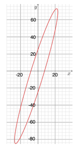


Problem of the Week

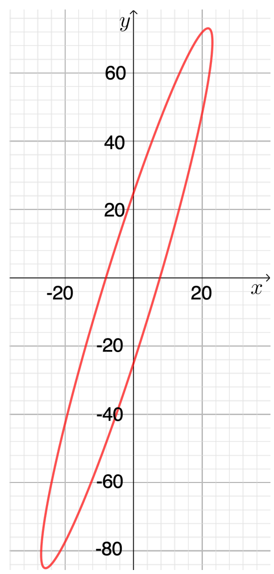
Problem E and Solution

Points on an Ellipse

Problem

The graph of $(x+1)^2 + (y-2)^2 = 100$ is a circle with centre $(-1, 2)$ and radius 10.

The graph of $10x^2 - 6xy + 4x + y^2 = 621$ is shown below. The shape of this curve is known as an ellipse.



List all the ordered pairs (x, y) of non-negative integers x and y that satisfy the equation $10x^2 - 6xy + 4x + y^2 = 621$.

Note: When solving this problem, it might be useful to use the following idea.

By completing the square,

$$x^2 + y^2 + 2x - 4y = 95$$

can be rewritten as

$$(x+1)^2 + (y-2)^2 = 100$$

One solution to this equation is $(x, y) = (5, 10)$.

Solution

Starting with the given equation, we obtain the following equivalent equations:

$$\begin{aligned} 10x^2 - 6xy + 4x + y^2 &= 621 \\ 9x^2 - 6xy + y^2 + x^2 + 4x &= 621 \\ 9x^2 - 6xy + y^2 + x^2 + 4x + 4 &= 621 + 4 \\ (3x - y)^2 + (x + 2)^2 &= 625 \end{aligned}$$

Notice that $625 = 25^2$.
Since x and y are both integers, then the left side of the given equation is the sum of two perfect squares. Since any perfect square is non-negative, then each of these perfect squares is at most $625 = 25^2$.

The pairs of perfect squares that sum to 625 are 625 and 0, 576 and 49, and 400 and 225.

Therefore, $(3x - y)^2$ and $(x + 2)^2$ are equal to 25^2 and 0^2 in some order, or 24^2 and 7^2 in some order, or 20^2 and 15^2 in some order.

Furthermore, $3x - y$ and $x + 2$ are equal to ± 25 and ± 0 in some order, or ± 24 and ± 7 in some order, or ± 20 and ± 15 in some order.

Since $x \geq 0$, then $x + 2 \geq 2$. So we need to consider when $x + 2$ is equal to 25, 24, 7, 20, or 15.

- If $x + 2 = 25$, then $x = 23$. Also, $3x - y = 0$. Thus, $y = 69$. Since $x \geq 0$ and $y \geq 0$, $(23, 69)$ is a valid ordered pair.
- If $x + 2 = 24$, then $x = 22$. Also, $3x - y = 7$ or $3x - y = -7$.
  When $3x - y = 7$, we find $y = 59$. Since $x \geq 0$ and $y \geq 0$, $(22, 59)$ is a valid ordered pair.
  When $3x - y = -7$, we find $y = 73$. Since $x \geq 0$ and $y \geq 0$, $(22, 73)$ is a valid ordered pair.
- If $x + 2 = 7$, then $x = 5$. Also, $3x - y = 24$ or $3x - y = -24$.
  When $3x - y = 24$, we find $y = -9$. Since $y < 0$, this does not lead to a valid ordered pair.
  When $3x - y = -24$, we find $y = 39$. Since $x \geq 0$ and $y \geq 0$, $(5, 39)$ is a valid ordered pair.
- If $x + 2 = 20$, then $x = 18$. Also, $3x - y = 15$ or $3x - y = -15$.
  When $3x - y = 15$, we find $y = 39$. Since $x \geq 0$ and $y \geq 0$, $(18, 39)$ is a valid ordered pair.
  When $3x - y = -15$, we find $y = 69$. Since $x \geq 0$ and $y \geq 0$, $(18, 69)$ is a valid ordered pair.
- If $x + 2 = 15$, then $x = 13$. Also, $3x - y = 20$ or $3x - y = -20$.
  When $3x - y = 20$, we find $y = 19$. Since $x \geq 0$ and $y \geq 0$, $(13, 19)$ is a valid ordered pair.
  When $3x - y = -20$, we find $y = 59$. Since $x \geq 0$ and $y \geq 0$, $(13, 59)$ is a valid ordered pair.

Therefore, the ordered pairs of non-negative integers that satisfy the equation are $(23, 69)$, $(22, 59)$, $(22, 73)$, $(5, 39)$, $(18, 39)$, $(18, 69)$, $(13, 19)$, and $(13, 59)$.