The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING

Problem of the Week
Problems and Solutions
2021 - 2022

Problem E (Grade 11/12)

Themes

(Click on a theme name to jump to that section.)

Number Sense (N)

Geometry & Measurement (G)

Algebra (A)

Data Management (D)

Computational Thinking (C)

The problems in this booklet are organized into themes.
A problem often appears in multiple themes.

Number Sense (N)

TAKE ME TO THE COVER

Problem of the Week
Problem E
Down to One

The *digit sum* of a positive integer is found by summing its digits.
The *digital root* is found by repeatedly calculating the digit sum until a single digit is achieved.

The digit sum of 413 is 8, since $4 + 1 + 3 = 8$ and 8 is a single-digit number. Note that the digital root is also 8, and this is calculated in one step.

The digit sum of 642 is $6 + 4 + 2 = 12$, which is not a single-digit number. The digit sum of 12 is $1 + 2 = 3$, which is a single-digit number. Therefore, the digital root of 642 is 3. This is calculated in two steps.

The digital root of 4 is 4. This is calculated in zero steps.

How many three-digit numbers have a digital root of 5 that is calculated in three or fewer steps?

$$4 + 1 + 3 = 8$$

4 + 1 + 3 = 8

Problem of the Week
Problem E and Solution
Down to One

Problem

The *digit sum* of a positive integer is found by summing its digits.
The *digital root* is found by repeatedly calculating the digit sum until a single digit is achieved.
The digit sum of 413 is 8, since $4 + 1 + 3 = 8$ and 8 is a single-digit number. Note that the digital root is also 8, and this is calculated in one step.
The digit sum of 642 is $6 + 4 + 2 = 12$, which is not a single-digit number. The digit sum of 12 is $1 + 2 = 3$, which is a single-digit number. Therefore, the digital root of 642 is 3. This is calculated in two steps.
The digital root of 4 is 4. This is calculated in zero steps.

How many three-digit numbers have a digital root of 5 that is calculated in three or fewer steps?

Solution

We will consider four cases for the three-digit numbers: a digital root of 5 in zero steps, a digital root of 5 in one step, a digital root of 5 in two steps, and a digital root of 5 in three steps.

Case 1: A digital root of 5 that is reached in zero steps.

This only occurs when the number is 5, which is not a three-digit number.
There are no three-digit numbers with a digital root of 5 that can be reached in zero steps.

Case 2: A digital root of 5 that is reached in one step.

Since the digital root is 5, then no digit in the three-digit number can be higher than 5. We use this to carefully generate the list below of all of the three-digit numbers whose digits sum to 5.

- $104, 113, 122, 131, 140$
- $203, 212, 221, 230$
- $302, 311, 320$
- $401, 410$
- 500

There are $5 + 4 + 3 + 2 + 1 = 15$ three-digit numbers with a digital root of 5 that can reached in exactly one step.

Case 3: A digital root of 5 that is reached in two steps.

The maximum sum of the digits of a three-digit number is $9 + 9 + 9 = 27$. In order to reach a digital root of 5 in two steps, the initial sum must be a two-digit number less than 28 whose digits sum to 5. There are only 2 two-digit numbers that satisfy this condition, namely 14 and 23.

If the sum of the digits of the three-digit number is 14, we can systematically generate the possible numbers. For example, if the first digit is 1, then the other two digits add to 13. This happens exactly when the last two digits are 4 and 9, 5 and 8, or 6 and 7. The resulting six three-digit numbers with first digit 1 are shown on the first line below. The remaining lines are generated in a similar manner and are also shown below.

- $149, 158, 167, 176, 185, 194$
- $239, 248, 257, 266, 275, 284, 293$
- $329, 338, 347, 356, 365, 374, 383, 392$
- $419, 428, 437, 446, 455, 464, 473, 482, 491$
- $509, 518, 527, 536, 545, 554, 563, 572, 581, 590$
- $608, 617, 626, 635, 644, 653, 662, 671, 680$
- $707, 716, 725, 734, 743, 752, 761, 770$
- $806, 815, 824, 833, 842, 851, 860$
- $905, 914, 923, 932, 941, 950$

There are $6 + 7 + 8 + 9 + 10 + 9 + 8 + 7 + 6 = 70$ three-digit numbers with a digital root of 5 that can reached in exactly two steps with the initial sum of 14.

If the sum of the digits of the three-digit number is 23, we can again systematically generate the possible numbers. If the final two digits of the three-digit number are 9 and 9, the first digit must be a 5. Therefore, no three-digit number less than 599 has digits that sum to 23. (For example if one of the digits is 4, then the sum of the other two digits must be 19 and this is impossible using two single digits.) The list below is generated in a similar way to the one shown above.

- 599
- $689, 698$
- $779, 788, 797$
- $869, 878, 887, 896$
- $959, 968, 977, 986, 995$

There are $1 + 2 + 3 + 4 + 5 = 15$ three-digit numbers with a digital root of 5 that can reached in exactly two steps with the initial sum of 23.

In total, there are $70 + 15 = 85$ three-digit numbers with a digital root of 5 that can be reached in exactly two steps.

Case 4: A digital root of 5 that is reached in three steps.

The maximum sum of the digits of a three-digit number is 27. The only number from 10 to 27 whose digits add to a two-digit number is 19. The digit sum of 19 is 10, whose digit sum is 1, not 5. Therefore, no three-digit number exists that has a digital root of 5 that is reached in three steps. It follows that all three-digit numbers with a digital root of 5 can be reached in at most two steps.

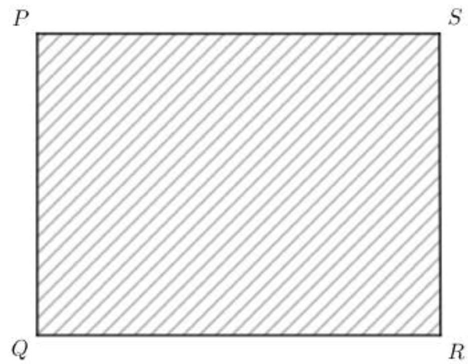
The above cases represent the only possible ways for a three-digit number to have a digital root of 5. In total there are $0 + 15 + 85 + 0 = 100$ three-digit numbers with a digital root of 5, all of which can be reached in two or fewer steps.

Problem of the Week

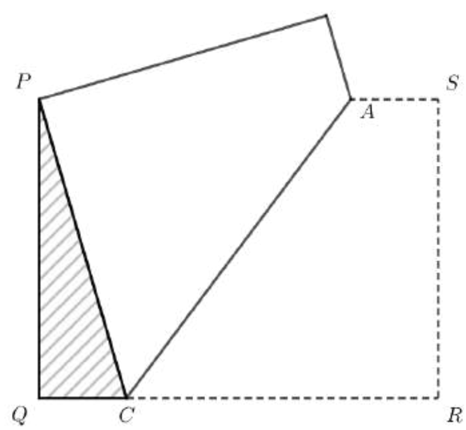
Problem E

Fold Once

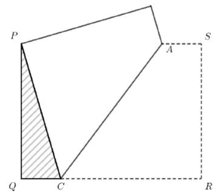
A rectangular piece of paper, $PQRS$, has $PQ = 30$ cm and $PS = 40$ cm. The paper has grey lines on one side and is plain white on the other.



The paper is folded so that the two diagonally opposite corners P and R coincide. This creates a crease along line segment AC, with A on PS and C on QR.



Determine the length of AC.



Problem of the Week
Problem E and Solution
Fold Once

Problem

A rectangular piece of paper, $PQRS$, has $PQ = 30$ cm and $PS = 40$ cm. The paper has grey lines on one side and is plain white on the other. The paper is folded so that the two diagonally opposite corners P and R coincide. This creates a crease along line segment AC, with A on PS and C on QR. Determine the length of AC.

Solution

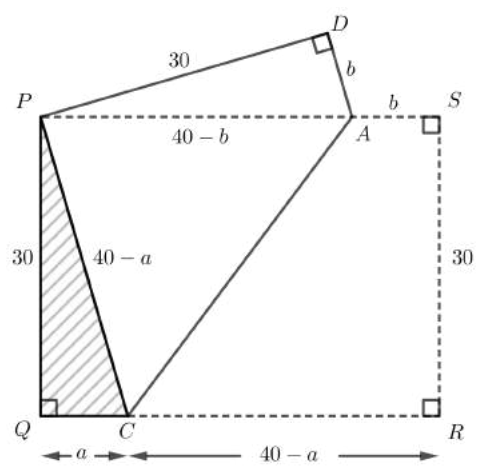
Since $PQRS$ is a rectangle, all angles inside $PQRS$ are $90°$. After the fold, R coincides with P. Label the point that S folds to as D. The angle at D is the same as the angle at S. Since $PQRS$ is a rectangle, $\angle PSR = 90°$ and $SR = 30$, and it follows that $\angle PDA = 90°$ and $PD = 30$.

Let a represent the length of QC and b represent the length of AS.
Then $CR = QR - QC = 40 - a$ and $PA = PS - AS = 40 - b$.

Since S folds to D, it follows that $AD = AS = b$.

Since R folds to P, it follows that $PC = CR = 40 - a$.

All of the information is recorded on the following diagram.



Since $\triangle PQC$ is a right-angled triangle, we can use the Pythagorean Theorem to find a.

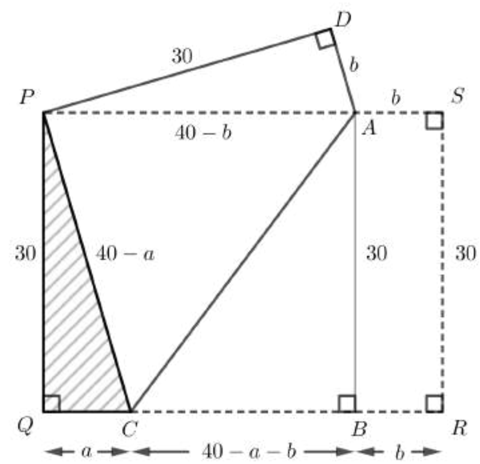
$$\begin{aligned} QC^2 + PQ^2 &= PC^2 \\ a^2 + 30^2 &= (40 - a)^2 \\ a^2 + 900 &= 1600 - 80a + a^2 \\ 80a &= 700 \\ a &= \frac{35}{4} \end{aligned}$$

Since $\triangle PDA$ is a right-angled triangle, we can use the Pythagorean Theorem to find b.

$$\begin{aligned} AD^2 + PD^2 &= PA^2 \\ b^2 + 30^2 &= (40-b)^2 \\ b^2 + 900 &= 1600 - 80b + b^2 \\ 80b &= 700 \\ b &= \frac{35}{4} \end{aligned}$$

Therefore, $a = b = \dfrac{35}{4}$ cm.

We still need to find the length of the crease. From A drop a perpendicular to QR intersecting QR at B. Then $ABRS$ is a rectangle. It follows that $AB = SR = 30$ and $BR = AS = b$.
Also, $CB = QR - QC - BR = 40 - a - b = 40 - \dfrac{35}{4} - \dfrac{35}{4} = \dfrac{90}{4} = \dfrac{45}{2}$ cm.



Using the Pythagorean Theorem in $\triangle CAB$,

$$\begin{aligned} AC^2 &= AB^2 + CB^2 \\ &= 30^2 + \left(\frac{45}{2}\right)^2 \\ &= 900 + \frac{2025}{4} \\ &= \frac{5625}{4} \end{aligned}$$

Since $AC > 0$, it follows that $AC = \dfrac{75}{2} = 37.5$ cm. Therefore, the length of the crease is $\dfrac{75}{2}$ cm.

Problem of the Week

Problem E

Hundred Deck 3

Hundred Deck is a deck consisting of 100 cards numbered from 1 to 100. Each card has the same number printed on both sides. One side of the card is red and the other side of the card is yellow.

Sarai places all of the cards on a table with each card's red side facing up.
She first flips over every card that has a number on it which is a multiple of 2.
She then flips over every card that has a number on it which is a multiple of 3.
Next, she flips over every card that has a number on it which is a multiple of 4.
Finally, she flips over every card that has a number on it which is a multiple of 5.

After Sarai has finished, how many cards have their red side facing up?

3

Problem of the Week
Problem E and Solution
Hundred Deck 3

Problem

Hundred Deck is a deck consisting of 100 cards numbered from 1 to 100. Each card has the same number printed on both sides. One side of the card is red and the other side of the card is yellow. Sarai places all of the cards on a table with each card's red side facing up.
She first flips over every card that has a number on it which is a multiple of 2. She then flips over every card that has a number on it which is a multiple of 3. Next, she flips over every card that has a number on it which is a multiple of 4. Finally, she flips over every card that has a number on it which is a multiple of 5. After Sarai has finished, how many cards have their red side facing up?

Solution

If a card is numbered with a multiple of $2, 3, 4$, and 5, it will be flipped four times. This card will flip from red facing up to yellow to red to yellow to red again. So the card will have its red side facing up once Sarai has finished.
If a card is numbered with a multiple of exactly three of $2, 3, 4$, and 5, it will be flipped three times. This card will flip from red facing up to yellow to red to yellow. So the card will have its yellow side facing up once Sarai has finished.
If a card is numbered with a multiple of exactly two of $2, 3, 4$, and 5, then it will be flipped twice. This card will flip from red facing up to yellow to red again. So the card will have its red side facing up once Sarai has finished.
If a card is numbered with a multiple of exactly one of $2, 3, 4$, and 5, it will be flipped once. This card will flip from red facing up to yellow. So the card will have its yellow side facing up once Sarai has finished.
If a card is numbered with a multiple of none of $2, 3, 4$, and 5, then this card will not be flipped and so the card will still have its red side facing up once Sarai has finished.
To determine how many cards have their red side facing up once Sarai has finished, let's determine how many cards have their yellow side facing up once Sarai has finished, and then subtract this number from the total number of cards. To determine how many cards have their yellow side facing up once Sarai has finished, we need to determine how many card numbers are multiples of exactly three of $2, 3, 4$, and 5 and how many card numbers are multiples of exactly one of $2, 3, 4$, and 5.

Let's consider the cases:

- Case 1: A card number is a multiple of $2, 3$, and 4, but not 5.
  If a card number is a multiple of $2, 3$, and 4, then it must be a multiple of 12, the lowest common multiple of $2, 3$, and 4. There are 8 cards with numbers which are multiples of 12. They are the cards numbered $12, 24, 36, 48, 60, 72, 84$, and 96. Only one of these numbers is also a multiple of 5, namely 60. Therefore, there are $8 - 1 = 7$ cards with numbers that are multiples of $2, 3$, and 4, but not 5.

- Case 2: A card number is a multiple of $2, 3$, and 5, but not 4.
  If a card number is a multiple of $2, 3$, and 5, then it must be a multiple of 30, the lowest common multiple of $2, 3$, and 5. There are 3 cards with numbers which are multiples of 30. They are the cards numbered $30, 60$, and 90. Only one of these numbers is also a multiple of 4, namely 60. Therefore, there are 2 cards with numbers that are multiples of $2, 3$, and 5, but not 4.

