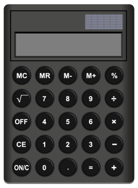

Problem of the Week
Problem E and Solution
Number Crunching

Problem

While waiting for the bus one day, Leo divided numbers on his calculator. He noticed that when 44 000 is divided by 18, the remainder is 8. He then noticed that the remainder is also 8 when 44 000 is divided by 24, and also when 44 000 is divided by 39. Leo then set out to find other numbers that had the same remainder (not necessarily 8), when divided by 18, 24, and 39. How many five-digit positive integers have the same remainder when divided by 18, 24, and 39?

Solution

Since $18 = 2 \times 3 \times 3$, $24 = 2 \times 2 \times 2 \times 3$, and $39 = 3 \times 13$, the lowest common multiple (LCM) of 18, 24, and 39 is $\text{LCM}(18, 24, 39) = 2 \times 2 \times 2 \times 3 \times 3 \times 13 = 936$.

Suppose n is a positive integer. Then the following statements are true:

Every integer of the form $936n$ will have a remainder of 0 when divided by 18, 24, and 39.
Every integer of the form $936n + 1$ will have a remainder of 1 when divided by 18, 24, and 39.
Every integer of the form $936n + 2$ will have a remainder of 2 when divided by 18, 24, and 39.
Every integer of the form $936n + 3$ will have a remainder of 3 when divided by 18, 24, and 39.

$$\vdots$$

Every integer of the form $936n + 16$ will have a remainder of 16 when divided by 18, 24, and 39.
Every integer of the form $936n + 17$ will have a remainder of 17 when divided by 18, 24, and 39.

However, every integer of the form $936n + 18$ will not have the same remainder when divided by 18, 24, and 39. The remainders will be 0, 18, and 18, respectively. Therefore, we need to find the number of five-digit integers that have the form $936n + r$ where $0 \leq r \leq 17$.

The smallest five-digit integer that is a multiple of 936 can be found by dividing 10 000 by 936. Since $\frac{10\,000}{936} \approx 10.68$, the first five-digit multiple is $936 \times 11 = 10\,296$. This means the integers from 10 296 to $10\,296 + 17 = 10\,313$ have the same remainder when divided by 18, 24, and 39.

The largest five-digit integer that is a multiple of 936 can be found by dividing 100 000 by 936. Since $\frac{100\,000}{936} \approx 106.84$, the largest five-digit multiple is $936 \times 106 = 99\,216$. This means the integers from 99 216 to $99\,216 + 17 = 99\,233$ have the same remainder when divided by 18, 24, and 39. We also note that these are all five-digit integers.

Thus, $936n$ is a positive five-digit integer for $11 \leq n \leq 106$. The number of positive five-digit integers that are divisible by 936 is $106 - 11 + 1 = 96$. For each of these multiples of 936, there are 18 integers that have the same remainder when divided by 18, 24, and 39. This gives a total of $96 \times 18 = 1728$ integers that have the same remainder when divided by 18, 24, and 39.

However, we need to check integers near 10 000. The largest multiple of 936 that is less than 10 000 is $936 \times 10 = 9360$. This means the integers between 9360 and $9360 + 18 = 9378$ have the same remainder when divided by 18, 24, and 39. However, none of these are five-digit integers.

Therefore, the number of five-digit positive integers that have the same remainder when divided by 18, 24, and 39 is 1728.