 2's, three 3's, four 4's, five 5's, six 1's, seven 2's, and so on.
The sequence appears as: 1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5, 1, 1, 1, 1, 1, 1, 2, 2, ...
The 100th digit in the sequence is

(**A**) 1 (**B**) 2 (**C**) 3 (**D**) 4 (**E**) 5

Solution

The total number of digits in n groups of the sequence is given by $1 + 2 + 3 + ... + n$. In order to determine the group containing the 100th digit in the sequence, we must find the positive integer n such that $1 + 2 + 3 + ... + (n-1) < 100$ and $1 + 2 + 3 + ... + n > 100$. By examining a few of these sums we find that $1 + 2 + 3 + ... + 13 = 91$ and $1 + 2 + 3 + ... + 13 + 14 = 105$. Thus the 100th digit in the sequence is in the 14th group. The 100th digit is a 4. ANSWER: (D)

20. Driving between two towns at 110 km/h instead of 100 km/h saves 9 minutes. The distance in kilometres between the two towns is

(**A**) 210 (**B**) 99 (**C**) 165 (**D**) 9900 (**E**) 150

Solution

Let x represent the distance, in kilometres, between the two towns. Driving at 100 km/h, it takes $\dfrac{x}{100}$ hours to travel between the towns. Driving at 110 km/h, it takes $\dfrac{x}{110}$ hours. We know that these two times differ by 9 minutes, or $\dfrac{9}{60}$ hours.

Thus, $\dfrac{x}{110} + \dfrac{9}{60} = \dfrac{x}{100}$

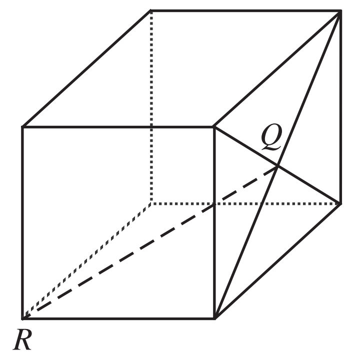
$$\frac{x}{11}+\frac{3}{2}=\frac{x}{10}$$
$$\frac{3}{2}=\frac{11x}{110}-\frac{10x}{110}$$
$$\frac{3}{2}=\frac{x}{110}$$
$$165=x$$

The two towns are 165 km apart. ANSWER: (C)

PART C:

21. Q is the point of intersection of the diagonals of one face of a cube whose edges have length 2 units. The length of QR is

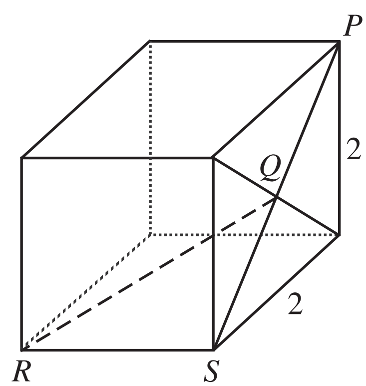
(**A**) 2 (**B**) $\sqrt{8}$ (**C**) $\sqrt{5}$
(**D**) $\sqrt{12}$ (**E**) $\sqrt{6}$



Solution

Label points P and S as shown. Since each face of the cube is a square with sides of length 2, use the Pythagorean Theorem to find the length of diagonal PS.

$$PS^2=2^2+2^2$$
$$=8$$
$$PS=2\sqrt{2}$$



Then QS has length $\sqrt{2}$, as Q is the midpoint of diagonal PS.

Because we are working with a cube, $\angle QSR=90^\circ$ and ΔQRS is a right-angled triangle. Use the Pythagorean Theorem in ΔQRS to get

$$QR^2=2^2+\left(\sqrt{2}\right)^2$$
$$=6$$
$$QR=\sqrt{6}$$

ANSWER: (E)

22. A deck of 100 cards is numbered from 1 to 100. Each card has the same number printed on both sides. One side of each card is red and the other side is yellow. Barsby places all the cards, red side up, on a table. He first turns over every card that has a number divisible by 2. He then examines all the cards, and turns over every card that has a number divisible by 3. How many cards have the red side up when Barsby is finished?

(**A**) 83 (**B**) 17 (**C**) 66 (**D**) 50 (**E**) 49

Solution

Initially, all 100 cards have the red side up. After Barsby's first pass only the 50 odd-numbered cards have the red side up, since he has just turned all the even-numbered cards from red to yellow.

During Barsby's second pass he turns over all cards whose number is divisible by 3. On this pass Barsby will turn any odd-numbered card divisible by 3 from red to yellow. Between 1 and 100, there are 17 odd numbers that are divisible by 3, namely 3, 9, 15, 21, ... , 93, and 99. Also on this pass, Barsby will turn any even-numbered card divisible by 3 from yellow to red. Between 1 and 100, there are 16 even numbers that are divisible by 3, namely 6, 12, 18, 24, ... , 90, and 96.