- Case 3: A card number is a multiple of $2, 4$, and 5, but not 3.
  If a card number is a multiple of $2, 4$, and 5, then it must be a multiple of 20, the lowest common multiple of $2, 4$, and 5. There are 5 cards with numbers which are multiples of 20. They are the cards numbered $20, 40, 60, 80$, and 100. Only one of these numbers is also a multiple of 3, namely 60. Therefore, there are 4 cards with numbers that are multiples of $2, 4$, and 5, but not 3.

- Case 4: A card number is a multiple of $3, 4$, and 5, but not 2.
  It is not possible for a card number to be a multiple of 4 but not 2. Therefore, there are no card numbers in this case.

- Case 5: A card number is a multiple of 2, but not $3, 4$, or 5.
  There are 50 cards with numbers which are multiples of 2, and 25 cards with numbers which are multiples of 4 (and thus 2). So there are $50 - 25 = 25$ cards with numbers that are multiples of 2 but not 4. They are the cards numbered
  $$2, 6, 10, 14, 18, 22, 26, 30, 34, 38, 42, 46, 50, 54, 58, 62, 66, 70, 74, 78, 82, 86, 90, 94, 98$$
  After removing numbers that are still a multiple of 3 or 5, we are left with
  $$2, 14, 22, 26, 34, 38, 46, 58, 62, 74, 82, 86, 94, 98$$
  So there are 14 cards with numbers that are multiples of 2, but not $3, 4$, or 5.

- Case 6: A card number is a multiple of 3, but not $2, 4$, or 5.
  There are 33 cards with numbers that multiples of 3. They are the cards numbered
  $$3, 6, 9, 12, 15, \ldots, 87, 90, 93, 96, 99$$
  Of these numbers, 17 are odd. So there are 17 cards with numbers that are multiples of 3 but not 2. They are the cards numbered
  $$3, 9, 15, 21, 27, 33, 39, 45, 51, 57, 63, 69, 75, 81, 87, 93, 99$$
  None of these numbers are a multiple of 4. After removing the multiples of 5, we are left with
  $$3, 9, 21, 27, 33, 39, 51, 57, 63, 69, 81, 87, 93, 99$$
  So there are 14 cards with numbers that are multiples of 3, but not $2, 4$, or 5.

- Case 7: A card number is a multiple of 4, but not $2, 3$, or 5.
  It is not possible for a card number to be a multiple of 4 but not 2. Therefore, there are no card numbers in this case.

- Case 8: A card number is a multiple of 5, but not $2, 3$, or 4.
  There are 20 cards with numbers which are multiples of 5, but half of those are multiples of 2. The card numbers that are multiples of 5 but not 2 are
  $$5, 15, 25, 35, 45, 55, 65, 75, 85, 95$$
  We still need to remove numbers that are multiples of 3. After doing so we are left with
  $$5, 25, 35, 55, 65, 85, 95$$
  So there are 7 cards with numbers that are multiples of 5 but not $2, 3$, or 4.

Thus, once she has finished, Sarai is left with $7 + 2 + 4 + 0 + 14 + 14 + 0 + 7 = 48$ cards with their yellow side facing up. Therefore, Sarai is left with $100 - 48 = 52$ cards with their red side facing up.

Extension:
Suppose Sarai continues flipping cards in this manner. That is, after she has flipped all cards whose number is a multiple of 5, she then flips all cards whose card number is a multiple of 6, then 7, then 8, and so on until she flips all cards whose number is a multiple of 100. Once Sarai has finished, how many cards will have their red side facing up?

Problem of the Week

Problem E

Mixture of Three

Kanza is making and selling trail mix. She makes three different blends, each consisting of a mixture of cashews, dark chocolate, and almonds. All of these blends are sold at the same price of \$18 per kg.

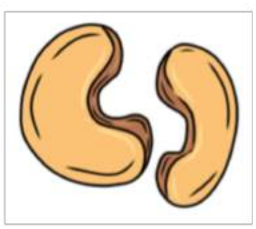
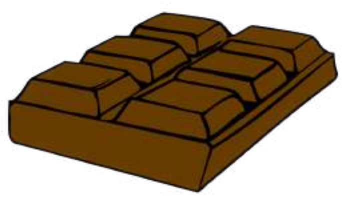
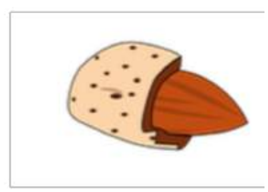
If she mixes cashews, dark chocolate, and almonds in the ratio of $1:1:1$, by mass, then she makes a profit of 20%.

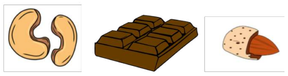
If she mixes cashews, dark chocolate, and almonds in the ratio of $3:2:1$, by mass, then she makes a profit of 8%.

If she mixes cashews, dark chocolate, and almonds in the ratio of $1:4:2$, by mass, then she makes a profit of 26%.

(a) What price, in dollars per kg, does Kanza pay for each of the cashews, dark chocolate, and almonds?

(b) What percentage of a profit would she make if she mixes cashews, dark chocolate, and almonds in the ratio of $2:3:4$, by mass?



Problem of the Week
Problem E and Solution
Mixture of Three

Problem

Kanza is making and selling trail mix. She makes three different blends, each consisting of a mixture of cashews, dark chocolate, and almonds. All of these blends are sold at the same price of \$18 per kg. If she mixes cashews, dark chocolate, and almonds in the ratio of 1 : 1 : 1, by mass, then she makes a profit of 20%. If she mixes cashews, dark chocolate, and almonds in the ratio of 3 : 2 : 1, by mass, then she makes a profit of 8%. If she mixes cashews, dark chocolate, and almonds in the ratio of 1 : 4 : 2, by mass, then she makes a profit of 26%.

(a) What price, in dollars per kg, does Kanza pay for each of the cashews, dark chocolate, and almonds?

(b) What percentage of a profit would she make if she mixes cashews, dark chocolate, and almonds in the ratio of 2 : 3 : 4, by mass?

Solution

Let c be the price Kanza pays for cashews, in dollars per kg.
Let d be the price Kanza pays for dark chocolate, in dollars per kg.
Let a be the price Kanza pays for almonds, in dollars per kg.

Consider the blend where she mixes cashews, dark chocolate, and almonds in the ratio of 1 : 1 : 1, by mass. In 1 kg of this blend, $\frac{1}{3}$ kg is cashews, $\frac{1}{3}$ kg is dark chocolate, and $\frac{1}{3}$ kg is almonds. Also, 1 kg of this blend will cost Kanza $\frac{1}{3}c + \frac{1}{3}d + \frac{1}{3}a$ and will be sold for \$18. Since she makes a profit of 20%, we have

$$1.2\left(\frac{1}{3}c + \frac{1}{3}d + \frac{1}{3}a\right) = 18$$

Multiplying by 3, we obtain

$$1.2(c + d + a) = 54$$

Dividing by 1.2, we obtain

$$c + d + a = 45 \qquad (1)$$

Consider the blend where she mixes cashews, dark chocolate, and almonds in the ratio of 3 : 2 : 1, by mass. In 1 kg of this blend, $\frac{1}{2}$ kg is cashews, $\frac{1}{3}$ kg is dark chocolate, and $\frac{1}{6}$ kg is almonds. Also, 1 kg of this blend will cost Kanza $\frac{1}{2}c + \frac{1}{3}d + \frac{1}{6}a$ and will be sold for \$18. Since she makes a profit of 8%, we have

$$1.08\left(\frac{1}{2}c + \frac{1}{3}d + \frac{1}{6}a\right) = 18$$

Multiplying by 6, we obtain

$$1.08(3c + 2d + a) = 108$$

Dividing by 1.08, we obtain

$$3c + 2d + a = 100 \qquad (2)$$

Consider the blend where she mixes cashews, dark chocolate, and almonds in the ratio of $1:4:2$, by mass. In 1 kg of this blend, $\frac{1}{7}$ kg is cashews, $\frac{4}{7}$ kg is dark chocolate, and $\frac{2}{7}$ kg is almonds. Also, 1 kg of this blend will cost Kanza $\frac{1}{7}c + \frac{4}{7}d + \frac{2}{7}a$ and will be sold for \$18. Since she makes a profit of 26%, we have

$$1.26\left(\frac{1}{7}c + \frac{4}{7}d + \frac{2}{7}a\right) = 18$$

Multiplying by 7, we obtain

$$1.26(c + 4d + 2a) = 126$$

Dividing by 1.26, we obtain

$$c + 4d + 2a = 100 \tag{3}$$

We now need to solve the following system of equations.

$$c + d + a = 45 \tag{1}$$
$$3c + 2d + a = 100 \tag{2}$$
$$c + 4d + 2a = 100 \tag{3}$$

First, subtracting equation (1) from equation (2) gives

$$2c + d = 55 \tag{4}$$

Second, doubling equation (1) and then subtracting equation (3) gives

$$c - 2d = -10 \tag{5}$$

We will now use equations (4) and (5) to solve for c and d. Doubling equation (4) and then adding equation (5) gives

$$5c = 100$$

or

$$c = 20$$

Substituting $c = 20$ into equation (4), we get $2(20) + d = 55$ or $d = 15$.

Substituting $c = 20$ and $d = 15$ into equation (1), we find $a = 10$.

(a) Therefore, Kanza pays \$20 per kg for cashews, \$15 per kg for dark chocolate, and \$10 per kg for almonds.

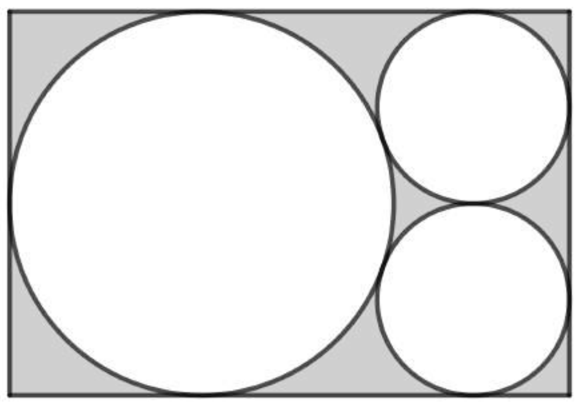
(b) Suppose she mixes cashews, dark chocolate, and almonds in the ratio of $2:3:4$, by mass. Then, in 1 kg of the blend, $\frac{2}{9}$ kg is cashews, $\frac{1}{3}$ kg is dark chocolate, and $\frac{4}{9}$ kg is almonds.

Also, 1 kg of this blend will cost Kanza $\frac{2}{9}c + \frac{1}{3}d + \frac{4}{9}a = \frac{2}{9}(20) + \frac{1}{3}(15) + \frac{4}{9}(10) = \frac{125}{9}$ dollars.

She sells 1 kg of this blend for \$18. Since $18 \div \frac{125}{9} = 1.296$, the percentage profit that Kanza makes on this mixture is 29.6%.

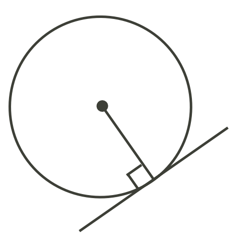
Problem of the Week
Problem E
Three Circles

Three circles are contained in a rectangle. Two of the circles have a radius of 1 and one of the circles has a radius of 2. The larger circle is tangent to three sides of the rectangle. The two smaller circles are each tangent to the larger circle and tangent to each other, and they are also each tangent to two sides of the rectangle. Determine the area of the rectangle that is not covered by the circles.

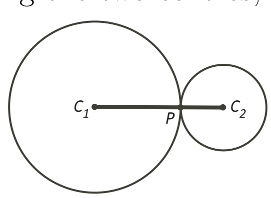


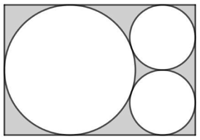
Note: For this problem, the following known results about circles may be useful:

- If a line is tangent to a circle, then the line is perpendicular to the radius drawn to the point of tangency.



- If two circles are tangent to each other at point P, then a line segment through the point of tangency can be drawn connecting the two centres, C_1 and C_2.





Problem of the Week
Problem E and Solution
Three Circles

Problem

Three circles are contained in a rectangle. Two of the circles have a radius of 1 and one of the circles has a radius of 2. The larger circle is tangent to three sides of the rectangle. The two smaller circles are each tangent to the larger circle and tangent to each other, and they are also each tangent to two sides of the rectangle. Determine the area of the rectangle that is not covered by the circles.

Solution

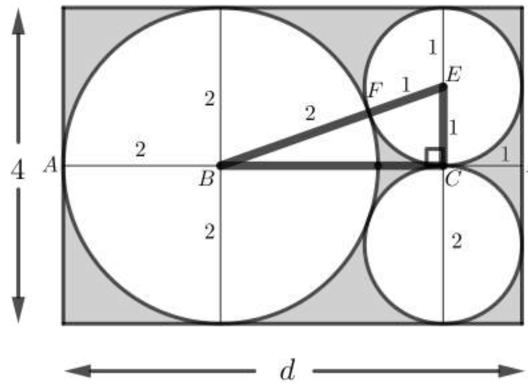
Let the centre of the larger circle be B and the centre of one of the smaller circles be E. Let C be the point of tangency of the two smaller circles. Let F be the point of tangency of the larger circle and the smaller circle with centre E. Position line segment AD so that it is parallel to the longer side of the rectangle, such that A and D are midpoints of the shorter sides of the rectangle and A lies on the larger circle. AD will pass through B and C.

Since the larger circle is tangent to two opposite sides of the rectangle and its radius is 2 m, then the length of the shorter side of the rectangle is equal to the diameter of the larger circle, or 4 m.

Let the length of the longer side of the rectangle be d. This is also the length of AD. We know that $AB = 2$ m, the length of a radius of the larger circle, and $CD = 1$ m, the same length as a radius of a smaller circle. We need to find the length of BC.

Since AD is tangent to the smaller circles at C, we know that EC is a radius of one of the smaller circles and is perpendicular to AD at C. Since the circles with centres B and E are tangent at F, EFB is a straight line segment and $EB = EF + FB = 1 + 2 = 3$ m.

Combining this information, $\triangle ECB$ is right-angled at C.



Using the Pythagorean Theorem in $\triangle ECB$, we have
$BC^2 = EB^2 - EC^2 = 3^2 - 1^2 = 8$. Thus, $BC = \sqrt{8}$, since $BC > 0$.

Therefore, the length of the longer side of the rectangle is

$$\begin{aligned} d &= AB + BC + CD \\ &= 2 + \sqrt{8} + 1 \\ &= 3 + \sqrt{8} \end{aligned}$$

To find the area not covered by the circles, we find the area of the rectangle and subtract the areas of the three circles.

$$\begin{aligned} \text{Shaded Area} &= \text{Area of Rectangle} - \text{Area of larger circle} - \text{Area of two smaller circles} \\ &= 4 \times (3 + \sqrt{8}) - \pi \times 2^2 - 2 \times (\pi \times 1^2) \\ &= 12 + 4\sqrt{8} - 4\pi - 2\pi \\ &= 12 + 4\sqrt{8} - 6\pi \end{aligned}$$

Students who have learned to simplify radicals will know that
$\sqrt{8} = \sqrt{4}\sqrt{2} = 2\sqrt{2}$. Therefore, the shaded area can then be written
$12 + 4 \times 2\sqrt{2} - 6\pi = 12 + 8\sqrt{2} - 6\pi$.

