Problem of the Week
Problem E and Solution
Bilal's Choices

Problem

Bilal chooses two distinct positive integers. He adds the product of the integers to the sum of the integers, and then adds 1. He finds that the result is equal to 196.

Determine all possible pairs of integers that Bilal could have chosen.

Solution

Let x and y represent the two positive integers that Bilal chooses. Since the integers are distinct, $x \neq y$. Let $x < y$. That is, let x represent the smaller of the two integers.
The product of the two integers is xy and the sum is $(x + y)$.
Bilal adds the product of the numbers to the sum of the numbers and then adds 1, and the result is 196. Thus,

$$xy + x + y + 1 = 196$$

Factoring the left side, by grouping the first two terms and the last two terms, we get

$$\begin{aligned} x(y + 1) + 1(y + 1) &= 196 \\ (x + 1)(y + 1) &= 196 \end{aligned}$$

Since x and y are positive integers, then $x + 1$ and $y + 1$ are positive integers. Thus, we are looking for a pair of positive integers whose product is 196. There are four ways to factor 196 as a product of two positive integers:

$$196 = 1 \times 196 = 2 \times 98 = 4 \times 49 = 7 \times 28 = 14 \times 14$$

For the product $196 = 1 \times 196$, we have $x + 1 = 1$ and $y + 1 = 196$. Thus, $x = 0$ and $y = 195$. Since the required numbers are positive integers, this solution is inadmissible.

For the product $196 = 2 \times 98$, we have $x + 1 = 2$ and $y + 1 = 98$. Thus, $x = 1$ and $y = 97$. This is a valid solution.

For the product $196 = 4 \times 49$, we have $x + 1 = 4$ and $y + 1 = 49$. Thus, $x = 3$ and $y = 48$. This is a valid solution.

For the product $196 = 7 \times 28$, we have $x + 1 = 7$ and $y + 1 = 28$. Thus, $x = 6$ and $y = 27$. This is a valid solution.

For the product $196 = 14 \times 14$, we have $x + 1 = 14$ and $y + 1 = 14$. Thus, $x = 13$ and $y = 13$. Since the required numbers are distinct, this solution is inadmissible.

Therefore, there are three pairs of distinct positive integers that Bilal could have chosen: 1 and 97, 3 and 48, or 6 and 27. It can be shown that these three pairs do indeed each give the required result.