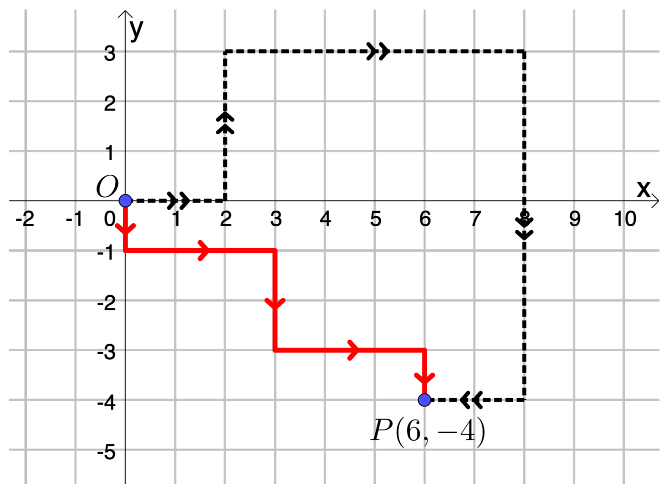
Problem of the Week

Problem E and Solution

The Shortest Path

Problem

On the Cartesian plane, we draw grid lines at integer points along the x and y axes. We can then draw paths along these grid lines between any two points with integer coordinates. The graph below shows two paths along these grid lines from $O(0, 0)$ to $P(6, -4)$. One path has length 10 and the other has length 20.



There are many different paths along the grid lines from O to P, but the smallest possible length of such a path is 10. Let's call this smallest possible length the *path distance* from O to P.

Determine the number of points with integer coordinates for which the path distance from O to that point is 10.

Solution

Solution 1

Let $Q(a, b)$ be a point that has path distance 10 from $O(0, 0)$.

Let's first suppose that Q lies on the x or y axis.
The only point along the positive x-axis that has path distance 10 from the origin is $(10, 0)$.
The only point along the negative x-axis that has path distance 10 from the origin is $(-10, 0)$.
The only point along the positive y-axis that has path distance 10 from the origin is $(0, 10)$.
The only point along the negative y-axis that has path distance 10 from the origin is $(0, -10)$.
Therefore, there are 4 points along the axes that have a path distance 10 from O.

Next, let's suppose $a > 0$ and $b > 0$, so Q is in the first quadrant.
Since the path distance from O to Q is 10, there must be a path from O to Q that moves a total of r units to the right and u units up (in some order) such that $r + u = 10$. This means that Q is r units to the right of O and u units up from O. In other words, $a = r$ and $b = u$, so $a + b = r + u = 10$.

The points (a, b) in the first quadrant that satisfy $a + b = 10$ where a and b are integers are $(1, 9)$, $(2, 8)$, $(3, 7)$, $(4, 6)$, $(5, 5)$, $(6, 4)$, $(7, 3)$, $(8, 2)$, $(9, 1)$. There are 9 such pairs. Therefore, there are 9 points in the first quadrant that have path distance 10 from O.

By symmetry, there are 9 points in each quadrant that have path distance 10 from O. In quadrant 2, the points are $(-1, 9)$, $(-2, 8)$, $(-3, 7)$, $(-4, 6)$, $(-5, 5)$, $(-6, 4)$, $(-7, 3)$, $(-8, 2)$, $(-9, 1)$. In quadrant 3, the points are $(-1, -9)$, $(-2, -8)$, $(-3, -7)$, $(-4, -6)$, $(-5, -5)$, $(-6, -4)$, $(-7, -3)$, $(-8, -2)$, $(-9, -1)$. In quadrant 4, the points are $(1, -9)$, $(2, -8)$, $(3, -7)$, $(4, -6)$, $(5, -5)$, $(6, -4)$, $(7, -3)$, $(8, -2)$, $(9, -1)$.

Therefore, there are a total of $4 + (4 \times 9) = 40$ points with integer coordinates that have path distance 10 from O.

Solution 2

We are permitted 10 moves to get from the origin to a point by travelling along the grid lines. These moves can be all horizontal (in one direction), all vertical (in one direction), or a combination of horizontal moves (in one direction) with vertical moves (in one direction).

We examine the cases based on the number of horizontal moves.

- **0 horizontal moves**: Since there are 0 horizontal moves, there are 10 vertical moves. There are two possible endpoints, $(0, 10)$ and $(0, -10)$.
- **1 horizontal move**: Since there is 1 horizontal move, there are 9 vertical moves. There are four possible endpoints, $(-1, 9)$, $(-1, -9)$, $(1, 9)$, and $(1, -9)$.
- **2 horizontal moves**: Since there are 2 horizontal moves, there are 8 vertical moves. There are four possible endpoints, $(-2, 8)$, $(-2, -8)$, $(2, 8)$, and $(2, -8)$.
- **3 horizontal moves**: Since there are 3 horizontal moves, there are 7 vertical moves. There are four possible endpoints, $(-3, 7)$, $(-3, -7)$, $(3, 7)$, and $(3, -7)$.
- **4 horizontal moves**: Since there are 4 horizontal moves, there are 6 vertical moves. There are four possible endpoints, $(-4, 6)$, $(-4, -6)$, $(4, 6)$, and $(4, -6)$.
- **5 horizontal moves**: Since there are 5 horizontal moves, there are 5 vertical moves. There are four possible endpoints, $(-5, 5)$, $(-5, -5)$, $(5, 5)$, and $(5, -5)$.
- **6 horizontal moves**: Since there are 6 horizontal moves, there are 4 vertical moves. There are four possible endpoints, $(-6, 4)$, $(-6, -4)$, $(6, 4)$, and $(6, -4)$.
- **7 horizontal moves**: Since there are 7 horizontal moves, there are 3 vertical moves. There are four possible endpoints, $(-7, 3)$, $(-7, -3)$, $(7, 3)$, and $(7, -3)$.
- **8 horizontal moves**: Since there are 8 horizontal moves, there are 2 vertical moves. There are four possible endpoints, $(-8, 2)$, $(-8, -2)$, $(8, 2)$, and $(8, -2)$.
- **9 horizontal moves**: Since there are 9 horizontal moves, there is 1 vertical move. There are four possible endpoints, $(-9, 1)$, $(-9, -1)$, $(9, 1)$, and $(9, -1)$.
- **10 horizontal moves**: Since there are 10 horizontal moves, there are 0 vertical moves. There are two possible endpoints, $(-10, 0)$ and $(10, 0)$.

Therefore, there are a total of $2 + (4 \times 9) + 2 = 40$ points with integer coordinates that have path distance 10 from O.