Therefore, the area of the rectangle not covered by the circles is
$(12 + 4\sqrt{8} - 6\pi)$ m^2 or $(12 + 8\sqrt{2} - 6\pi)$ m^2, which is approximately 4.5 m^2.

Problem of the Week

Problem E

Just an Average Sum

Faisal chooses four numbers. When each number is added to the mean (average) of the other three, the following sums are obtained: 25, 37, 43, and 51.

Determine the mean of the four numbers Faisal chose.

Extra Problem: Can you interpret the following picture puzzle? You may need to research the meanings of some mathematical symbols used in the puzzle.

$$B > \frac{1}{n}\sum_{i=1}^{n} x_i$$

$$B > \frac{1}{n}\sum_{i=1}^{n} x_i$$

Problem of the Week

Problem E and Solution

Just an Average Sum

Problem

Faisal chooses four numbers. When each number is added to the mean (average) of the other three, the following sums are obtained: 25, 37, 43, and 51. Determine the mean of the four numbers Faisal chose.

Extra Problem: Can you interpret the picture puzzle above? You may need to research the meanings of some mathematical symbols used in the puzzle.

Solution

Let a, b, c, and d represent the four numbers. It is possible to precisely determine the four numbers, but the problem asks for only their average, which is $\frac{a+b+c+d}{4}$.

When the first number is added to the average of the other three numbers, the result is 25. Thus,

$$a + \frac{b+c+d}{3} = 25$$

which can be rewritten as

$$3a + b + c + d = 75 \qquad (1)$$

When the second number is added to the average of the other three numbers, the result is 37. Thus,

$$b + \frac{a+c+d}{3} = 37$$

which can be rewritten as

$$a + 3b + c + d = 111 \qquad (2)$$

When the third number is added to the average of the other three numbers, the result is 43. Thus,

$$c + \frac{a+b+d}{3} = 43$$

which can be rewritten as

$$a + b + 3c + d = 129 \qquad (3)$$

When the fourth number is added to the average of the other three numbers, the result is 51. Thus,

$$d + \frac{a+b+c}{3} = 51$$

which can be rewritten as

$$a + b + c + 3d = 153 \qquad (4)$$

Adding equations (1), (2), (3), and (4), we obtain $6a + 6b + 6c + 6d = 468$. Dividing this equation by 6 gives $a + b + c + d = 78$. It follows that $\frac{a+b+c+d}{4} = 19.5$.

Therefore, the average of the four numbers is 19.5.

Although it is not required, we could solve the system of equations to determine that the numbers are: -1.5, 16.5, 25.5, and 37.5.

Extra Problem Solution:

The notation $\frac{1}{n}\sum_{i=1}^{n} x_i$ is a mathematical short form which represents the average of the n numbers $x_1, x_2, \ldots, x_n$. So the picture puzzle can be interpreted as "Be greater than average".

Problem of the Week
Problem E
Picture This

Eight people, Alex, Braiden, Christine, Gary, Henry, Mary, Sam, and Zachary are lining up in a row for a picture. Due to the dynamics of the people involved, there are certain restrictions in the way the people will line up. Anyone with a name that ends in ‘y’ will not stand next to anyone else with a name that ends in ‘y’. (Notice that four names end in a ‘y’: Gary, Henry, Mary, and Zachary.) Also the twins, Alex and Gary, will not stand beside each other.

If the photographer randomly organizes the people, what is the probability that she arranges the people in a valid order?

The diagram below illustrates one possible valid arrangement of the people.

Problem of the Week
Problem E and Solution
Picture This

Problem

Eight people, Alex, Braiden, Christine, Gary, Henry, Mary, Sam, and Zachary are lining up in a row for a picture. Due to the dynamics of the people involved, there are certain restrictions in the way the people will line up. Anyone with a name that ends in 'y' will not stand next to anyone else with a name that ends in 'y'. (Notice that four names end in a 'y': Gary, Henry, Mary, and Zachary.) Also the twins, Alex and Gary, will not stand beside each other.

If the photographer randomly organizes the people, what is the probability that she arranges the people in a valid order?

Solution

Four of the eight people have a name that ends in 'y', and these people will not stand next to each other. The problem is further complicated by the fact that Gary and Alex cannot stand together. Since Gary also has a name that ends in 'y', we will break the problem into cases, based on Gary's position. We will number the positions from 1 to 8, starting on the left.

Let Gary's position in the line be marked with a G. Let the positions of the other people with names that end in 'y' be marked with a Y. Positions that have not yet been filled with a person will be marked with a _.

1. **Gary is in position 1.**

   We place Gary in position 1 and systematically list out the possible placements of the three other people with names that end in 'y', and see that there are 4 possible configurations:

   $$G \_ Y \_ Y \_ Y \_$$
   $$G \_ Y \_ Y \_ \_ Y$$
   $$G \_ Y \_ \_ Y \_ Y$$
   $$G \_ \_ Y \_ Y \_ Y$$

   For each of these 4 configurations, there are 3 possible people to fill in the empty spot immediately next to Gary (these people do not have a name that ends in 'y', nor are they Alex). For each of these, there are $3 \times 2 \times 1$ ways to place the other people with a name that ends in 'y' and $3 \times 2 \times 1$ ways to place the remaining people, including Alex. Therefore, there are $4 \times 3 \times (3 \times 2 \times 1) \times (3 \times 2 \times 1) = 432$ ways to place the people properly with Gary in position 1.

2. **Gary is in position 8.**

   The possible placements of Gary and the three other people with names that end in 'y' will be the reverse of the configurations above when Gary is in position 1. Using a similar argument, we see that there are 432 ways to place the people properly with Gary in position 8.

3. **Gary is in position 2.**

   We place Gary in position 2 and systematically list out the possible placements of the three other people with names that end in 'y', and see that there is only 1 possible configuration:

   $$\_\ G\ \_\ Y\ \_\ Y\ \_\ Y$$

   For this configuration, there are 3 possible people to fill in the empty spot to the left of Gary (these people do not have a name that ends in 'y', nor are they Alex). Once that person has been placed, there are 2 possible people to put in the empty spot immediately to the right of Gary. For each of these arrangements, there are $3 \times 2 \times 1$ ways to place the other people with a name that ends in 'y' and 2×1 ways to place the remaining people, including Alex.

   Therefore, there are $1 \times (3 \times 2) \times (3 \times 2 \times 1) \times (2 \times 1) = 72$ ways to place the people properly with Gary in position 2.

4. **Gary is in position 7.**

   The possible placements of Gary and the three other people with names that end in 'y' will be the reverse of the configurations above when Gary is in position 2. Using a similar argument, we see that there are 72 ways to place the people properly with Gary in position 7.

5. **Gary is in position 3.**

   We place Gary in position 3 and systematically list out the possible placements of the three other people with names that end in 'y', and see that there are only 3 possible configurations:

   $$Y\ \_\ G\ \_\ \_\ Y\ \_\ Y$$
   $$Y\ \_\ G\ \_\ Y\ \_\ Y\ \_$$
   $$Y\ \_\ G\ \_\ Y\ \_\ \_\ Y$$

   For each of these 3 configurations, there are 3 possible people to fill in the empty spot to the left of Gary (these people do not have a name that ends in 'y', nor are they Alex). Once that person has been placed, there are 2 possible people to put in the empty spot immediately to the right of Gary. For each of these arrangements, there are $3 \times 2 \times 1$ ways to place the other people with a name that ends in 'y' and 2×1 ways to place the remaining people, including Alex.

   Therefore, there are $3 \times (3 \times 2) \times (3 \times 2 \times 1) \times (2 \times 1) = 216$ ways to place the people properly with Gary in position 3.

6. **Gary is in position 6.**

   The possible placements of Gary and the three other people with names that end in 'y' will be the reverse of the configurations above when Gary is in position 3. Using a similar argument, we see that there are 216 ways to place the people properly with Gary in position 6.

7. **Gary is in position 4.**

   We place Gary in position 4 and systematically list out the possible placements of the three other people with names that end in 'y', and see that there are only 2 possible configurations:

$$\_\,Y\,\_\,G\,\_\,Y\,\_\,Y$$
$$Y\,\_\,\_\,G\,\_\,Y\,\_\,Y$$

   Using an analysis similar to that in previous cases, we see that there are $2 \times (3 \times 2) \times (3 \times 2 \times 1) \times (2 \times 1) = 144$ ways to place the people properly with Gary in position 4.

8. **Gary is in position 5.**

   The possible placements of Gary and the three other people with names that end in 'y' will be the reverse of the configurations above when Gary is in position 4. Therefore, there are 144 ways to place the people properly with Gary in position 5.

The cases have no overlapping possibilities and we have considered all of the possible placements of Gary. Therefore, there are

$$432 + 432 + 72 + 72 + 216 + 216 + 144 + 144 = 1728$$

ways for the people to line up correctly.

If the people could stand in any position in the line, the number of possible ways to line up is

$$8 \times 7 \times 6 \times \times 5 \times 4 \times 3 \times 2 \times 1 = 40\,320$$

Therefore, the probability that the photographer randomly lines up the people in a valid order is

$$\frac{1728}{40\,320} = \frac{3}{70} \approx 0.043$$

Therefore, there is approximately a 4.3% chance of the photographer arranges the people in a valid order.

Problem of the Week
Problem E
Ducks in a Row

There are 10 rubber ducks arranged in a row on a desk. On the bottom of each duck there is one of the following letters: A, B, C, D, E, F, G, H, I, or J. Each letter occurs exactly once.

Amina has written the following algorithm to find a particular letter.

Pick up the leftmost duck and look on the bottom. If it's the letter you're looking for, put the duck back down and stop. Otherwise pick up the next duck and look on the bottom. If it's the letter you're looking for, put the duck back down, swap it with the duck to its left, and stop. Repeat this until you find the letter you're looking for.

For example, suppose the ducks were in the following order.

G, B, F, J, A, I, E, H, C, D

Amina uses her algorithm to find the letter E. She looks at the first 6 ducks and returns each of them to the desk. When she looks at the seventh duck and sees that it is the E, she swaps ducks E and I. So to locate the letter E, Amina looked at 7 ducks, and the ducks would now be in the following order.

G, B, F, J, A, E, I, H, C, D

Suppose Amina now wants to find the letter J. She looks at the G, B, and F and puts them back down. She then looks at the fourth duck, sees the J, and swaps ducks F and J. The ducks would now be in the following order.

G, B, J, F, A, E, I, H, C, D

After searching for the E and the J, Amina has looked at at a total of $7 + 4 = 11$ ducks.

If the 10 ducks begin in some unknown order and Amina uses her algorithm to search for each of the ten letters exactly once, what is the maximum possible number of ducks that Amina picks up to look on the bottom?

Problem of the Week
Problem E and Solution
Ducks in a Row

Problem

There are 10 rubber ducks arranged in a row on a desk. On the bottom of each duck there is one of the following letters: A, B, C, D, E, F, G, H, I, or J. Each letter occurs exactly once. Amina has written the following algorithm to find a particular letter.

Pick up the leftmost duck and look on the bottom. If it's the letter you're looking for, put the duck back down and stop. Otherwise pick up the next duck and look on the bottom. If it's the letter you're looking for, put the duck back down, swap it with the duck to its left, and stop. Repeat this until you find the letter you're looking for.

If the 10 ducks begin in some unknown order and Amina uses her algorithm to search for each of the ten letters exactly once, what is the maximum possible number of ducks that Amina picks up to look on the bottom?

Solution

If no swaps were required as a result of finding a duck, how many ducks would Amina have to look at in total?

Let's number the positions 1 through 10, starting with the leftmost position as number 1. At some point Amina is looking for the duck in position 1. She would have to look at 1 duck to find it. At some point she is looking for the duck in position 2. She would have to look at 2 ducks to find it. At some point she is looking for the duck in position 3. She would have to look at 3 ducks to find it. This continues until at some point she is looking for the duck in position 10. She would have to look at 10 ducks to find it. To locate all 10 ducks, Amina would have to look at $1 + 2 + 3 + \cdots + 10 = 55$ ducks.

Since Amina looks for each letter exactly once, swapping the position of one letter with the position of another letter can only have the effect of increasing the number of ducks looked at for the letter on the preceding duck by one. The number of ducks looked at to find other letters would not be affected. Therefore, swapping can only increase the number of ducks looked at (by one) for all but the first search. This means swapping can increase the number of ducks looked at by at most 9 in total making the maximum total number of ducks looked at equal to $55 + 9 = 64$.

On the next page, an illustration of how this maximum can be achieved is shown.

Is 64 an achievable maximum?

First we will put the ducks in order, left to right, from A to J.

A, B, C, D, E, F, G, H, I, J,

Now we will search for each letter in order from B to J and search for A last.

Since B is in the second position, we must look at 2 ducks to find it. We then swap A and B.

B, A, C, D, E, F, G, H, I, J

Since C is in the third position, we must look at 3 ducks to find it. We then swap A and C.

B, C, A, D, E, F, G, H, I, J

Since D is in the fourth position, we must look at 4 ducks to find it. We then swap A and D.

B, C, D, A, E, F, G, H, I, J

Since E is in the fifth position, we must look at 5 ducks to find it. We then swap A and E.

B, C, D, E, A, F, G, H, I, J

Since F is in the sixth position, we must look at 6 ducks to find it. We then swap A and F.

B, C, D, E, F, A, G, H, I, J

Since G is in the seventh position, we must look at 7 ducks to find it. We then swap A and G.

B, C, D, E, F, G, A, H, I, J

Since H is in the eighth position, we must look at 8 ducks to find it. We then swap A and H.

B, C, D, E, F, G, H, A, I, J

Since I is in the ninth position, we must look at 9 ducks to find it. We then swap A and I.

B, C, D, E, F, G, H, I, A, J

Since J is in the tenth position, we must look at 10 ducks to find it. We then swap A and J.

B, C, D, E, F, G, H, I, J, A

Finally, since A is in the tenth position, we must look at 10 ducks to find it. We then swap A and J (again).

B, C, D, E, F, G, H, I, A, J

We have looked at a total of $2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 10 = 64$ ducks to find each of the letters.

On the next page, we look at a possible extension and a connection to Computer Science.

Extension:

Suppose you have n ducks, each with something different on them. You lay the ducks out in a similar manner to how we handled the 10 different ducks. You search for each of the different ducks, one at a time. What is the maximum number of ducks you must look at in order to locate all of the ducks using the search described in the problem?

Connection to Computer Science:

This problem was inspired by a problem from the Beaver Computer Challenge in 2012. You can see the problem and others like it at http://www.cemc.uwaterloo.ca/contests/bcc.html.

One of the fundamental problems in computer science is how to organize data in order to search within it quickly. There are many ways to do this: using binary trees, splay trees, skip lists, sorted arrays, etc. The technique outlined in this problem is the idea of moving found items closer to the "front," with the assumption that if we search for something once, it is quite likely that the same item will be searched for again. The transpose (swap) heuristic used by Amina in this problem is one technique for doing this. Other heuristics include move-to-front, which moves a found element to the very front of the list. Moreover, this problem highlights the process of performing worst-case analysis for an algorithm. Computer scientists care about "what is the worst possible input for this algorithm, and how long will it take to execute on that input?" In this question, we are asking about the worst-case performance of the transpose heuristic on a list of size 10.

