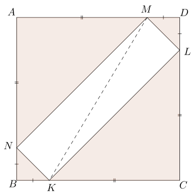


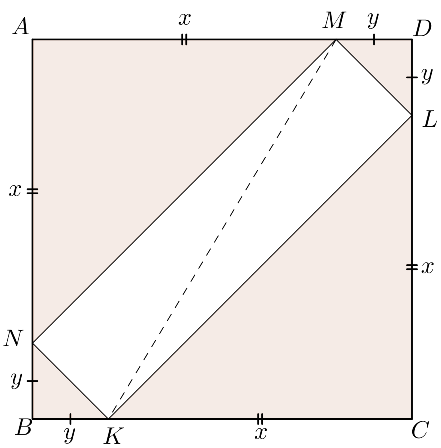
Problem of the Week
Problem E and Solution
Diagonal Distance

Problem

Square $ABCD$ has K on BC, L on DC, M on AD, and N on AB such that $KLMN$ forms a rectangle, $\triangle AMN$ and $\triangle LKC$ are congruent isosceles triangles, and also $\triangle MDL$ and $\triangle BNK$ are congruent isosceles triangles. If the total area of the four triangles is 50 cm^2, what is the length of MK?

Solution

Let x represent the lengths of the equal sides of $\triangle AMN$ and $\triangle LKC$, and let y represent the lengths of the equal sides of $\triangle MDL$ and $\triangle BNK$.



Thus, area $\triangle AMN =$ area $\triangle LKC = \frac{1}{2}x^2$, and area $\triangle MDL =$ area $\triangle BNK = \frac{1}{2}y^2$.

Therefore, the total area of the four triangles is equal to $\frac{1}{2}x^2 + \frac{1}{2}x^2 + \frac{1}{2}y^2 + \frac{1}{2}y^2 = x^2 + y^2$. Since we're given that this area is 50 cm^2, we have $x^2 + y^2 = 50$.

Three different solutions to find the length of MK are provided.

Solution 1

In $\triangle AMN$, $MN^2 = AM^2 + AN^2 = x^2 + x^2$, and in $\triangle BNK$, $NK^2 = BN^2 + BK^2 = y^2 + y^2$.

Since MK is a diagonal of rectangle $KLMN$, then by the Pythagorean Theorem we have

$$\begin{aligned} MK^2 &= MN^2 + NK^2 \\ &= x^2 + x^2 + y^2 + y^2 \\ &= x^2 + y^2 + x^2 + y^2 \\ &= 50 + 50 \\ &= 100 \end{aligned}$$

Since $MK > 0$, we have $MK = 10$ cm.

Solution 2

In $\triangle AMN$, $MN^2 = x^2 + x^2 = 2x^2$. Therefore, $MN = \sqrt{2}x$, since $x > 0$.
In $\triangle BNK$, $NK^2 = y^2 + y^2 = 2y^2$. Therefore, $NK = \sqrt{2}y$, since $y > 0$.

Since MK is a diagonal of rectangle $KLMN$, then by the Pythagorean Theorem we have

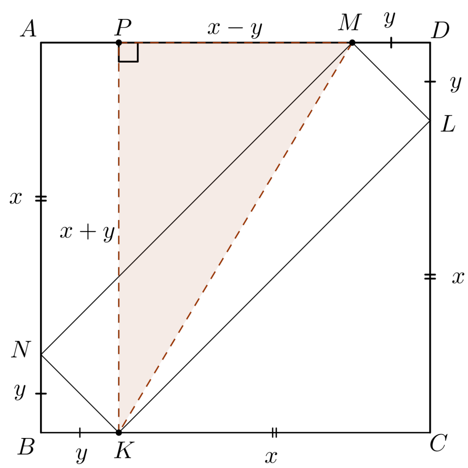
$$\begin{aligned} MK^2 &= MN^2 + NK^2 \\ &= (\sqrt{2}x)^2 + (\sqrt{2}y)^2 \\ &= 2x^2 + 2y^2 \\ &= 2(x^2 + y^2) \\ &= 2(50) \\ &= 100 \end{aligned}$$

Since $MK > 0$, we have $MK = 10$ cm.

Solution 3

We construct the line segment KP, where P lies on AD such that KP is perpendicular to AD.

Then $APKB$ is a rectangle. Furthermore, $AP = BK = y$, $PK = AB = x + y$, and
$PM = AM - AP = x - y$.



Since $\triangle PKM$ is a right-angled triangle, by the Pythagorean Theorem we have

$$\begin{aligned} MK^2 &= PM^2 + PK^2 \\ &= (x-y)^2 + (x+y)^2 \\ &= x^2 - 2xy + y^2 + x^2 + 2xy + y^2 \\ &= 2x^2 + 2y^2 \\ &= 2(x^2 + y^2) \\ &= 2(50) \\ &= 100 \end{aligned}$$

Since $MK > 0$, we have $MK = 10$ cm.