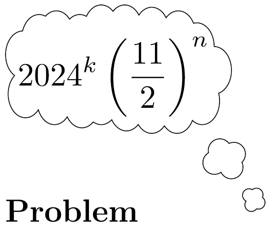

Problem of the Week

Problem E and Solution

Looking for Integers

Problem

Suppose n and k are integers and $4^k < 2024$. For how many (n, k) pairs is $2024^k\left(\frac{11}{2}\right)^n$ equal to an integer?

Solution

First we write 2024 as a product of prime factors: $2024 = 2^3 \times 11 \times 23$.

We can then substitute this into our expression.

$$\begin{aligned} 2024^k\left(\frac{11}{2}\right)^n &= \left(2^3 \times 11 \times 23\right)^k\left(\frac{11}{2}\right)^n \\ &= 2^{3k} \times 11^k \times 23^k \times \frac{11^n}{2^n} \\ &= 2^{3k-n} \times 11^{k+n} \times 23^k \end{aligned}$$

Since $2024^k\left(\frac{11}{2}\right)^n$ is equal to an integer, it follows that none of the exponents can be negative. Thus, $3k - n \geq 0$, $k + n \geq 0$, and $k \geq 0$.

From $3k - n \geq 0$, we can determine that $n \leq 3k$. Similarly, from $k + n \geq 0$, we can determine that $n \geq -k$. Thus, n is an integer between $-k$ and $3k$, inclusive.

Since $4^5 = 1024$, $4^6 = 4096$, and $4^k < 2024$, it follows that $k \leq 5$. Since $k \geq 0$ and k is an integer, the possible values of k are $0,\ 1,\ 2,\ 3,\ 4$, and 5.

In the table below, we summarize the number of values of n for each possible value of k.

k	Minimum value of n	Maximum value of n	Number of values of n
0	0	0	1
1	-1	3	5
2	-2	6	9
3	-3	9	13
4	-4	12	17
5	-5	15	21

Thus, the total number of (n, k) pairs is $1 + 5 + 9 + 13 + 17 + 21 = 66$.