Problem of the Week
Problem E
Sum View

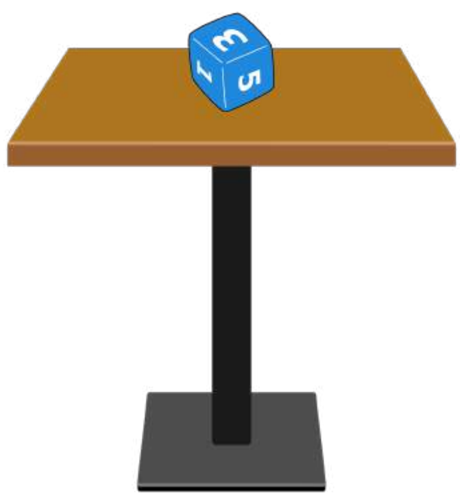
The six faces of a cube have each been marked with one of the numbers 1, 2, 3, 4, 5, and 6, with each number being used exactly once.

Three people, Paul, Lee, and Jenny, are seated around a square table.

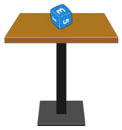
The cube is placed on the table so that from their different seat locations, each one can see the top face and two adjacent side faces. No two people see the same pair of adjacent side faces.

When Paul adds the three numbers that he can see, his total is 9. When Lee adds the three numbers that they can see, their total is 14. When Jenny adds the three numbers that she can see, her total is 15.

Determine all possibilities for the number on the bottom face of the cube.



Note that the three faces that are visible on the above cube add to 9. The picture is for illustration only. Do not assume anything from the above diagram.



Problem of the Week
Problem E and Solution
Sum View

Problem

The six faces of a cube have each been marked with one of the numbers 1, 2, 3, 4, 5, and 6, with each number being used exactly once.

Three people, Paul, Lee, and Jenny, are seated around a square table.

The cube is placed on the table so that from their different seat locations, each one can see the top face and two adjacent side faces. No two people see the same pair of adjacent side faces.

When Paul adds the three numbers that he can see, his total is 9. When Lee adds the three numbers that they can see, their total is 14. When Jenny adds the three numbers that she can see, her total is 15.

Determine all possibilities for the number on the bottom face of the cube.

Solution

We will look at the six possible cases for the number on the top face of the cube, and then decide if there are possible solutions given that top number.

Let t be the number on the top face of the cube.

- Case 1: $t = 1$

  The two side faces that Jenny sees must sum to $15 - 1 = 14$. Since the largest face value is 6, the maximum sum of the numbers on two faces is $5 + 6 = 11$. Therefore, it is not possible for the numbers on two side faces to sum to 14, and so the number on the top face cannot be 1.

- Case 2: $t = 2$

  The two side faces that Jenny sees must sum to $15 - 2 = 13$. Again, the maximum sum of the numbers on two faces is $5 + 6 = 11$. Therefore, it is not possible for the numbers on two side faces to sum to 13, and so the number on the top face cannot be 2.

- Case 3: $t = 3$

  The two side faces that Jenny sees must sum to $15 - 3 = 12$. Again, the maximum sum of the numbers on two faces is $5 + 6 = 11$. Therefore, it is not possible for the numbers on two side faces to sum to 12, and so the number on the top face cannot be 3.

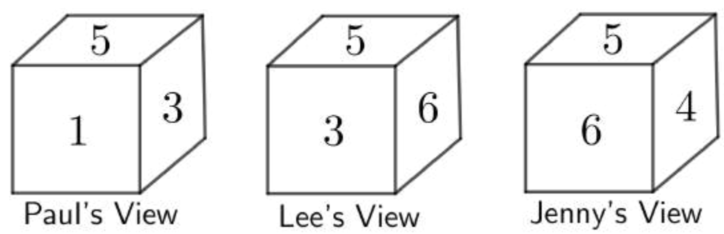
- Case 4: $t = 4$

  The two side faces that Jenny sees must sum to $15 - 4 = 11$. It is possible to get a sum of 11, and the only possible way is with the numbers on the two faces being 5 and 6. Now, the two side faces that Lee sees must sum to $14 - 4 = 10$. The only possible way to get a sum of 10 is with the numbers on the two faces being 4 and 6. But the number on the top face is also 4, so it is not possible to have a 4 on a side face.
  Therefore, the number on the top face cannot be 4.

- Case 5: $t = 5$

  The two side faces that Jenny sees must sum to $15 - 5 = 10$. The only possible way to get a sum of 10 is with the two faces being 4 and 6.
  Now, the two side faces that Lee sees must sum to $14 - 5 = 9$. It is possible to get a sum of 9, and the only possible ways are with the numbers on the two faces being 4 and 5 or 3 and 6. Since the number on the top face is 5, then the numbers on the two side faces cannot be 4 and 5. Therefore, the numbers on two side faces that Lee sees must be 3 and 6.
  Finally, the two side faces that Paul sees must sum to $9 - 5 = 4$. It is possible to get a sum of 4, and the only possible way is with the numbers on the faces being 1 and 3.

  Therefore, it is possible for the number on the top face to be 5. Then the numbers on the four side faces must be 4, 6, 3, and 1. Each person's view is shown below.

  The visible numbers are 1, 3, 4, 5, and 6. Therefore, the number on the bottom of the cube is 2.

- Case 6: $t = 6$

  The two side faces that Jenny sees must sum to $15 - 6 = 9$. It is possible to get a sum of 9, and the only possible ways are with the numbers on the two faces being 3 and 6, or 4 and 5. Since the number on the top face is 6, then the numbers on the two side faces cannot be 3 and 6. Therefore, the numbers on the two side faces that Jenny sees must be 4 and 5.

  Now, the two side faces that Lee sees must sum to $14 - 6 = 8$. It is possible to get a sum of 8, and the only possible ways are with the numbers on the two faces being 2 and 6 or 3 and 5. Since the number on the top face is 6, then the numbers on the two side faces cannot be 2 and 6. Therefore, the numbers on the two side faces that Lee sees must be 3 and 5.

  Finally, the two side faces that Paul sees must sum to $9 - 6 = 3$. The only possible way to get a sum of 3 is with the faces being 1 and 2.

  From Jenny's, Lee's and Paul's views, we know that the side faces must have numbers 4, 5, 3, 1, and 2. But there are only four side faces, so it is impossible for each of these five numbers to appear on a side face.

  Therefore, it is not possible for the number on the top face of the cube to be 6.

Therefore, the only possible number on the bottom face of the cube is 2.

Problem of the Week
Problem E
Pool Season

Philip plans to use small and large hoses to fill a swimming pool. They know that it takes four hours for nine large hoses to fill the pool. They also know that it takes eight hours for six small hoses to fill the pool.

How long will it take to fill the pool if Philip uses four large hoses and eight small hoses?

Problem of the Week

Problem E and Solution

Pool Season

Problem

Philip plans to use small and large hoses to fill a swimming pool. They know that it takes four hours for nine large hoses to fill the pool. They also know that it takes eight hours for six small hoses to fill the pool.

How long will it take to fill the pool if Philip uses four large hoses and eight small hoses?

Solution

Solution 1

We know 9 large hoses can fill 1 swimming pool in 4 hours.
Therefore, 9 large hoses can fill $\frac{1}{4}$ of the swimming pool in 1 hour.
Therefore, 1 large hose can fill $\frac{1}{9} \times \frac{1}{4} = \frac{1}{36}$ of the swimming pool in 1 hour.

We know 6 small hoses can fill 1 swimming pool in 8 hours.
Therefore, 6 small hoses can fill $\frac{1}{8}$ of the swimming pool in 1 hour.
Therefore, 1 small hose can fill $\frac{1}{6} \times \frac{1}{8} = \frac{1}{48}$ of the swimming pool in 1 hour.

Thus, the fraction of the pool that is filled in one hour using 4 large hoses and 8 small hoses is

$$4\left(\frac{1}{36}\right) + 8\left(\frac{1}{48}\right) = \frac{1}{9} + \frac{1}{6} = \frac{2}{18} + \frac{3}{18} = \frac{5}{18}$$

Since the pool is $\frac{5}{18}$ full in 1 hour, it will be completely full in $\frac{18}{5} = 3\frac{3}{5}$ hours or 3 hours and 36 minutes.

Solution 2

We know 9 large hoses can fill 1 swimming pool in 4 hours.
Therefore, 1 large hose can fill $\frac{1}{9}$ of the swimming pool in 4 hours.
Therefore, 4 large hoses can fill $\frac{1}{9}$ of the swimming pool in 1 hour.

We know 6 small hoses can fill 1 swimming pool in 8 hours.
Therefore, 1 small hose can fill $\frac{1}{6}$ of the swimming pool in 8 hours.
Therefore, 8 small hoses can fill $\frac{1}{6}$ of the swimming pool in 1 hour.

Thus, the fraction of the pool that is filled in one hour using 4 large hoses and 8 small hoses is

$$\frac{1}{9} + \frac{1}{6} = \frac{2}{18} + \frac{3}{18} = \frac{5}{18}$$

Since the pool is $\frac{5}{18}$ full in 1 hour, it will be completely full in $\frac{18}{5} = 3\frac{3}{5}$ hours or 3 hours and 36 minutes.

Problem of the Week

Problem E

Odd Sum

A sequence consists of 2022 terms. Each term after the first term is 1 greater than the previous term. The sum of the 2022 terms is 31 341.

Determine the sum of the terms in the odd-numbered positions. That is, determine the sum of every second term starting with the first term and ending with the second last term.

NOTE:
In solving the above problem, it may be helpful to use the fact that the sum of the first n positive integers is equal to $\frac{n(n+1)}{2}$. That is,

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

Problem of the Week
Problem E and Solution
Odd Sum

Problem
A sequence consists of 2022 terms. Each term after the first term is 1 greater than the previous term. The sum of the 2022 terms is 31 341.

Determine the sum of the terms in the odd-numbered positions. That is, determine the sum of every second term starting with the first term and ending with the second last term.

NOTE:
In solving the above problem, it may be helpful to use the fact that the sum of the first n positive integers is equal to $\frac{n(n+1)}{2}$. That is,

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

Solution
Solution 1

Let t_n denote the n^{th} term in the sequence.

Let S_O represent the sum of the terms in the odd-numbered positions. That is,

$$S_O = t_1 + t_3 + t_5 + \cdots + t_{2021}$$

Let S_E represent the sum of the terms in the even-numbered positions. That is,

$$S_E = t_2 + t_4 + t_6 + \cdots + t_{2022}$$

Since there are 2022 terms, and half of the terms of the sequence are in even-numbered positions and half are in odd-numbered positions, there are 1011 terms in S_O and 1011 terms in S_E.

Let S represent the sum of the 2022 terms. That is,

$$S = S_O + S_E = 31\,341 \tag{1}$$

Since each term after the first term is 1 greater than the term before,

$$t_2 = t_1 + 1$$
$$t_4 = t_3 + 1$$
$$t_6 = t_5 + 1$$

and so on, until

$$t_{2022} = t_{2021} + 1$$

Now,

$$\begin{aligned} S_E &= t_2 + t_4 + t_6 + \cdots + t_{2016} + t_{2022} \\ &= (t_1 + 1) + (t_3 + 1) + (t_5 + 1) + \cdots + (t_{2019} + 1) + (t_{2021} + 1) \\ &= (t_1 + t_3 + t_5 + \cdots + t_{2019} + t_{2021}) + 1011 \\ &= S_O + 1011 \end{aligned}$$

Substituting $S_E = S_O + 1011$ into equation (1),

$$\begin{aligned} S_O + S_E &= 31\,341 \\ S_O + S_O + 1011 &= 31\,341 \\ 2S_O &= 31\,341 - 1011 \\ 2S_O &= 30\,330 \\ S_O &= 15\,165 \end{aligned}$$

Therefore, the sum of the terms in the odd-numbered positions is 15 165.

Notice that this solution did not need the formula given in the note after the problem.

Solution 2

Let t_1 represent the first term in the sequence. Every term in the sequence can be written in terms of t_1. The second term is 1 more than the first term, the third term is 2 more than the first term, the fourth term is 3 more than the first term, and so on. Thus,

$$\begin{aligned} t_1 + t_2 + t_3 + t_4 + \cdots + t_{2021} + t_{2022} &= 31\,341 \\ t_1 + (t_1 + 1) + (t_1 + 2) + (t_1 + 3) + \cdots + (t_1 + 2020) + (t_1 + 2021) &= 31\,341 \\ 2022t_1 + (1 + 2 + 3 + \cdots + 2020 + 2021) &= 31\,341 \end{aligned}$$

Using the formula for the sum of the first n positive integers with $n = 2021$,

$$2022t_1 + \frac{2021(2022)}{2} = 34\,341$$

Dividing by 2022,

$$\begin{aligned} t_1 + \frac{2021}{2} &= \frac{31\,341}{2022} \\ t_1 + 1010.5 &= 15.5 \\ t_1 &= -995 \end{aligned}$$

Since the first term in the sequence is -995, we know that the original sum is

$$\begin{aligned} &t_1 + t_2 + t_3 + t_4 + \cdots + t_{2020} + t_{2021} + t_{2022} \\ &\qquad = t_1 + (t_1 + 1) + (t_1 + 2) + (t_1 + 3) + \cdots + (t_1 + 2020) + (t_1 + 2021) \\ &\qquad = -995 - 994 - 993 - 992 - \cdots + 1025 + 1026 \end{aligned}$$

We are interested in the sum of the terms in the odd-numbered positions. That is, we're interested in the sum

$$-995 - 993 - 991 - \cdots + 991 + 993 + 995 + 997 + 999 + \cdots + 1023 + 1025$$

From this point, we will present two different methods for determining this sum.

- *Method 1:*

  Notice that this sum includes all of the odd integers from -995 to 995, inclusive. This sum is 0. Thus,

$$\begin{aligned} -995 - 993 - 991 - \cdots + 991 + 993 + 995 + 997 + 999 + \cdots + 1023 + 1025 \\ &= 0 + 997 + 999 + \cdots + 1023 + 1025 \\ &= 997(15) + 2 + 4 + \cdots + 28 \\ &= 14\,955 + 2(1 + 2 + \cdots + 14) \\ &= 14\,955 + 2\left[\frac{14(15)}{2}\right] \\ &= 14\,955 + 210 \\ &= 15\,165 \end{aligned}$$

- *Method 2:*

  This is an arithmetic series with $n = 1011$ terms, first term $t_1 = -995$, and last term $t_n = 1025$.

  Then, using the formula $S_n = n\left[\frac{t_1 + t_n}{2}\right]$ for the sum of a series,

$$\begin{aligned} -995 - 993 - 991 - 989 - \cdots + 1023 + 1025 &= 1011\left[\frac{-995 + 1025}{2}\right] \\ &= 1011(15) \\ &= 15\,165 \end{aligned}$$

Therefore, the sum of the terms in the odd-numbered positions is $15\,165$.

