# Problem of the Week
## Problem D and Solution
## Sum New Year!

**Problem**

The positive integers are written consecutively in rows, with seven integers in each row. That is, the first row contains the integers 1, 2, 3, 4, 5, 6, and 7. The second row contains the integers 8, 9, 10, 11, 12, 13, and 14. The third row contains the integers 15, 16, 17, 18, 19, 20, and 21, and so on.

$$\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 8 & 9 & 10 & 11 & 12 & 13 & 14 \\ 15 & 16 & 17 & 18 & 19 & 20 & 21 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{array}$$

The *row sum* of a row is the sum of the integers in the row. For example, the row sum of the first row is $1 + 2 + 3 + 4 + 5 + 6 + 7 = 28$.

Determine the numbers in the row that has a row sum closest to 2024.

## Solution
## Solution 1

The last number in row 1 is 7, the last number in row 2 is 14, and the last number in row 3 is 21. Observe that the last number in each row is a multiple of 7. Furthermore, the last number in row $n$ is $7n$. Since the last number in row $n$ is $7n$, the six preceding numbers in the row are $7n - 1$, $7n - 2$, $7n - 3$, $7n - 4$, $7n - 5$, and $7n - 6$.

The sum of the numbers in row $n$ is

$$(7n - 6) + (7n - 5) + (7n - 4) + (7n - 3) + (7n - 2) + (7n - 1) + 7n = 49n - 21$$

We want to find the integer value of $n$ so that $49n - 21$ is as close to 2024 as possible.

$$\begin{aligned} 49n - 21 &= 2024 \\ 49n &= 2045 \\ n &\approx 41.7 \end{aligned}$$

The closest integer to 41.7 is 42. The row sum of row 42 is $49n - 21 = 49(42) - 21 = 2037$. The last number in row 42 is $7 \times 42 = 294$. The seven integers in the row 42 are 288, 289, 290, 291, 292, 293, and 294. Row 41 contains the integers 281, 282, 283, 284, 285, 286, and 287, and has row sum equal to 1988. This row sum is farther from 2024 than 2037, the row sum of row 42.

Therefore, row 42 has the row sum closest to 2024. This row contains the integers 288, 289, 290, 291, 292, 293, and 294.

The second solution approaches the problem by establishing a linear relationship.

## Solution 2

Let $x$ represent the row number and $y$ represent the sum of the integers in the row. Observe that the seventh integer in any row is a multiple of 7. In fact, the seventh integer in any row is 7 times the row number or $7x$. The following table of values summarizes the row sums for the first three rows.

| Row Number ($x$) | Row Sum ($y$) |
|---|---|
| 1 | 28 |
| 2 | 77 |
| 3 | 126 |

Notice that the $y$ values increase by 49 as the $x$ values increase by 1. We will verify that this is true. In Solution 1 we saw that the sum of the numbers in row $n$ is $49n - 21$. Therefore, the sum of the numbers in row $x$ is $49x - 21$ and the sum of the numbers in row $(x + 1)$ is $49(x + 1) - 21 = (49x - 21) + 49$. Thus, the $y$ values increase by 49 as the $x$ values increase by 1. This tells us that the sum of the fourth row should be $126 + 49 = 175$. We can verify this by adding $22 + 23 + 24 + 25 + 26 + 27 + 28$, the numbers in the fourth row. The sum is indeed 175.

As the values of $x$ increase by 1, the values of $y$ increase by 49. The relation is linear. The slope is $\dfrac{\Delta y}{\Delta x} = \dfrac{49}{1} = 49$. Substituting $x = 1$, $y = 28$, $m = 49$ into the equation $y = mx + b$, we get

$$\begin{aligned} 28 &= 49(1) + b \\ -21 &= b \end{aligned}$$

Thus, the equation of the line which passes through the points in the relation is $y = 49x - 21$. Note that $x$ and $y$ are positive integers. We want to find the value of $x$, the row number, so that the value of $y$, the row sum, is as close to 2024 as possible.

$$\begin{aligned} 49x - 21 &= 2024 \\ 49x &= 2045 \\ x &\approx 41.7 \end{aligned}$$

The closest integer to 41.7 is 42. When $x = 42$, the row sum is $y = 49(42) - 21 = 2037$. The row sum when $x = 41$ is $y = 49(41) - 21 = 1988$. The row sum 2037 is closer to 2024 than the row sum 1988.

Therefore, row 42 has the row sum closest to 2024. The seventh number in row 42 is $7 \times 42 = 294$. Thus, this row contains the integers 288, 289, 290, 291, 292, 293, and 294.