Hundreds	Tens	Ones
9	4	6

Problem of the Week
Problem E and Solution
It's the Ones that We Want

Problem

The sum of the first n positive integers is $1 + 2 + 3 + \cdots + n$.
We define a_n to be the ones digit of the sum of the first n positive integers.

For example,

$$\begin{aligned} 1 &= 1 \text{ and } a_1 = 1,\\ 1+2 &= 3 \text{ and } a_2 = 3,\\ 1+2+3 &= 6 \text{ and } a_3 = 6,\\ 1+2+3+4 &= 10 \text{ and } a_4 = 0,\\ 1+2+3+4+5 &= 15 \text{ and } a_5 = 5. \end{aligned}$$

Thus, $a_1 + a_2 + a_3 + a_4 + a_5 = 1 + 3 + 6 + 0 + 5 = 15$.

Determine the smallest value of n such that $a_1 + a_2 + a_3 + \cdots + a_n \geq 2024$.

Solution

Let's start by examining the values of a_n until we start to see a pattern.

We know $a_1 = 1$, $a_2 = 3$, $a_3 = 6$, $a_4 = 0$, and $a_5 = 5$.

Unfortunately, we do not have a pattern yet. We need to keep calculating values of a_n. Since $15 + 6 = 21$, $a_6 = 1$.

Notice that we can determine the ones digit of the sum of the first n integers from the ones digit from the sum of the first $n - 1$ integers and the ones digit of n. For example, to calculate a_7, we simply need to know that $a_6 = 1$ and the sum $1 + 7 = 8$ has ones digit 8. So $a_7 = 8$. Thus, continuing on, we know

$$\begin{aligned} a_8 &= 6, \text{ since } a_7 + 8 = 16\\ a_9 &= 5, \text{ since } a_8 + 9 = 15\\ a_{10} &= 5, \text{ since } a_9 + 0 = 5\\ a_{11} &= 6, \text{ since } a_{10} + 1 = 6\\ a_{12} &= 8, \text{ since } a_{11} + 2 = 8\\ a_{13} &= 1, \text{ since } a_{12} + 3 = 11\\ a_{14} &= 5, \text{ since } a_{13} + 4 = 5\\ a_{15} &= 0, \text{ since } a_{14} + 5 = 10\\ a_{16} &= 6, \text{ since } a_{15} + 6 = 6\\ a_{17} &= 3, \text{ since } a_{16} + 7 = 13\\ a_{18} &= 1, \text{ since } a_{17} + 8 = 11\\ a_{19} &= 0, \text{ since } a_{18} + 9 = 10\\ a_{20} &= 0, \text{ since } a_{19} + 0 = 0\\ a_{21} &= 1, \text{ since } a_{20} + 1 = 1 \end{aligned}$$

The values of a_n should repeat now. Can you see why?
Since $a_{21} = a_1$ and the ones digit of 22 equals the ones digit of 2, $a_{22} = a_2$.
Similarly, since $a_{22} = a_2$ and the ones digit of 23 equals the ones digit of 3, $a_{23} = a_3$.
We will also have $a_{24} = a_4$, and so on.

Therefore, the values of a_n will repeat every 20 values of n.

We can calculate

$$\begin{aligned} & a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9 + a_{10} + a_{11} + a_{12} + a_{13} + a_{14} + a_{15} + a_{16} + a_{17} + a_{18} + a_{19} + a_{20} \\ & \quad = 1 + 3 + 6 + 0 + 5 + 1 + 8 + 6 + 5 + 5 + 6 + 8 + 1 + 5 + 0 + 6 + 3 + 1 + 0 + 0 \\ & \quad = 70 \end{aligned}$$

Since the values of a_n repeat every 20 values of n, it is also true that
$a_{21} + a_{22} + a_{23} + \cdots + a_{39} + a_{40} = 70$, and $a_{41} + a_{42} + a_{43} + \cdots + a_{59} + a_{60} = 70$, and so on.
Since $\dfrac{2024}{70} = 28\dfrac{32}{35}$, there are 28 complete cycles of the 20 repeating values of a_n.
Therefore, the sum of the first $28 \times 20 = 560$ values of a_n sum to $28 \times 70 = 1960$.

In other words, $a_1 + a_2 + a_3 + \cdots + a_{559} + a_{560} = 1960$.

Let's keep adding values of a_n until we reach 2024.

$$\begin{aligned} a_{561} + a_{562} + a_{563} + a_{564} + a_{565} + a_{566} + a_{567} + a_{568} + a_{569} + a_{570} & \\ = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9 + a_{10} & \\ = 1 + 3 + 6 + 0 + 5 + 1 + 8 + 6 + 5 + 5 & \\ = 40 & \end{aligned}$$

Therefore, $a_1 + a_2 + a_3 + \cdots + a_{569} + a_{570} = 1960 + 40 = 2000$.

We also know that $a_{571} = a_{11} = 6$, $a_{572} = a_{12} = 8$, $a_{573} = a_{13} = 1$, $a_{574} = a_{14} = 5$, and $a_{575} = a_{15} = 0$.

Thus,

$$\begin{aligned} a_1 + a_2 + a_3 + \cdots + a_{569} + a_{570} + a_{571} + a_{572} + a_{573} + a_{574} + a_{575} &= 2000 + 6 + 8 + 1 + 5 + 0 \\ &= 2020 \leq 2024 \end{aligned}$$

and

$$\begin{aligned} a_1 + a_2 + a_3 + \cdots + a_{569} + a_{570} + a_{571} + a_{572} + a_{573} + a_{574} + a_{575} + a_{576} &= 2020 + 6 \\ &= 2026 \geq 2024 \end{aligned}$$

Therefore, the smallest value of n such that $a_1 + a_2 + a_3 + \cdots + a_n \geq 2024$ is $n = 576$.