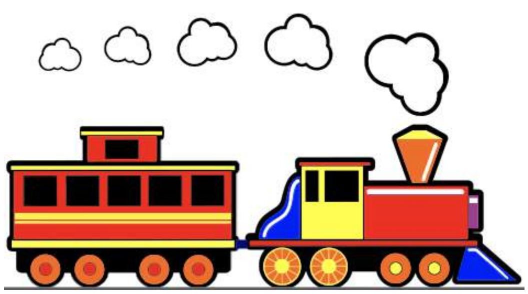
Problem of the Week
Problem E
Passing Trains

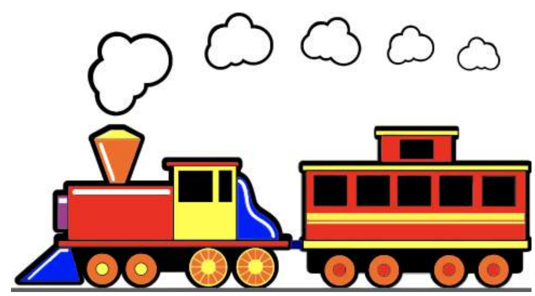
Two trains of equal length are on parallel tracks. One train travels at 40 km/h and the other train travels at 20 km/h.

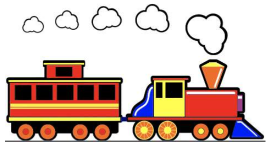
One day, the two trains are travelling in the same direction, and the front end of the faster train is at the same place as the back end of the slower train. The faster train then completely passes the slower train so that the back end of the faster train is now at the same place as the front end of the slower train.

Another day, the two trains are travelling in opposite directions, and the front end of the faster train is at the same place as the front end of the slower train. The trains then completely pass each other so that the back end of the faster train is at the same place as the back end of the slower train.

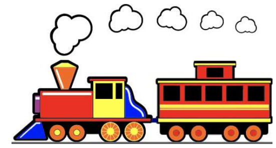
If it takes 2 minutes longer for the trains to completely pass one another when travelling in the same direction than it does when they are travelling in opposite directions, determine the length of each train.







Problem of the Week
Problem E and Solution
Passing Trains



Problem

Two trains of equal length are on parallel tracks. One train travels at 40 km/h and the other train travels at 20 km/h.

One day, the two trains are travelling in the same direction, and the front end of the faster train is at the same place as the back end of the slower train. The faster train then completely passes the slower train so that the back end of the faster train is now at the same place as the front end of the slower train.

Another day, the two trains are travelling in opposite directions, and the front end of the faster train is at the same place as the front end of the slower train. The trains then completely pass each other so that the back end of the faster train is at the same place as the back end of the slower train.

If it takes 2 minutes longer for the trains to completely pass one another when travelling in the same direction than it does when they are travelling in opposite directions, determine the length of each train.

Solution

Solution 1

Let L represent the length, in km, of each train. Let t_1 represent the time, in hours, required for the faster train to completely pass the slower train when they are travelling in the same direction. Let t_2 represent the time, in hours, required for the faster train to completely pass the slower train when they are travelling in opposite directions.

In order to completely pass one another when travelling in the same direction, the faster train must travel two lengths of the train plus whatever distance the slower train travels. Therefore,

$$\begin{aligned}40t_1 &= 20t_1 + 2L\\ 20t_1 &= 2L\\ t_1 &= \frac{L}{10}\end{aligned}$$

In order to completely pass one another when travelling in opposite directions, the total distance travelled by the two trains must be $2L$. Therefore,

$$\begin{aligned}40t_2 + 20t_2 &= 2L\\ 60t_2 &= 2L\\ t_2 &= \frac{L}{30}\end{aligned}$$

We know it takes 2 minutes or $\frac{2}{60}$ hours longer for the trains to completely pass one another when travelling in the same direction than when travelling in opposite directions. Thus,

$$\begin{aligned} t_1 - t_2 &= \frac{2}{60} \\ \frac{L}{10} - \frac{L}{30} &= \frac{1}{30} \\ 3L - L &= 1 \\ 2L &= 1 \\ L &= 0.5 \end{aligned}$$

Therefore, the length of each train is 0.5 km.

Solution 2

Let L represent the length, in km, of each train. When travelling in the same direction, the faster train is travelling at $40 - 20 = 20$ km/h relative to the slower train. In order to completely pass the slower train, the faster train must travel $2L$ km. Therefore, it takes $\frac{2L}{20} = \frac{L}{10}$ hours to completely pass the slower train.

When travelling in opposite directions, the faster train is travelling at $40 + 20 = 60$ km/h relative to the slower train. In order to completely pass the slower train, the faster train must travel $2L$ km. Therefore, it takes $\frac{2L}{60} = \frac{L}{30}$ hours to completely pass the slower train.

We know it takes 2 minutes or $\frac{2}{60} = \frac{1}{30}$ hours longer for the trains to completely pass one another when travelling in the same direction than when travelling in opposite directions. Thus,

$$\begin{aligned} \frac{L}{10} - \frac{L}{30} &= \frac{1}{30} \\ 3L - L &= 1 \\ 2L &= 1 \\ L &= 0.5 \end{aligned}$$

Therefore, the length of each train is 0.5 km.

Solution 3

Let L represent the length, in km, of each train. While the trains are travelling in opposite directions, let y km be the distance travelled by the slower train from the time the faster train begins to pass until it completely passes. The slower train travels y km and the faster train travels $(2L - y)$ km. We know that the time travelled will be the same, so

$$\begin{aligned} \frac{y}{20} &= \frac{2L - y}{40} \\ 2y &= 2L - y \\ 3y &= 2L \end{aligned} \tag{1}$$

While the trains are travelling in the same direction, let x km be the distance travelled by the slower train from the time the faster train begins to pass until it completely passes. The slower train travels x km and the faster train travels $(x + 2L)$ km. We know that the time travelled will be the same, so

$$\begin{aligned} \frac{x}{20} &= \frac{x+2L}{40} \\ 2x &= x+2L \\ x &= 2L \end{aligned} \tag{2}$$

We know that it takes 2 minutes or $\frac{2}{60}$ hours longer for the trains to completely pass one another when travelling in the same direction than when travelling in opposite directions. Thus,

$$\begin{aligned} \frac{x}{20} - \frac{y}{20} &= \frac{2}{60} \\ 3x - 3y &= 2 \end{aligned}$$

Substituting $2L$ for x from equation (2) and $2L$ for $3y$ from equation (1),

$$\begin{aligned} 3(2L) - 2L &= 2 \\ 4L &= 2 \\ L &= 0.5 \end{aligned}$$

Therefore, the length of each train is 0.5 km.

Problem of the Week
Problem E
Three Lists

Ameya has two lists, List 1 and List 2, which each have six entries that are consecutive positive integers. The smallest entry in List 1 is a and the smallest entry in List 2 is b, and $a < b$.

Ameya creates a third list, List 3. The thirty-six entries in List 3 come from the product of each number in List 1 with each number of List 2. (There could be repeated numbers in List 3.)

Suppose that List 3 has 49 as an entry, has no entry that is multiple of 64, and has an entry larger than 75. Determine all possible pairs (a, b).

$$1 \times 2 = 3\,?$$

$1 \times 2 = 3\,?$

Problem of the Week
Problem E and Solution
Three Lists

Problem

Ameya has two lists, List 1 and List 2, which each have six entries that are consecutive positive integers. The smallest entry in List 1 is a and the smallest entry in List 2 is b, and $a < b$.

Ameya creates a third list, List 3. The thirty-six entries in List 3 come from the product of each number in List 1 with each number of List 2. (There could be repeated numbers in List 3.)

Suppose that List 3 has 49 as an entry, has no entry that is multiple of 64, and has an entry larger than 75. Determine all possible pairs (a, b).

Solution

We will start by considering what the first condition tells us about the values of a and b, as it seems to be the most restrictive of the three conditions.

The first condition tells us that 49 must be the product of an integer from List 1 and an integer from List 2. Since $49 = 7^2$, 7 is prime, and all integers in the two lists are positive, these integers must be either 1 and 49, or 7 and 7.

Note: It is not possible for 49 to be obtained in both of these ways at once because if a list contains 49, then it cannot also contain 7. However, knowing this will not be important for our solution.

We will find all possible values of a and b by considering the two cases separately:

- Case 1: 49 was obtained in the third list by multiplying 1 and 49.

  Since the number 1 is in one of the lists, we must have either $a = 1$ or $b = 1$. The condition of $a < b$ means we must have $a = 1$. This means that List 1 must be

  $$1, 2, 3, 4, 5, 6$$

  and the number 49 must appear somewhere in List 2.

  Therefore, List 2 is one of the following six lists:

  $$44, 45, 46, 47, 48, 49$$
  $$45, 46, 47, 48, 49, 50$$
  $$46, 47, 48, 49, 50, 51$$
  $$47, 48, 49, 50, 51, 52$$
  $$48, 49, 50, 51, 52, 53$$
  $$49, 50, 51, 52, 53, 54$$

  Notice that $4 \times 48 = 192 = 64 \times 3$. Since 4 is in List 1, and no number in the third list can be a multiple of 64, then List 2 cannot contain the number 48. This leaves just one possibility for List 2:

  $$49, 50, 51, 52, 53, 54$$

This case gives exactly one possibility for the pair (a, b), namely $(1, 49)$.

We can verify that the third list for the pair $(a, b) = (1, 49)$ actually satisfies the second and third conditions. For the second condition, we note that $64 = 2^6$ and that we can get at most two factors of 2 from a number in List 1 and at most two factors of 2 from a number in List 2. It follows that any product in the third list will have at most 4 factors of 2, and hence cannot be a multiple of 64. For the third condition, we note that $2 \times 49 = 98$ is in the third list and is greater than 75.

- Case 2: 49 was obtained in the third list by multiplying 7 and 7.

  In this case, we know that the number 7 must appear in both List 1 and List 2. In order for this to happen we need to have $2 \leq a \leq 7$ and $2 \leq b \leq 7$. Since $a < b$, we actually must have $3 \leq b \leq 7$. (The smallest a can be is 2 and so b must be at least one more than that.)

  Since $3 \leq b \leq 7$, List 2 *must* contain the number 8. This means that to satisfy the second condition, List 1 *cannot* contain the number 8. Therefore, we must have $a = 2$. This means that List 1 must be

  $$2, 3, 4, 5, 6, 7$$

  Since $7 \times 10 = 70$ and $7 \times 11 = 77$, the third list can only satisfy the third condition if List 2 contains a number at least as large as 11. This means we cannot have $b = 3$, $b = 4$, or $b = 5$, leaving the only possible values to be $b = 6$ or $b = 7$. These values produce the following possibilities for List 2:

  $$6, 7, 8, 9, 10, 11$$

  or

  $$7, 8, 9, 10, 11, 12$$

  Therefore, this case gives two additional possibilities for the pair (a, b), namely $(2, 6)$ and $(2, 7)$.

  We can verify that the third list for each of the the pairs $(a, b) = (2, 6)$ and $(a, b) = (2, 7)$ satisfies the second and third conditions using an argument similar to the one given in Case 1.

Combining the two cases, we conclude that there are exactly three pairs, (a, b), that satisfy all three conditions. They are $(1, 49)$, $(2, 6)$, and $(2, 7)$.

Problem of the Week

Problem E

A Very Large Prime

A *prime number* is an integer greater than 1 that has only two positive divisors: 1 and itself.

For some six-digit positive integer $216\,09d$ with ones (units) digit d, $2^{21609d} - 1$ is a very large prime number. In fact, the number contains 65 050 digits. The number begins with 746 093 103 064 661 343 and ends with the digit 7.

Determine the value of d.

Here are some facts which may be helpful when solving this problem:

1. If n is a positive integer and divisible by 3, then $2^n - 1$ is divisible by 7.
2. If n is a positive integer and divisible by 5, then $2^n - 1$ is divisible by 31.

Did you know?

One use for very large prime numbers is in the area of *cryptography*, the study of coding and decoding information so that it can be securely transmitted. This area of study is very important because of its application to areas like online banking, email, and general internet security, to list just a few.

Problem of the Week
Problem E and Solution
A Very Large Prime

Problem

A *prime number* is an integer greater than 1 that has only two positive divisors: 1 and itself.

For some six-digit positive integer $216\,09d$ with ones (units) digit d, $2^{21609d} - 1$ is a very large prime number. In fact, the number contains 65 050 digits. The number begins with 746 093 103 064 661 343 and ends with the digit 7.

Determine the value of d.

Here are some facts which may be helpful when solving this problem:

1. If n is a positive integer and divisible by 3, then $2^n - 1$ is divisible by 7.
2. If n is a positive integer and divisible by 5, then $2^n - 1$ is divisible by 31.

Solution

To start, we will look for a pattern in the ones digit of powers of 2.

$$2^1 = 2,\ 2^2 = 4,\ 2^3 = 8,\ 2^4 = 16,\ 2^5 = 32,\ 2^6 = 64,\ 2^7 = 128,\ 2^8 = 256$$

It appears that the ones digit of powers of 2 repeat in the cycle $2,\ 4,\ 8,\ 6$. The next four powers of 2, $2^9,\ 2^{10},\ 2^{11}$, and 2^{12}, end with ones digits $2,\ 4,\ 8$, and 6, respectively, as expected. It turns out that this pattern continues, can you convince yourself of this?

Since $2^{21609d} - 1$ has ones digit 7, we will be interested in finding powers of 2 with ones digit 8.

From the pattern, we know that 2^{216088} has ones digit 6 since 216 088 is divisible by 4. It then follows that 2^{216089} has ones digit 2, 2^{216090} has ones digit 4, and 2^{216091} has ones digit 8. Since 2^{216091} has ones digit 8, it follows that 2^{216095} and 2^{216099} also each have ones digits 8.

Then $2^{216091} - 1$, $2^{216095} - 1$ and $2^{216099} - 1$ each have ones digit 7, and the only possible values of d are $1,\ 5$, and 9.

- If $d = 5$, then our six-digit number 216 095 is divisible by 5. From the facts given in the problem, it follows that $2^{216095} - 1$ is divisible by 31 and is therefore not a prime number. Thus, $d \neq 5$.

- If $d = 9$, then the sum of the digits of our six-digit positive integer $216\,099$ is 27. If the sum of the digits of an integer is divisible by 3, then the integer itself is divisible by 3. Since 27 is divisible by 3, it follows that $216\,099$ is divisible by 3. From the facts given in the problem, it follows that $2^{216099} - 1$ is divisible by 7 and is therefore not a prime number. Thus $d \neq 9$.

- It follows that we must have $d = 1$, and thus $2^{216091} - 1$ is a prime number ending in 7. In fact, this prime number is from a group of prime numbers called *Mersenne primes*. This number is the 31^{st} Mersenne prime and it was discovered in September of 1985. For more on Mersenne Primes, check out the Great Internet Mersenne Prime Search (GIMPS) at `www.mersenne.org`. According to GIMPS, as of March 2022, 51 Mersenne Primes are known. Perhaps you will be part of a team that will discover the next Mersenne Prime. There are prizes awarded when new discoveries are found and verified.

Extension: Can you prove the two facts given in the problem? To do so, you may need to look up the proof technique called "Proof by Mathematical Induction".

