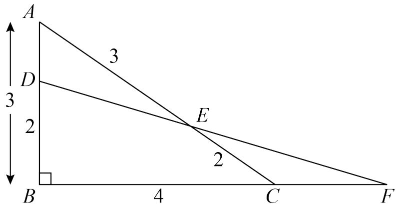
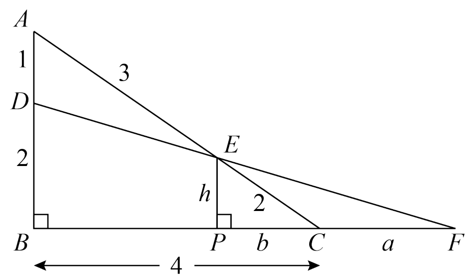


Problem of the Week
Problem E and Solution
Overlapping Shapes 3

Problem

Austin draws $\triangle ABC$ with $AB = 3$ cm, $BC = 4$ cm, and $\angle ABC = 90^\circ$. Lachlan then draws $\triangle DBF$ on top of $\triangle ABC$ so that D lies on AB, F lies on the extension of BC, $DB = 2$ cm, and sides AC and DF meet at E. If $AE = 3$ cm and $EC = 2$ cm, determine the length of CF.

Solution



Since $AB = 3$ and $DB = 2$, it follows that $AD = 1$ cm.
Draw a perpendicular from E to BF.
Let P be the point where the perpendicular intersects BF.
Let $CF = a$, $PC = b$, and $EP = h$.
We will now proceed with three solutions. The first two solutions depend on this setup. The first uses similar triangles, the second uses trigonometry, and the third uses coordinate geometry.

Solution 1

Since EP is perpendicular to BF, we know $\angle EPF = 90^\circ$. Also, $\angle ECP = \angle ACB$ (same angle). Therefore, $\triangle ABC \sim \triangle EPC$ (by angle-angle triangle similarity).

From the similarity, $\dfrac{AC}{BC} = \dfrac{EC}{PC}$, so $\dfrac{5}{4} = \dfrac{2}{b}$ or $b = \dfrac{8}{5}$. Also, $\dfrac{AC}{AB} = \dfrac{EC}{EP}$, so $\dfrac{5}{3} = \dfrac{2}{h}$ or $h = \dfrac{6}{5}$.

Now let's calculate PF. We know $\angle EPF = \angle DBF = 90^\circ$ and $\angle EFP = \angle DFB$ (same angle).

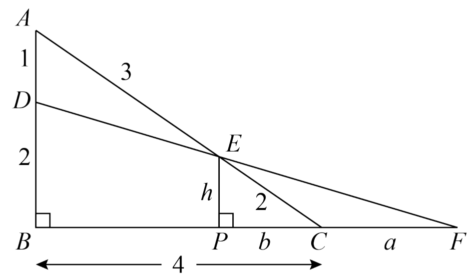
Therefore, $\triangle DBF \sim \triangle EPF$ (by angle-angle triangle similarity). This tells us $\dfrac{DB}{BF} = \dfrac{EP}{PF}$.

Since $BF = BC + CF = 4 + a$ and $PF = PC + CF = \frac{8}{5} + a$, we have

$$\begin{aligned}
\frac{DB}{BF} &= \frac{EP}{PF} \\
\frac{2}{4+a} &= \frac{\frac{6}{5}}{\frac{8}{5}+a} \\
\frac{16}{5} + 2a &= \frac{24}{5} + \frac{6}{5}a \\
2a - \frac{6}{5}a &= \frac{24}{5} - \frac{16}{5} \\
\frac{4}{5}a &= \frac{8}{5} \\
a &= 2
\end{aligned}$$

Therefore, $CF = 2$ cm.

Solution 2



In $\triangle EPC$, $\sin(\angle ECP) = \frac{h}{2}$. In $\triangle ABC$, $\sin(\angle ACB) = \frac{3}{5}$.
Since $\angle ECP = \angle ACB$ (same angle),

$$\begin{aligned}
\sin(\angle ECP) &= \sin(\angle ACB) \\
\frac{h}{2} &= \frac{3}{5} \\
h &= \frac{6}{5}
\end{aligned}$$

Since $\triangle EPC$ is a right-angled triangle,

$$\begin{aligned}
EP^2 + PC^2 &= EC^2 \\
h^2 + b^2 &= 2^2 \\
\left(\frac{6}{5}\right)^2 + b^2 &= 4 \\
b^2 &= 4 - \frac{36}{25} \\
b^2 &= \frac{64}{25} \\
b &= \frac{8}{5}, \quad \text{since} \quad b > 0
\end{aligned}$$

In $\triangle EPF$, $\tan(\angle EFP) = \dfrac{EP}{PF} = \dfrac{h}{a+b} = \dfrac{\frac{6}{5}}{a + \frac{8}{5}}$.

