Problem of the Week
Problem E and Solution
The Weightiest Problem

Problem

Pauline has five rocks, each of a different mass. She weighs the rocks in pairs and records the mass of each pair of rocks. Later, she realizes that she forgot to weigh the rocks individually, but no longer has access to a scale. The ten recorded masses are 728 g, 757 g, 771 g, 783 g, 797 g, 817 g, 826 g, 831 g, 860 g, and 886 g. Determine the combined mass of all five rocks. Then determine the mass of the lightest rock.

Solution

Let a, b, c, d, and e represent the masses of the rocks, in grams, from lightest to the heaviest.

Notice that
$(a+b)+(a+c)+(a+d)+(a+e)+(b+c)+(b+d)+(b+e)+(c+d)+(c+e)+(d+e)$
simplifies to $4a+4b+4c+4d+4e$. It is also equal to the sum of the 10 recorded masses.

Thus,

$$\begin{aligned} 4a+4b+4c+4d+4e &= 728+757+771+783+797+817+826+831+860+886 \\ &= 8056 \end{aligned}$$

Therefore, $4a+4b+4c+4d+4e=8056$. Dividing each side of the equation by 4, we obtain $a+b+c+d+e=2014$. That is, the combined mass of all five rocks is 2014 g.

The smallest recorded mass is created by weighing the two lightest rocks. Therefore, $a+b=728$. The largest recorded mass is created by weighing the two heaviest rocks. Therefore, $d+e=886$. Thus,

$$\begin{aligned} (a+b+c+d+e)-(a+b)-(d+e) &= 2014-728-886 \\ c &= 400 \end{aligned}$$

The second smallest recorded mass is created by adding the mass of the lightest object, a, to the mass of the object with the mass in the middle, c. (A justification of this statement is given at the end of this solution.) Therefore, $a+c=757$ and $a=757-c=757-400=357$.

Therefore, the combined mass of all five rocks is 2014 g and the lightest rock has mass 357 g.

Although we are not asked to, from here, we could go on to determine the masses of all five rocks. Doing so, we would find that the masses are 357 g, 371 g, 400 g, 426 g, and 460 g. It is left as an exercise for you to verify the correctness of this list.

Why is $a+c$ the second smallest recorded mass?

We will repeatedly use the fact that $a<b<c<d<e$ to show that $a+c$ is the second smallest recorded mass. First, since $c>b$, $a+c>a+b$. That is, $a+c$ is not the smallest recorded mass. We'll now show that $a+c$ is smaller than all other pairs.

- Since $c<d$, we have $a+c<a+d$.
- Since $c<e$, we have $a+c<a+e$.
- Since $a<b$, we have $a+c<b+c$.
- Since $a<b$ and $c<d$, we have $a+c<b+d$.
- Since $a<b$ and $c<e$, we have $a+c<b+e$.
- Since $a<d$, we have $a+c<d+c=c+d$.
- Since $a<e$, we have $a+c<e+c=c+e$.
- Since $a<d$ and $c<e$, we have $a+c<d+e$.

We have shown that $a+c>a+b$ and that $a+c$ is smaller than the other eight recorded sums. Therefore, $a+c$ is the second smallest recorded mass. In a similar manner, it is also possible to show that $c+e$ is the second largest recorded mass.