Geometry & Measurement (G)

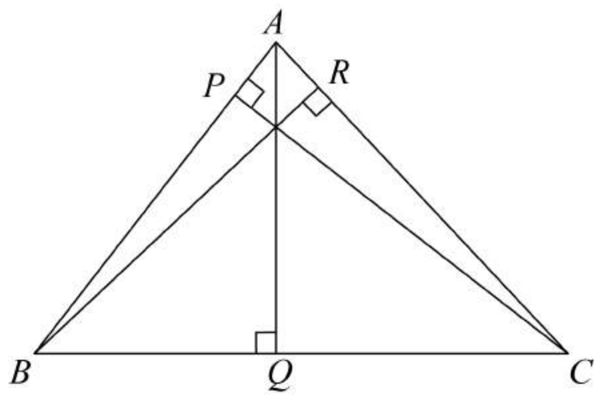
TAKE ME TO THE COVER

Problem of the Week

Problem E

Now I Know My ABCs

In triangle ABC, point P lies on AB, point Q lies on BC, and point R lies on AC such that AQ, BR, and CP are altitudes with lengths 21 cm, 24 cm, and 56 cm, respectively.



Determine the measure, in degrees, of $\angle ABC$, and the lengths, in centimetres, of AB, BC, and CA.

Note the diagram is not drawn to scale.

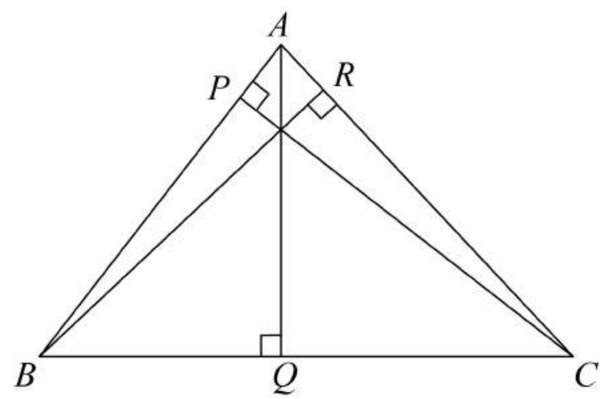
Problem of the Week

Problem E and Solution

Now I Know My ABCs

Problem

In triangle ABC, point P lies on AB, point Q lies on BC, and point R lies on AC such that AQ, BR, and CP are altitudes with lengths 21 cm, 24 cm, and 56 cm, respectively.



Determine the measure, in degrees, of $\angle ABC$, and the lengths, in centimetres, of AB, BC, and CA.

Note the diagram is not drawn to scale.

Solution

Let $BC = a$, $AC = b$, and $AB = c$. We will present two methods for determining that $21a = 24b = 56c$, and then continue on with the rest of the solution.

- *Method 1: Use Areas*

  We can find the area of $\triangle ABC$ by multiplying the length of the altitude (the height) by the corresponding base and dividing by 2. Therefore,

  $$\frac{AQ \times BC}{2} = \frac{BR \times AC}{2} = \frac{CP \times AB}{2}$$

  Substituting $AQ = 21$, $BR = 24$, and $CP = 56$, and multiplying through by 2 gives us $21a = 24b = 56c$.

- *Method 2: Use Trigonometry*

  In right-angled $\triangle ARB$, $\sin A = \frac{BR}{AB} = \frac{24}{c}$. In $\triangle APC$, $\sin A = \frac{CP}{AC} = \frac{56}{b}$. Putting these together gives $\frac{24}{c} = \frac{56}{b}$, or $24b = 56c$.

  In right-angled $\triangle BAQ$, $\sin B = \frac{AQ}{AB} = \frac{21}{c}$. In $\triangle BPC$, $\sin B = \frac{CP}{BC} = \frac{56}{a}$. Putting these together gives $\frac{21}{c} = \frac{56}{a}$, or $21a = 56c$.

  Combining these gives $21a = 24b = 56c$.

We now will continue on with the rest of the solution.

From $21a = 24b$ we obtain $b = \frac{21}{24}a = \frac{7}{8}a$, and from $21a = 56c$ we obtain $c = \frac{21}{56}a = \frac{3}{8}a$. The ratio of the sides in $\triangle ABC$ is therefore $a : b : c = a : \frac{7}{8}a : \frac{3}{8}a = 8 : 7 : 3$. Let $BC = 8x$, $AC = 7x$, and $AB = 3x$, where $x > 0$.

Using the cosine law,

$$\begin{aligned} AC^2 &= AB^2 + BC^2 - 2(AB)(BC)\cos(\angle ABC) \\ (7x)^2 &= (3x)^2 + (8x)^2 - 2(3x)(8x)\cos(\angle ABC) \\ 49x^2 &= 9x^2 + 64x^2 - 48x^2\cos(\angle ABC) \end{aligned}$$

Since $x > 0$, we know $x^2 \neq 0$. So dividing by x^2,

$$49 = 73 - 48\cos(\angle ABC)$$

Rearranging,

$$\begin{aligned} 48\cos(\angle ABC) &= 24 \\ \cos(\angle ABC) &= \frac{1}{2} \end{aligned}$$

Therefore, $\angle ABC = 60^\circ$.

In right $\triangle BPC$,

$$\begin{aligned} \frac{CP}{BC} &= \sin 60^\circ \\ BC &= \frac{CP}{\sin 60^\circ} \\ BC &= \frac{56}{\frac{\sqrt{3}}{2}} \\ BC &= \frac{112}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} \\ BC &= \frac{112\sqrt{3}}{3} \end{aligned}$$

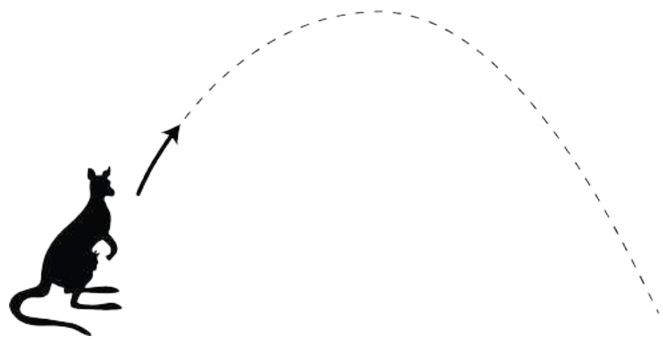
However, $BC = 8x$. Therefore,

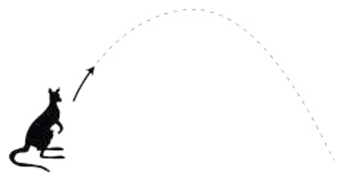
$$\begin{aligned} 8x &= \frac{112\sqrt{3}}{3} \\ x &= \frac{14\sqrt{3}}{3} \\ 3x &= 14\sqrt{3} \\ 7x &= \frac{98\sqrt{3}}{3} \end{aligned}$$

Therefore, $\angle ABC = 60^\circ$, and the side lengths of $\triangle ABC$ are $AB = 3x = 14\sqrt{3}$ cm, $AC = 7x = \dfrac{98\sqrt{3}}{3}$ cm, and $BC = \dfrac{112\sqrt{3}}{3}$ cm.

Problem of the Week
Problem E
Reach for the Sky

The equation $y = -5x^2 + ax + b$, where a and b are real numbers and $a \neq b$, represents a parabola. If this parabola passes through the points with coordinates (a, b) and (b, a), determine the maximum value of the parabola.





Problem of the Week
Problem E and Solution
Reach for the Sky

Problem

The equation $y = -5x^2 + ax + b$, where a and b are real numbers and $a \neq b$, represents a parabola. If this parabola passes through the points with coordinates (a, b) and (b, a), determine the maximum value of the parabola.

Solution

Since (a, b) lies on the parabola, it satisfies the equation of the parabola. We can substitute $x = a$ and $y = b$ into the equation $y = -5x^2 + ax + b$.

$$\begin{aligned} b &= -5a^2 + a^2 + b \\ b &= -4a^2 + b \\ 0 &= -4a^2 \\ 0 &= a^2 \\ 0 &= a \end{aligned}$$

The equation becomes $y = -5x^2 + 0x + b$, or simply $y = -5x^2 + b$.

Since (b, a) lies on the parabola, it satisfies the equation of the parabola. We can substitute $x = b$ and $y = a = 0$ into the equation $y = -5x^2 + b$.

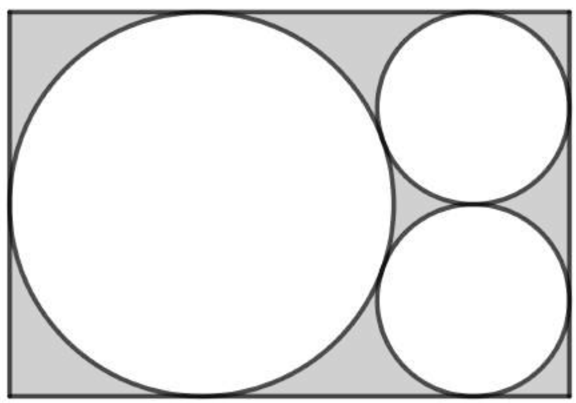
$$\begin{aligned} 0 &= -5b^2 + b \\ 0 &= b(-5b + 1) \end{aligned}$$

This means that $b = 0$ or $-5b + 1 = 0$. Therefore, $b = 0$ or $b = \frac{1}{5}$.
Since $a \neq b$ and $a = 0$, then $b = 0$ is inadmissible.

Therefore, $b = \frac{1}{5}$ and the equation representing the parabola is $y = -5x^2 + \frac{1}{5}$. The parabola opens down and the vertex of the parabola is $\left(0, \frac{1}{5}\right)$, and so the maximum value of the parabola is $\frac{1}{5}$.

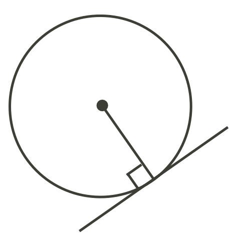
Problem of the Week
Problem E
Three Circles

Three circles are contained in a rectangle. Two of the circles have a radius of 1 and one of the circles has a radius of 2. The larger circle is tangent to three sides of the rectangle. The two smaller circles are each tangent to the larger circle and tangent to each other, and they are also each tangent to two sides of the rectangle. Determine the area of the rectangle that is not covered by the circles.

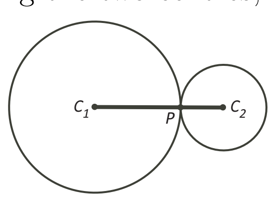


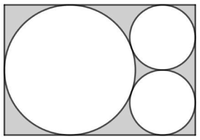
NOTE: For this problem, the following known results about circles may be useful:

- If a line is tangent to a circle, then the line is perpendicular to the radius drawn to the point of tangency.



- If two circles are tangent to each other at point P, then a line segment through the point of tangency can be drawn connecting the two centres, C_1 and C_2.





Problem of the Week
Problem E and Solution
Three Circles

Problem

Three circles are contained in a rectangle. Two of the circles have a radius of 1 and one of the circles has a radius of 2. The larger circle is tangent to three sides of the rectangle. The two smaller circles are each tangent to the larger circle and tangent to each other, and they are also each tangent to two sides of the rectangle. Determine the area of the rectangle that is not covered by the circles.

Solution

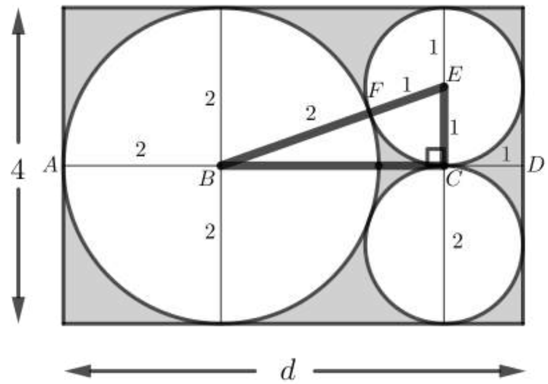
Let the centre of the larger circle be B and the centre of one of the smaller circles be E. Let C be the point of tangency of the two smaller circles. Let F be the point of tangency of the larger circle and the smaller circle with centre E. Position line segment AD so that it is parallel to the longer side of the rectangle, such that A and D are midpoints of the shorter sides of the rectangle and A lies on the larger circle. AD will pass through B and C.

Since the larger circle is tangent to two opposite sides of the rectangle and its radius is 2 m, then the length of the shorter side of the rectangle is equal to the diameter of the larger circle, or 4 m.

Let the length of the longer side of the rectangle be d. This is also the length of AD. We know that $AB = 2$ m, the length of a radius of the larger circle, and $CD = 1$ m, the same length as a radius of a smaller circle. We need to find the length of BC.

Since AD is tangent to the smaller circles at C, we know that EC is a radius of one of the smaller circles and is perpendicular to AD at C. Since the circles with centres B and E are tangent at F, EFB is a straight line segment and $EB = EF + FB = 1 + 2 = 3$ m.

Combining this information, $\triangle ECB$ is right-angled at C.



Using the Pythagorean Theorem in $\triangle ECB$, we have
$BC^2 = EB^2 - EC^2 = 3^2 - 1^2 = 8$. Thus, $BC = \sqrt{8}$, since $BC > 0$.

Therefore, the length of the longer side of the rectangle is

$$\begin{aligned} d &= AB + BC + CD \\ &= 2 + \sqrt{8} + 1 \\ &= 3 + \sqrt{8} \end{aligned}$$

To find the area not covered by the circles, we find the area of the rectangle and subtract the areas of the three circles.

$$\begin{aligned} \text{Shaded Area} &= \text{Area of Rectangle} - \text{Area of larger circle} - \text{Area of two smaller circles} \\ &= 4 \times (3 + \sqrt{8}) - \pi \times 2^2 - 2 \times (\pi \times 1^2) \\ &= 12 + 4\sqrt{8} - 4\pi - 2\pi \\ &= 12 + 4\sqrt{8} - 6\pi \end{aligned}$$

Students who have learned to simplify radicals will know that
$\sqrt{8} = \sqrt{4}\sqrt{2} = 2\sqrt{2}$. Therefore, the shaded area can then be written
$12 + 4 \times 2\sqrt{2} - 6\pi = 12 + 8\sqrt{2} - 6\pi$.

Therefore, the area of the rectangle not covered by the circles is
$(12 + 4\sqrt{8} - 6\pi)$ m^2 or $(12 + 8\sqrt{2} - 6\pi)$ m^2, which is approximately 4.5 m^2.

Problem of the Week

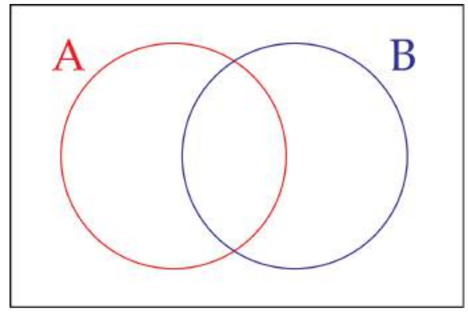
Problem E

Everything in its Place 3

(a) A Venn diagram has two circles, labelled A and B. Each circle contains functions, $f(x)$, that satisfy the following criteria.

A: $f(2) = -3$

B: $f(-2) = -1$



The overlapping region in the middle contains functions that are in both A and B, and the region outside both circles contains functions that are neither in A nor B.