When Barsby is finished, the cards that have the red side up are the 50 odd-numbered cards from the first pass, minus the 17 odd-numbered cards divisible by 3 from the second pass, plus the 16 even-numbered cards divisible by 3, also from the second pass.

Thus, $50-17+16=49$ cards have the red side up. ANSWER: (E)

23. The numbers $123\,456\,789$ and $999\,999\,999$ are multiplied. How many of the digits in the final result are 9's?

(**A**) 0 (**B**) 1 (**C**) 2 (**D**) 3 (**E**) 17

Solution

Rewrite the product as follows:

$$\begin{aligned}(123\ 456\ 789)(999\ 999\ 999) &= (123\ 456\ 789)\left(10^9-1\right)\\ &= (123\ 456\ 789)\times 10^9-(123\ 456\ 789)\end{aligned}$$

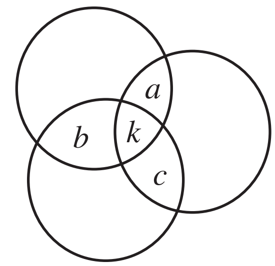
When 123 456 789 is subtracted from $(123\ 456\ 789)\times 10^9$ the result is 123 456 788 876 543 211. None of the digits are 9's. ANSWER: (A)

24. Three rugs have a combined area of 200 m^2. By overlapping the rugs to cover a floor area of 140 m^2, the area which is covered by exactly two layers of rug is 24 m^2. What area of floor is covered by three layers of rug?

(**A**) 12 m^2 (**B**) 18 m^2 (**C**) 24 m^2 (**D**) 36 m^2 (**E**) 42 m^2

Solution

Draw the rugs in the following manner, where $a+b+c$ represents the amount of floor covered by exactly two rugs and k represents the amount of floor covered by exactly three rugs. We are told that $a+b+c=24$ (1).



Since the total amount of floor covered when the rugs do not overlap is 200 m^2 and the total covered when they do overlap is 140 m^2, then 60 m^2 of rug is wasted on double or triple layers. Thus, $a+b+c+2k=60$ (2). Subtract equation (1) from equation (2) to get $2k=36$ and solve for $k=18$. Thus, the area of floor covered by exactly three layers of rug is 18 m^2. ANSWER: (B)

25. One way to pack a 100 by 100 square with 10 000 circles, each of diameter 1, is to put them in 100 rows with 100 circles in each row. If the circles are repacked so that the centres of any three tangent circles form an equilateral triangle, what is the maximum number of additional circles that can be packed?

(**A**) 647 (**B**) 1442 (**C**) 1343 (**D**) 1443 (**E**) 1344

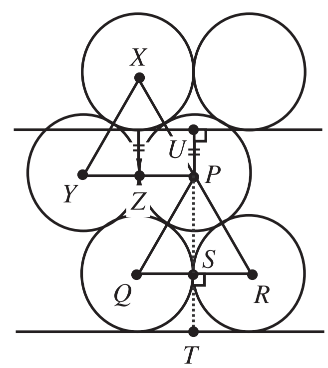
Solution

Remove one circle from every second row and shift to form the given configuration. Label the diagram as shown. Since each circle has diameter 1, ΔPQR and ΔPXY are equilateral triangles with sides of length 1.

In ΔPQR, altitude PS bisects side QR. Use the Pythagorean Theorem to find PS.

$$PS^2 = (1)^2 - \left(\tfrac{1}{2}\right)^2$$
$$= \tfrac{3}{4}$$
$$PS = \tfrac{\sqrt{3}}{2}$$

Also, $XZ = \frac{\sqrt{3}}{2}$



Since all radii have length $\frac{1}{2}$, then $PU = \frac{\sqrt{3}}{2} - \frac{1}{2}$ and $TU = \frac{1}{2} + \frac{\sqrt{3}}{2} + \left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right) = \sqrt{3}$. This tells us that two rows of circles require a height of $\sqrt{3}$ before a third row begins.

Since $\frac{100}{\sqrt{3}} = 57.7$, we can pack 57 double rows, each containing 100+99=199 circles.

Can we pack one final row of 100 circles? Yes. The square has sides of length 100 and our configuration of 57 double rows requires a height of $57\sqrt{3}$ before the next row begins. Since $100 - 57\sqrt{3} > 1$, and since the circles each have diameter 1, there is room for one final row of 100 circles.

The number of circles used in this new packing is $57(199)+100=11\ 443$
Thus, the maximum number of extra circles that can be packed into the square is
$11\ 443-10\ 000=1443$. ANSWER: (D)