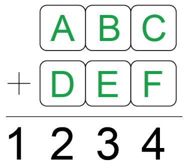


Problem of the Week
Problem E and Solution
Arranging Tiles 3

Problem

Eliana has a box of tiles, each with an integer from 0 to 9 on it. Each integer appears on at least three tiles. Eliana creates larger numbers by placing tiles side by side. For example, using the tiles 3 and 7, Eliana can create the 2-digit number 37 or 73. Using six of her tiles, Eliana forms two 3-digit numbers, ABC and DEF, that add to 1234. Eliana then notices that $A > D$, $B > E$, and $C > F$. How many possible 6-tuples $(A,\ B,\ C,\ D,\ E,\ F)$ could she have chosen?

Solution

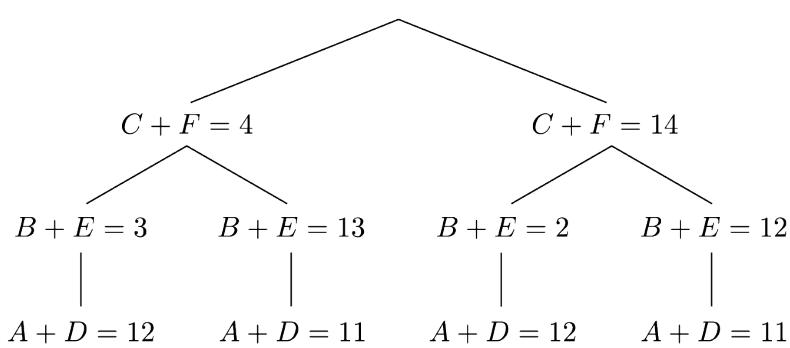
Since $C + F$ ends in a 4, then $C + F = 4$ or $C + F = 14$. The value of $C + F$ cannot be 20 or more, because C and F are digits. In the case that $C + F = 14$, we "carry" a 1 to the tens column. Now we will look at the tens column for these two cases.

- **Case 1:** $C + F = 4$
  Since the result in the tens column is 3 and there was no "carry" from the units column, it follows that $B + E$ ends in a 3. Then $B + E = 3$ or $B + E = 13$. The value of $B + E$ cannot be 20 or more, because B and E are digits. In the case that $B + E = 13$, we "carry" a 1 to the hundreds column.

- **Case 2:** $C + F = 14$
  Since the result in the tens column is 3 and there was a "carry" from the units column, it follows that $1 + B + E$ ends in a 3, so $B + E$ ends in a 2. Then $B + E = 2$ or $B + E = 12$. The value of $B + E$ cannot be 20 or more, because B and E are digits. In the case that $B + E = 12$, we "carry" a 1 to the hundreds column.

Since the result in the hundreds column is 12, then $A + D = 12$, or in the case when there was a "carry" from the tens column, $1 + A + D = 12$, so $A + D = 11$.

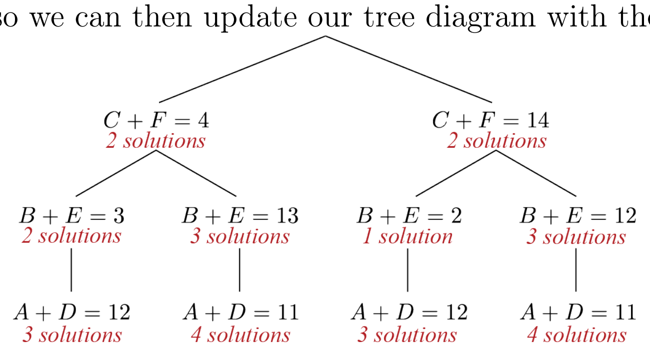
We summarize this information in the following tree.



We now look at the different possibilities for digits $A,\ B,\ C,\ D,\ E$, and F for each individual sum, with the restriction that $A > D$, $B > E$, and $C > F$.

Sum	Solutions	Number of Solutions
$C+F=4$	$C=4,\ F=0$ $C=3,\ F=1$	2
$C+F=14$	$C=9,\ F=5$ $C=8,\ F=6$	2
$B+E=2$	$B=2,\ E=0$	1
$B+E=3$	$B=3,\ E=0$ $B=2,\ E=1$	2
$B+E=12$	$B=9,\ E=3$ $B=8,\ E=4$ $B=7,\ E=5$	3
$B+E=13$	$B=9,\ E=4$ $B=8,\ E=5$ $B=7,\ E=6$	3
$A+D=11$	$A=9,\ D=2$ $A=8,\ D=3$ $A=7,\ D=4$ $A=6,\ D=5$	4
$A+D=12$	$A=9,\ D=3$ $A=8,\ D=4$ $A=7,\ D=5$	3

Since we can use each number at least 3 times, all combinations of solutions outlined in the table are possible, and so we can then update our tree diagram with the total number of solutions for each sum.



The first (leftmost) path through the tree corresponds to the sums $C+F=4$ (2 solutions), $B+E=3$ (2 solutions), and $A+D=12$ (3 solutions). Since there are 2 ways to achieve the first sum, and for each of these possibilities there are 2 ways to achieve the second sum, and for each of these possibilities there are 3 ways to achieve the third sum, the number of 6-tuples this path corresponds to is equal to $2\times2\times3=12$.

Similarly, the second path through the tree corresponds to the sums $C+F=4$ (2 solutions), $B+E=13$ (3 solutions), and $A+D=11$ (4 solutions). So the number of 6-tuples this path corresponds to is equal to $2\times3\times4=24$.

The third path through the tree corresponds to the sums $C+F=14$ (2 solutions), $B+E=2$ (1 solution), and $A+D=12$ (3 solutions). So the number of 6-tuples this path corresponds to is equal to $2\times1\times3=6$.

The fourth path through the tree corresponds to the sums $C+F=14$ (2 solutions), $B+E=12$ (3 solutions), and $A+D=11$ (4 solutions). So the number of 6-tuples this path corresponds to is equal to $2\times3\times4=24$.

Therefore, the total number of 6-tuples is $12+24+6+24=66$.