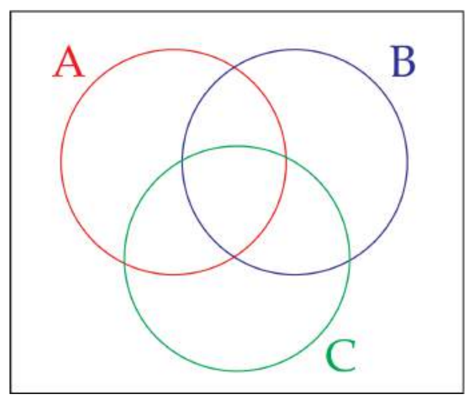
In total this Venn diagram has four regions. Place functions in as many of the regions as you can. Is it possible to find a function for each region?

(b) A Venn diagram has three circles, labelled A, B, and C. Each circle contains ordered pairs, (x, y), where x and y are real numbers, that satisfy the following criteria.

A: $y = (x + 3)^3 + 2$

B: $y = \frac{1}{2}x^2 + 1$

C: $y = |x + 1|$



In total this Venn diagram has eight regions. Place ordered pairs in as many of the regions as you can. Is it possible to find an ordered pair for each region?

Problem of the Week

Problem E and Solution

Everything in its Place 3

Problem

(a) A Venn diagram has two circles, labelled A and B. Each circle contains functions, $f(x)$, that satisfy the following criteria.

A: $f(2) = -3$

B: $f(-2) = -1$

The overlapping region in the middle contains functions that are in both A and B, and the region outside both circles contains functions that are neither in A nor B. In total this Venn diagram has four regions. Place functions in as many of the regions as you can. Is it possible to find a function for each region?

(b) A Venn diagram has three circles, labelled A, B, and C. Each circle contains ordered pairs, (x, y), where x and y are real numbers, that satisfy the following criteria.

A: $y = (x + 3)^3 + 2$

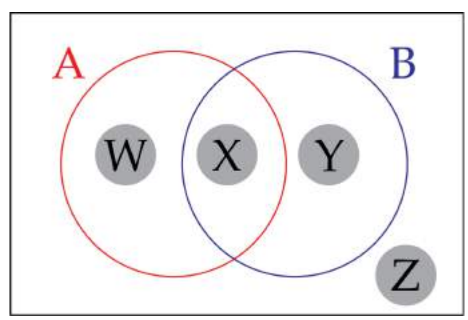
B: $y = \frac{1}{2}x^2 + 1$

C: $y = |x + 1|$

In total this Venn diagram has eight regions. Place ordered pairs in as many of the regions as you can. Is it possible to find an ordered pair for each region?

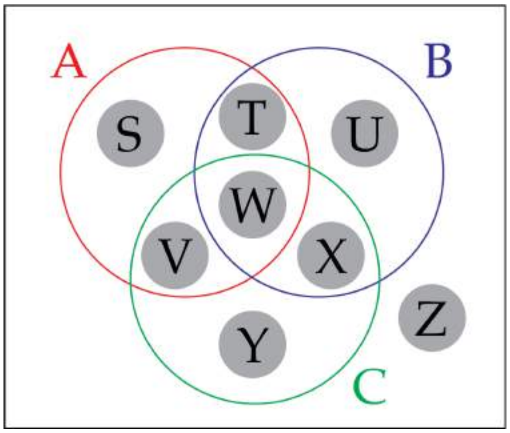
Solution

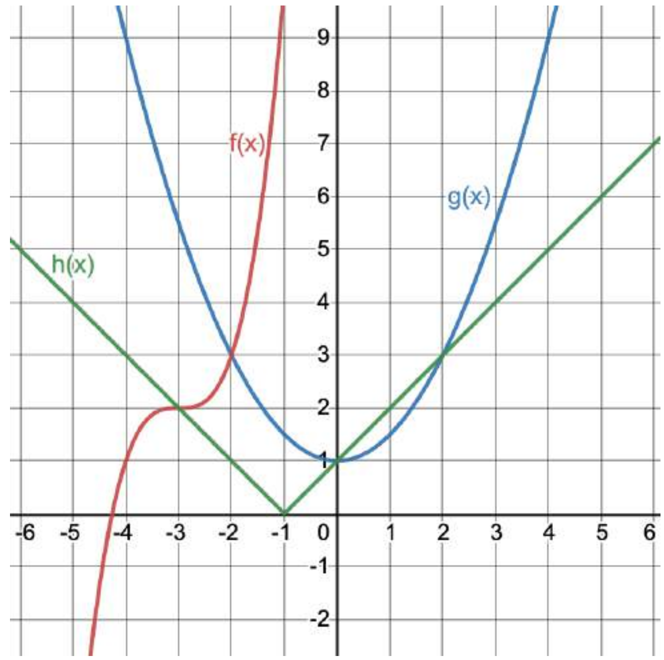
(a) We have marked the four regions W, X, Y, and Z.
When creating functions, you can think of the problem algebraically, or graphically. When thinking algebraically, a function that satisfies $f(2) = -3$ is one that evaluates to -3 when 2 is substituted for x. When thinking graphically, a function that satisfies $f(2) = -3$ is one whose graph goes through the point $(2, -3)$.



- Any function in region W must satisfy $f(2) = -3$ but *not* $f(-2) = -1$. There are infinitely many possibilities. Some examples are $f(x) = -x - 1$ and $f(x) = x^2 - 7$.
- Any function in region X must satisfy both $f(2) = -3$ and $f(-2) = -1$. In other words, the graph of the function must pass through both $(2, -3)$ and $(-2, -1)$. There are infinitely many possibilities. Some examples are $f(x) = -\frac{1}{2}x - 2$ and $f(x) = -\frac{1}{8}x^3 - 2$.
- Any function in region Y must satisfy $f(-2) = -1$ but *not* $f(2) = -3$. There are infinitely many possibilities. Some examples are $f(x) = \frac{1}{2}x$ and $f(x) = x^2 - 5$.
- Any function in region Z must satisfy neither $f(2) = -3$ nor $f(-2) = -1$. There are again infinitely many possibilities. Some examples are $f(x) = x$ and $f(x) = x^2$.

(b) We have marked the eight regions S, T, U, V, W, X, Y, and Z. We will name the functions as follows: $f(x) = (x+3)^3+2$, $g(x) = \frac{1}{2}x^2+1$, and $h(x) = |x+1|$. We have also provided a graph of the functions.



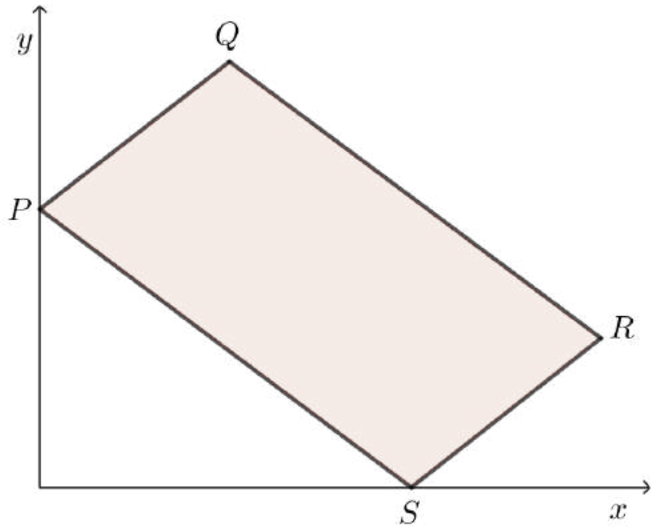


Note that the graphs of f and g intersect at $(-2, 3)$, the graphs of f and h intersect at $(-3, 2)$, and the graphs of g and h intersect at $(0, 1)$ and $(2, 3)$.

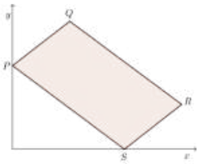
- Any ordered pair in region S must satisfy the equation $y = f(x)$, but not $y = g(x)$ or $y = h(x)$. Since the point $(-4, 1)$ is on the graph of f, but not on the graph of g or h, the ordered pair $(-4, 1)$ works. There are infinitely many others as well.
- Any ordered pair in region T must satisfy the equations $y = f(x)$ and $y = g(x)$, but not $y = h(x)$. Since the point $(-2, 3)$ is on the graph of f and on the graph of g, but not on the graph of h, the ordered pair $(-2, 3)$ works. In fact, since $(-2, 3)$ is the only point of intersection of the graphs of f and g, this is the only possible choice for region T.
- Any ordered pair in region U must satisfy the equation $y = g(x)$, but not $y = f(x)$ or $y = h(x)$. The ordered pair $(4, 9)$ works. There are infinitely many others as well.
- Any ordered pair in region V must satisfy the equations $y = f(x)$ and $y = h(x)$, but not $y = g(x)$. The ordered pair $(-3, 2)$ works. In fact, since $(-3, 2)$ is the only point of intersection of the graphs of f and h, this is the only possible choice for region V.
- Any ordered pair in region W must satisfy the equations $y = f(x)$, $y = g(x)$, and $y = h(x)$. There are no ordered pairs that satisfy this as the graphs of the three functions do not have a common point of intersection. So this region must remain empty.
- Any ordered pair in region X must satisfy the equations $y = g(x)$ and $y = h(x)$, but not $y = f(x)$. The ordered pairs $(0, 1)$ and $(2, 3)$ work. In fact, since these are the only points of intersection of the graphs of g and h, these are the only possible choices for region X.
- Any ordered pair in region Y must satisfy the equation $y = h(x)$, but not $y = f(x)$ or $y = g(x)$. The ordered pair $(-2, 1)$ works. There are infinitely many others as well.
- Any ordered pair in region Z must not satisfy the equations $y = f(x)$, $y = g(x)$, or $y = h(x)$. The ordered pair $(0, 0)$ works. There are infinitely many others as well.

Problem of the Week
Problem E
Parallelogram Askew

Parallelogram $PQRS$ is positioned such that P lies on the positive y-axis, S lies on the positive x-axis, and Q and R lie in the first quadrant.



If vertices P, Q, and S are located at $(0, 30)$, $(k, 50)$ and $(40, 0)$, respectively, and the area of $PQRS$ is 1340 units^2, determine the coordinates of Q and R.



Problem of the Week
Problem E and Solution
Parallelogram Askew

Problem

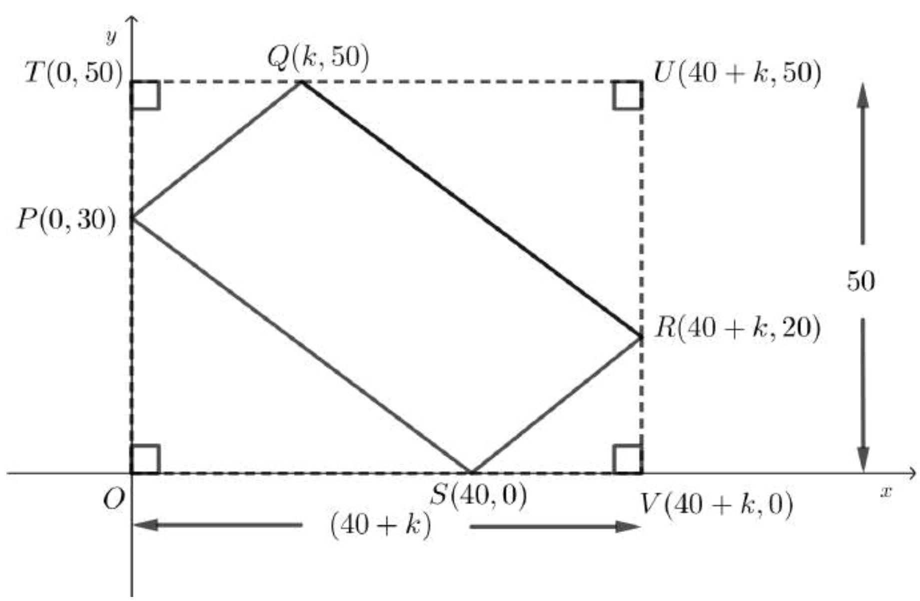
Parallelogram $PQRS$ is positioned such that P lies on the positive y-axis, S lies on the positive x-axis, and Q and R lie in the first quadrant. If vertices P, Q, and S are located at $(0, 30)$, $(k, 50)$ and $(40, 0)$, respectively, and the area of $PQRS$ is 1340 units^2, determine the coordinates of Q and R.

Solution
Solution 1:

In this solution we will use a method known commonly as "*completing the rectangle*".

Since $PQRS$ is a parallelogram, $PQ = SR$ and PQ is parallel to SR. We can use this to find the coordinates of R. To get from P to Q, we go up 20 units and right k units. Therefore, to get from S to R we do the same. Therefore, R is located at $(40 + k, 20)$.

Enclose $PQRS$ in rectangle $OTUV$ such that OT is on the positive y-axis passing through P, TU is parallel to the positive x-axis passing through Q, UV is parallel to the positive y-axis passing through R, and OV lies along the positive x-axis passing through S. The y-coordinate of Q is the distance from the x-axis to TU and also the height, UV, of rectangle $OTUV$. It follows that $OT = UV = 50$ units. Therefore, the coordinates of T are $(0, 50)$. Similarly, the x-coordinate of R is the distance from the y-axis to UV and also the width, OV, of rectangle $OTUV$. It follows that $TU = OV = (40 + k)$ units. Therefore, the coordinates of V are $(40 + k, 0)$ and the coordinates of U are $(40 + k, 50)$.



We can now put the information together using areas to determine the value of k.

$$\begin{aligned}
\text{Area } OTUV &= \text{Area } \triangle PTQ + \text{Area } \triangle QUR + \text{Area } \triangle RVS + \text{Area } \triangle SOP + \text{Area } PQRS \\
UV \times OV &= \frac{PT \times TQ}{2} + \frac{QU \times UR}{2} + \frac{RV \times VS}{2} + \frac{OS \times OP}{2} + 1340 \\
50 \times (40 + k) &= \frac{(50-30) \times k}{2} + \frac{((40+k)-k) \times (50-20)}{2} + \frac{20 \times ((40+k)-40)}{2} + \frac{40 \times 30}{2} + 1340 \\
50 \times (40 + k) &= \frac{20 \times k}{2} + \frac{40 \times 30}{2} + \frac{20 \times k}{2} + \frac{40 \times 30}{2} + 1340 \\
2000 + 50k &= 10k + 600 + 10k + 600 + 1340 \\
2000 + 50k &= 20k + 2540 \\
30k &= 540 \\
k &= 18
\end{aligned}$$

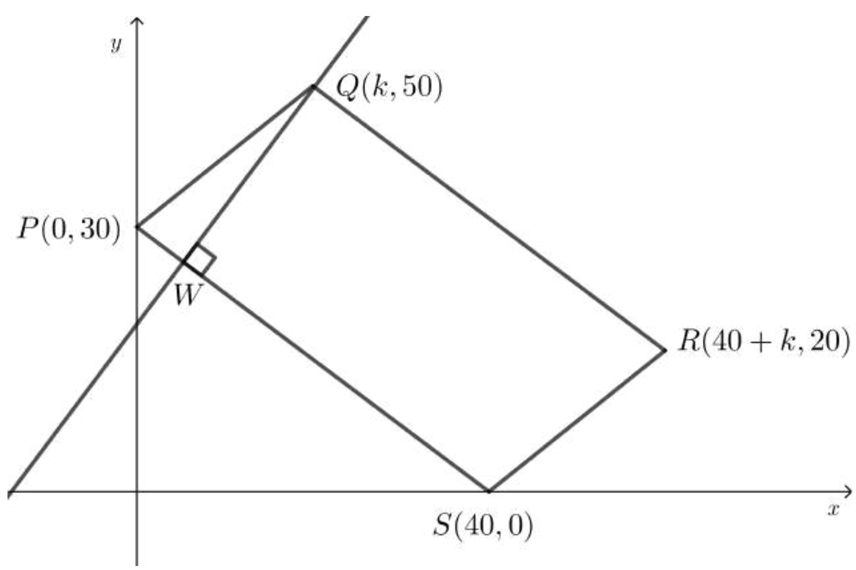
Therefore, the value of k is 18 and coordinates of Q and R are $Q(18, 50)$ and $R(58, 20)$.

