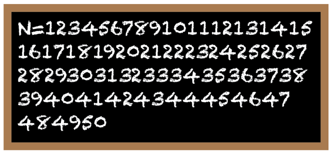

Problem of the Week

Problem E and Solution

Discarding Digits

Problem

Stef forms the integer N by writing the integers from 1 to 50 in order.

That is,

$N = 1234567891011121314151617181920212223242526272829303132333435363738394041424344454647484950.$

Stef then selects some of the digits in N and discards them, so that the remaining digits, in their original order, form a new integer. The sum of the digits in this new integer is 200.

If M is the largest integer that Stef could have formed, what are the first ten digits of M?

Solution

We start by determining the sum of the digits of N. This is the same as determining the sum of the digits of the numbers from 1 to 50. The digits 0 to 9 sum to
$0 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45$.

The digits in the numbers 10 to 19 sum to
$(1 + 0) + (1 + 1) + (1 + 2) + (1 + 3) + (1 + 4) + (1 + 5) + (1 + 6) + (1 + 7) + (1 + 8) + (1 + 9)$
$= 10(1) + (0 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9) = 10 + 45 = 55$.

The digits in the numbers 20 to 29 sum to
$(2 + 0) + (2 + 1) + (2 + 2) + (2 + 3) + (2 + 4) + (2 + 5) + (2 + 6) + (2 + 7) + (2 + 8) + (2 + 9)$
$= 10(2) + (0 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9) = 20 + 45 = 65$.

Similarly, the digits in the numbers 30 to 39 sum to $10(3) + 45 = 75$ and the digits in the numbers 40 to 49 sum to $10(4) + 45 = 85$.

We must add 5 and 0 in order to account for the number 50 at the end of N. Therefore, the sum of the digits of N is $45 + 55 + 65 + 75 + 85 + (5 + 0) = 330$.

Since the digits of M sum to 200, the digits that are removed and discarded must sum to $330 - 200 = 130$.

In order for M to be as large as possible, we need M to have as many digits as possible. So we need to remove as few digits as possible such that the digits that are removed sum to 130. To determine the fewest number of digits to remove, we remove the largest digits in N.

We notice that in N there are five 9s, five 8s, and five 7s. These 15 digits have a sum of $5 \times 9 + 5 \times 8 + 5 \times 7 = 120$. After removing these digits, we would still need to remove additional digits that have a sum of $130 - 120 = 10$. We would need at least two more digits to do this. So the fewest number of digits that we can remove that will have a sum of 130 is 17.

So which 17 digits do we remove? We are left with the following four options.

1. Remove five 9s, five 8s, five 7s, one 6, and one 4.
2. Remove five 9s, five 8s, five 7s, and two 5s.
3. Remove five 9s, five 8s, four 7s, two 6s, and one 5.
4. Remove five 9s, four 8s, five 7s, and three 6s.

These are the only ways to remove 17 digits from N that have a sum of 130. Thus, each option will result in a number that is exactly 17 digits shorter than N. So to determine which option results in the largest possible number, we can look at how each affects the first few digits of N.

Option 1: Remove five 9s, five 8s, five 7s, one 6, and one 4.
After removing all the 9s, 8s, and 7s, the remaining digits start $123456101112\ldots$.

- Removing a 6 and a 4 from anywhere after the first six digits will result in a number whose first six digits are 123456.
- Removing the first 6, and a 4 from anywhere past this 6 will result in a number whose first 6 digits are 123451.
- Removing the first 4, and a 6 from anywhere else in the number will result in a number whose first 6 digits are 123510 or 123561.

Since $123561 > 123510 > 123456 > 123451$, removing the first 4, and a 6 from anywhere else in the number can result in a number whose first six digits are 123561. This would result in the largest possible number so far.

Option 2: Remove five 9s, five 8s, five 7s, and two 5s.
After removing all the 9s, 8s, and 7s, the remaining digits start $123456101112\ldots$. No matter how we remove two 5s, we are not able to get a number whose first six digits are larger than 123561. Thus, this option will not result in the largest possible value of M.

Option 3: Remove five 9s, five 8s, four 7s, two 6s, and one 5.
After removing all the 9s and 8s, the remaining digits start $1234567101112\ldots$. No matter how we remove four 7s, two 6s, and one 5, we are not able to get a number whose first six digits are larger than 123561. Thus, this option will not result in the largest possible value of M.

Option 4: Remove five 9s, four 8s, five 7s, and three 6s.
After removing all the 9s and 7s, the remaining digits start $1234568101112\ldots$. No matter how we remove four 8s, and three 6s, we are not able to get a number whose first six digits are larger than 123561. Thus, this option will not result in the largest possible value of M.

Therefore, in order to form the largest possible value of M, we should remove all 9s, 8s, and 7s, the first 4, and a 6 from anywhere else in the number.

After removing the 9s, 8s, 7s, and the first 4, we are left with

$$1235610111213141516112021222324252622230313233343536333404142434445464450$$

We still must remove a 6 to bring our digit sum to 200. We want whatever 6 we remove to affect the size of the final number in the least possible way. We need to therefore remove the 6 with the lowest place value. The 6 to be removed is therefore the hundred thousands digit in the number shown just above. After removing this 6,

$$M = 1235610111213141516112021222324252622230313233343536333404142434445444450$$

It follows that the first 10 digits of M are 1235610111.