Problem of the Week
Problem E and Solution
A New Pair of Dice

Problem

A standard six-sided die has its faces marked with the numbers 1, 2, 3, 4, 5, and 6. The die is *fair*, which means that when it is rolled each of its faces has the same probability of being the top face. When two standard six-sided dice are rolled and the numbers on the top faces are added together, the sums range from 2 to 12.

Noemi creates two special six-sided dice that are also fair, but have non-standard numbers on their faces. Numbers on these special dice are positive integers and may appear more than once. The largest number on one of her special dice is 8. When the two special dice are rolled and the numbers on the top faces are added together, the sums range from 2 to 12 and the probability of obtaining each sum is the same as it would be if two standard dice had been used.

Determine all possible pairs of special dice that Noemi could have created.

Solution

We first examine what happens when two standard dice are rolled. To do so, we create a table where the columns show the possible numbers on the top face for one die, the rows show the possible numbers on the top face for the other die, and each cell in the body of the table gives the sum of the corresponding numbers.

		Die 1					
		1	**2**	**3**	**4**	**5**	**6**
Die 2	**1**	2	3	4	5	6	7
	2	3	4	5	6	7	8
	3	4	5	6	7	8	9
	4	5	6	7	8	9	10
	5	6	7	8	9	10	11
	6	7	8	9	10	11	12

From this table we can determine the probability of each sum by counting the number of times each sum appears in the table, and dividing by 36, the total number of possible outcomes.

Sum	2	3	4	5	6	7	8	9	10	11	12
Probability	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

We need the pair of special dice to give these same probabilities when rolled together. Since the dice each have 6 sides, the total number of possible outcomes will still be $6 \times 6 = 36$.

We need the smallest possible sum to be 2. The only way to get a sum of 2 is if each die has a 1 on it. Thus, the smallest number on each die is 1. Furthermore, we need the probability that a sum of 2 is rolled to be $\frac{1}{36}$. Since there are 36 possible outcomes, this means that we need exactly 1 way to get a sum of 2. Therefore, exactly one face on each die must have a 1.

The largest possible sum must be 12 and we need the probability that this sum is rolled to be $\frac{1}{36}$. This means that we need exactly 1 way to get a sum of 12. Since the largest number on one die is 8, it follows that the other die must have exactly one 4. Furthermore, the die with the 8 must contain exactly one 8 and the die with the 4 must have exactly one 4 and no larger number.

Thus, we know that the numbers on one die, from least to greatest, must be (1, ___, ___, ___, ___, 4) and the numbers on the other die, from least to greatest, must be (1, ___, ___, ___, ___, 8), where ___ represents a number we still have to determine.

On the die where the largest number is 4, the remaining numbers must be all 2s and 3s. Since we need the probability that a sum of 3 is rolled to be $\frac{2}{36}$, it follows that there cannot be more than two 2s on this die, otherwise there would be more than 2 ways to get a sum of 3. Similarly, we need the probability that the sum of 11 is rolled to be $\frac{2}{36}$. Since $8 + 3 = 11$, it follows that there can not be more than two 3s on this die, otherwise there would be more than 2 ways to get a sum of 11.

Thus, on the die where the largest number is 4, we have concluded that there is exactly one 1, no more than two 2s, no more than two 3s, and exactly one 4. It follows that the numbers on that die must be (1, 2, 2, 3, 3, 4). We will now find possible numbers for the other die.

We need the probability that a sum of 4 is rolled to be $\frac{3}{36}$. This means that there must be exactly 3 ways to get a sum of 4. Currently we have 2 ways, so we need 1 more. If the die with the 8 has a 2 on a face, then there will be 2 more ways to get a sum of 4. Since we know that this die has exactly one 1, then it must have a 3 on it. Then we will have 3 ways to get a sum of 4, as desired. Thus, we know that the numbers on the second die must be (1, 3, ___, ___, ___, 8).

We need the probability that a sum of 10 is rolled to be $\frac{3}{36}$. That means that there must be exactly 3 ways to get a sum of 10. Currently we have 2 ways, so we need 1 more. If we had a 7 then we would have 2 more ways to get a sum of 10, which is too many. Thus, we must have one 6 on the die so that we can have 3 ways to get a sum of 10, as desired. Thus, we know that the numbers on the second die must be (1, 3, ___, ___, 6, 8).

We need the probability that a sum of 5 is rolled to be $\frac{4}{36}$. That means that there must be exactly 4 ways to get a sum of 5. Currently we have 3 ways, so we need one more. Thus, we must have one 4 on the die so that we can have 4 ways to get a sum of 5, as desired. Thus, we know that the numbers on the second die must be (1, 3, 4, ___, 6, 8).

Since we can't add another number that is currently on the die without changing the probability of rolling a sum we have already looked at, it follows that the remaining number must be a 5. Thus, it must be the case that the numbers on the first die are (1, 2, 2, 3, 3, 4) and the numbers on the second die are (1, 3, 4, 5, 6, 8).

We now need to check that this pair of dice satisfies the conditions of the problem. To do so, we create a table where the columns show the possible numbers on the top face for one die, the rows show the possible numbers on the top face for the other die, and each cell in the body of the table gives the sum of the corresponding numbers.

		Special Die 1					
		1	**2**	**2**	**3**	**3**	**4**
Special Die 2	**1**	2	3	3	4	4	5
	3	4	5	5	6	6	7
	4	5	6	6	7	7	8
	5	6	7	7	8	8	9
	6	7	8	8	9	9	10
	8	9	10	10	11	11	12

From this table we can determine the probability of each sum by counting the number of times each sum appears in the table, and dividing by 36, the total number of possible outcomes.

It turns out that these probabilities match the probabilities when two standard dice are rolled, so we can conclude that there is only one possible pair of special dice that Noemi could have created. The numbers on this pair of dice must be (1, 2, 2, 3, 3, 4) and (1, 3, 4, 5, 6, 8).