In $\triangle DBF$, $\tan(\angle DFB) = \dfrac{DB}{BF} = \dfrac{2}{4+a}$.

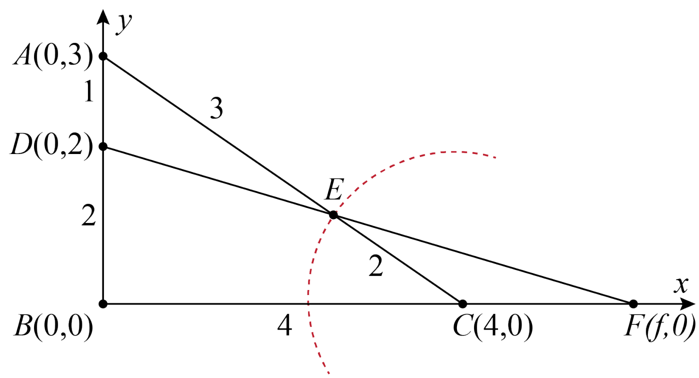
Since $\angle EFP = \angle DFB$ (same angle),

$$\begin{aligned}
\tan(\angle EFP) &= \tan(\angle DFB) \\
\frac{\frac{6}{5}}{a + \frac{8}{5}} &= \frac{2}{4+a} \\
\frac{24}{5} + \frac{6}{5}a &= 2a + \frac{16}{5} \\
2a - \frac{6}{5}a &= \frac{24}{5} - \frac{16}{5} \\
\frac{4}{5}a &= \frac{8}{5} \\
a &= 2
\end{aligned}$$

Therefore, $CF = 2$ cm.

Solution 3

We will use coordinate geometry in this solution, and place B at the origin. Using the given information, D is at $(0,2)$, A is at $(0,3)$, C is at $(4,0)$, and F is on the positive x-axis at $(f,0)$ with $f>4$. Consider the circle through E with centre $C(4,0)$. Since $CE=2$, the radius of this circle is 2. Thus, the equation of this circle is $(x-4)^2+y^2=4$.



The line passing through $A(0,3)$ and $C(4,0)$ has y-intercept 3 and slope $-\frac{3}{4}$, and so has equation $y=-\frac{3}{4}x+3$. Since E lies on the line with equation $y=-\frac{3}{4}x+3$ and the circle with equation $(x-4)^2+y^2=4$, to find the coordinates of E, we substitute $y=-\frac{3}{4}x+3$ for y in $(x-4)^2+y^2=4$. Note that E is in the first quadrant so $x>0$ and $y>0$.

Doing so, we get
$$(x-4)^2+\left(-\frac{3}{4}x+3\right)^2=4$$

Expanding the left side, we get
$$x^2-8x+16+\frac{9}{16}x^2-\frac{9}{2}x+9=4$$

Multiplying by 16, we get
$$16x^2-128x+256+9x^2-72x+144=64$$

Simplifying, we get
$$25x^2-200x+336=0$$

Factoring, we then get
$$(5x-12)(5x-28)=0$$

It follows that $x=\frac{12}{5}$ or $x=\frac{28}{5}$. Substituting $x=\frac{12}{5}$ in $y=-\frac{3}{4}x+3$, we obtain $y=\frac{6}{5}$. Substituting $x=\frac{28}{5}$ in $y=-\frac{3}{4}x+3$, we obtain $y=-\frac{6}{5}$. But E is in the first quadrant so $y>0$, and this second possibility is inadmissible. It follows that E has coordinates $\left(\frac{12}{5},\frac{6}{5}\right)$.

We can now find the equation of the line containing $D(0,2)$, $E\left(\frac{12}{5},\frac{6}{5}\right)$, and $F(f,0)$. This line has y-intercept 2, slope equal to $\dfrac{\frac{6}{5}-2}{\frac{12}{5}-0}=\dfrac{-\frac{4}{5}}{\frac{12}{5}}=-\frac{1}{3}$, and thus has equation $y=-\frac{1}{3}x+2$.

The point $F(f,0)$ lies on this line, so $0=-\frac{1}{3}(f)+2$, which leads to $f=6$. Thus, the point F has coordinates $(6,0)$. Since C is at $(4,0)$ and F is at $(6,0)$, $CF=2$. It turns out that F also lies on the circle through E.

Therefore, $CF=2$ cm.