Solution 2:

In this solution we will use linear equations, intersections, and lengths to find k.

Since $PQRS$ is a parallelogram, $PQ = SR$ and PQ is parallel to SR. We can use this to find the coordinates of R. To get from P to Q, we go up 20 units and right k units. Therefore, to get from S to R we do the same. Therefore, R is located at $(40 + k, 20)$.

Construct a line perpendicular to PS that passes through Q and meets PS at W.



We are going to find the coordinates of W in terms of k.

The line through PS has a slope of $-\frac{3}{4}$ and a y-intercept of 30. Therefore, the equation of this line is

$$y = -\frac{3}{4}x + 30 \tag{1}$$

The line though QW is perpendicular to PS and so has slope $\frac{4}{3}$. The equation of this line is $4x - 3y = C$. Substituting the coordinates of Q into this equation, we get $4k - 3(50) = C$ or $C = 4k - 150$. Therefore, the line through QW has equation

$$4x - 3y = 4k - 150 \tag{2}$$

W is the intersection point of the lines with equations (1) and (2).

Substituting equation (1) into equation (2), we get:

$$\begin{aligned} 4x - 3\left(-\frac{3}{4}x + 30\right) &= 4k - 150 \\ 4x + \frac{9}{4}x - 90 &= 4k - 150 \\ 16x + 9x - 360 &= 16k - 600 \\ 25x &= 16k - 240 \\ x &= 0.64k - 9.6 \end{aligned} \tag{3}$$

Substituting equation (3) into equation (1) we get:

$$\begin{aligned} y &= -\frac{3}{4}(0.64k - 9.6) + 30 \\ y &= -0.48k + 37.2 \end{aligned}$$

Therefore, the point W has coordinates $(0.64k - 9.6, -0.48k + 37.2)$.

We will now find two expressions for the length of QW.

Using the distance formula we know

$$QW = \sqrt{(0.64k - 9.6 - k)^2 + (-0.48k + 37.2 - 50)^2} \tag{4}$$

Another way to find the length QW is using the area of the parallelogram.

The length of $PS = \sqrt{(30-0)^2 + (0-40)^2} = \sqrt{2500} = 50$, since $PS > 0$.

PS is the base of the parallelogram and QW is the height. Therefore,

$$\begin{aligned} PS \times QW &= 1340 \\ 50QW &= 1340 \\ QW &= 26.8 \end{aligned} \tag{5}$$

Now equating equations (4) and (5), we can solve for k.

$$\begin{aligned} \sqrt{(0.64k - 9.6 - k)^2 + (-0.48k + 37.2 - 50)^2} &= 26.8 \\ (-0.36k - 9.6)^2 + (-0.48k - 12.8)^2 &= 718.24 \\ 0.1296k^2 + 6.912k + 92.16 + 0.2034k^2 + 12.288k + 163.84 &= 718.24 \\ 0.36k^2 + 19.2k - 462.24 &= 0 \end{aligned}$$

Using the quadratic formula, we find $k = 18$ or $k = -\frac{214}{3}$. Since $Q(k, 50)$ is in the first quadrant, we must have $k > 0$ and so $k = 18$.

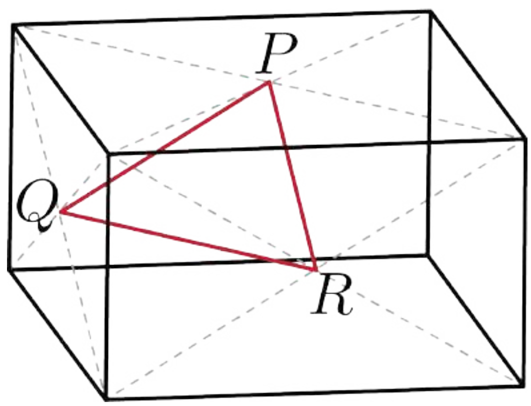
Therefore, the coordinates of Q and R are $Q(18, 50)$ and $R(58, 20)$.

Problem of the Week

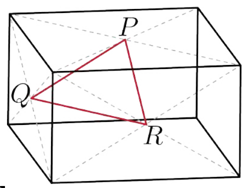
Problem E

I Want More Volume

A rectangular prism is placed on a table. Points P, Q, and R lie on three different faces of the prism with P on the top face and Q and R on two adjacent side faces. Each point is located where the diagonals of the particular face intersect. Connecting these three points gives us $\triangle PQR$.



If $PQ = 4$ cm, $QR = 5$ cm, and $RP = 6$ cm, determine the volume of the rectangular prism.



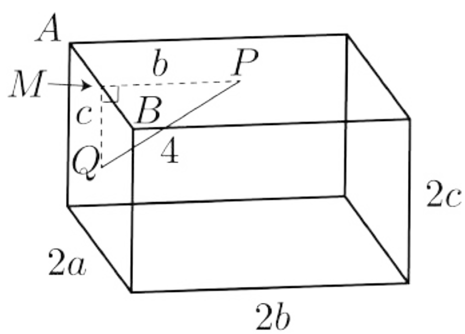
Problem of the Week
Problem E and Solution
I Want More Volume

Problem

A rectangular prism is placed on a table. Points P, Q, and R lie on three different faces of the prism with P on the top face and Q and R on two adjacent side faces. Each point is located where the diagonals of the particular face intersect. Connecting these three points gives us $\triangle PQR$. If $PQ = 4$ cm, $QR = 5$ cm, and $RP = 6$ cm, determine the volume of the rectangular prism.

Solution

First, we label the top edge of the face containing point Q as AB and its midpoint as M. Let $AM = a$, $PM = b$, and $MQ = c$. Since the centre of a rectangle is where its diagonals intersect, P and Q are at the centres of their respective faces. Further, since the centre of a rectangle is also where the perpendicular bisectors of the sides of the rectangle meet, the rectangular prism has dimensions $2a$, $2b$, and $2c$, as shown in the following diagram.



Since $\angle PMQ = 90°$, it follows that

$$b^2 + c^2 = 16 \quad (1)$$

Similarly, we can conclude the following.

$$a^2 + c^2 = 36 \quad (2)$$

$$a^2 + b^2 = 25 \quad (3)$$

Adding equations (1), (2), and (3) gives us the following.

$$2a^2 + 2b^2 + 2c^2 = 77$$

$$a^2 + b^2 + c^2 = \frac{77}{2} \quad (4)$$

Now, we subtract each of equations (1), (2), and (3) from equation (4) to obtain

$$a^2 = \frac{45}{2},\ b^2 = \frac{5}{2},\ \text{and } c^2 = \frac{27}{2}$$

Multiplying a^2, b^2, and c^2 together gives the product

$$a^2b^2c^2 = \frac{45}{2} \times \frac{5}{2} \times \frac{27}{2} = \frac{6075}{8}$$

Then, taking the positive square root,

$$abc = \sqrt{\frac{6075}{8}} = \frac{45\sqrt{3}}{2\sqrt{2}} = \frac{45\sqrt{3}}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{45\sqrt{6}}{4}$$

To determine the volume of the rectangular prism, we multiply the side lengths together.

$$\begin{aligned} V &= (2a)(2b)(2c) \\ &= 8abc \\ &= 8\left(\frac{45\sqrt{6}}{4}\right) \\ &= 90\sqrt{6}\ \text{cm}^3 \end{aligned}$$

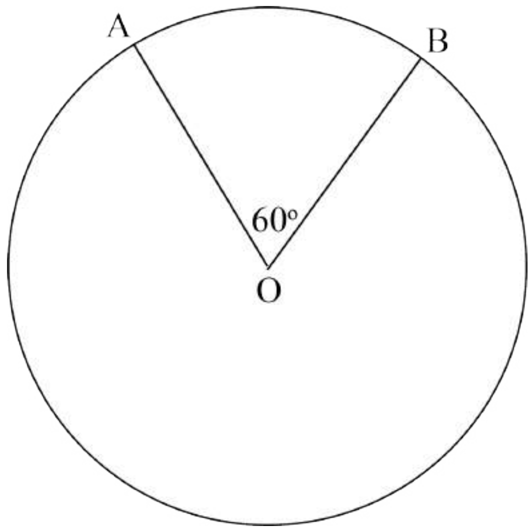
Therefore, the volume of the rectangular prism is $90\sqrt{6}$ cm^3.

Problem of the Week

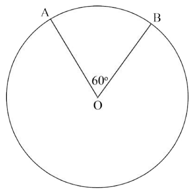
Problem E

Another Circle

Points A and B are on a circle with centre O and radius 6 cm, such that $\angle AOB = 60^\circ$.



Determine the radius of the circle which passes through points A, B, and O.



Problem of the Week

Problem E and Solution

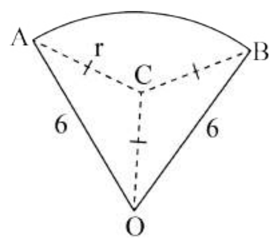
Another Circle

Problem

Points A and B are on a circle with centre O and radius 6 cm, such that $\angle AOB = 60°$. Determine the radius of the circle which passes through points $A,\ B$, and O.

Solution

Let C be the centre of the circle that passes through $A,\ B$, and O. Then $CA,\ CB$, and CO are radii. Therefore, $CA = CB = CO = r$, where r is the radius of the circle.



In $\triangle CAO$ and $\triangle CBO$, $CA = CB$, CO is common, and $OA = OB$. Therefore, $\triangle CAO \cong \triangle CBO$ and it follows that $\angle COA = \angle COB$. But $\angle AOB = 60°$. Therefore, $\angle COA = \angle COB = 30°$.

In $\triangle CAO$, $CA = CO = r$ and $\triangle CAO$ is isosceles. Therefore, $\angle CAO = \angle COA = 30°$ and $\angle ACO = 180° - 30° - 30° = 120°$.

From here we can find the length of r using either the sine law or the cosine law.

Method 1: Using the sine law,

$$\begin{aligned}
\frac{CA}{\sin(\angle COA)} &= \frac{OA}{\sin(\angle ACO)} \\
\frac{r}{\sin 30°} &= \frac{6}{\sin 120°} \\
r &= \frac{6}{\sin 120°} \times \sin 30° \\
r &= \frac{6}{\frac{\sqrt{3}}{2}} \times \frac{1}{2} \\
r &= 6 \times \frac{2}{\sqrt{3}} \times \frac{1}{2} \\
r &= \frac{6}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} \\
r &= 2\sqrt{3} \text{ cm}
\end{aligned}$$

Therefore, the radius of the circle that passes through $A,\ B$, and O is $2\sqrt{3}$ cm.

Method 2: Using the cosine law,

$$\begin{aligned}
CA^2 &= CO^2 + AO^2 - 2 \times CO \times AO \times \cos(\angle COA) \\
r^2 &= r^2 + 6^2 - 2(6)(r)\cos 30^\circ \\
12r\cos 30^\circ &= 36 \\
r\cos 30^\circ &= 3 \\
r \times \frac{\sqrt{3}}{2} &= 3 \\
r \times \sqrt{3} &= 6 \\
r \times \sqrt{3} \times \sqrt{3} &= 6 \times \sqrt{3} \\
3r &= 6\sqrt{3} \\
r &= 2\sqrt{3} \text{ cm}
\end{aligned}$$

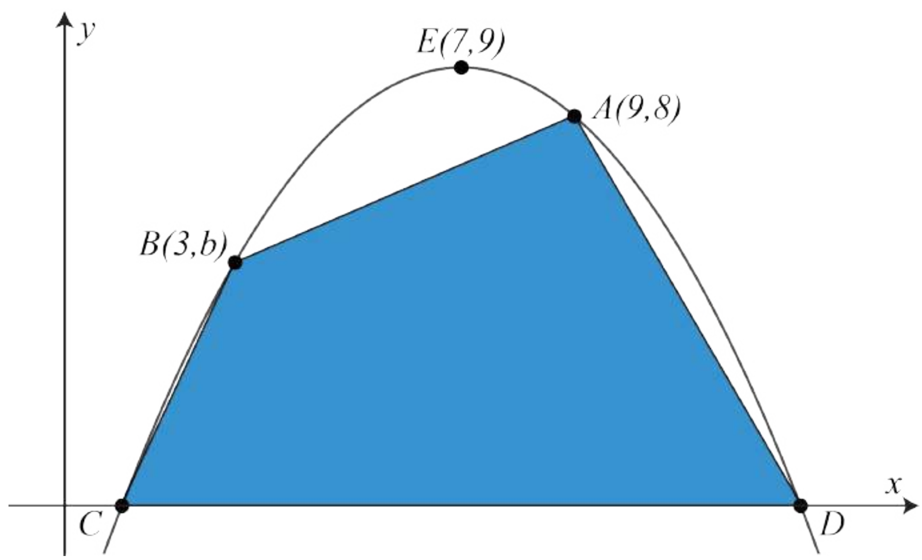
Therefore, the radius of the circle that passes through A, B, and O is $2\sqrt{3}$ cm.

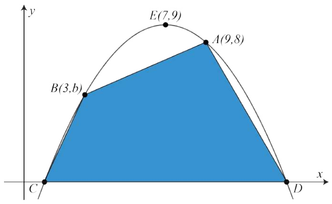
Problem of the Week

Problem E

Parabolic Art

Kenna likes making artistic creations using parabolas, to put on the walls of her math classroom. She drew a parabola with vertex $E(7, 9)$ and plotted points $A(9, 8)$ and $B(3, b)$ on the parabola as well as points C and D where the parabola intersects the x-axis, with C to the left of D. Then she connected points A, B, C, and D to form quadrilateral $ABCD$, and painted it blue. What is the area of quadrilateral $ABCD$?





Problem of the Week
Problem E and Solution
Parabolic Art

Problem

Kenna likes making artistic creations using parabolas, to put on the walls of her math classroom. She drew a parabola with vertex $E(7,9)$ and plotted points $A(9,8)$ and $B(3,b)$ on the parabola as well as points C and D where the parabola intersects the x-axis, with C to the left of D. Then she connected points A, B, C, and D to form quadrilateral $ABCD$, and painted it blue. What is the area of quadrilateral $ABCD$?

Solution

First we need to find the equation of the parabola. Then, we can find the x-intercepts of the parabola and the y-coordinate of point B on the parabola.

We are given the vertex of the parabola, $E(7,9)$. Using the vertex form of the equation of a parabola, $y = a(x-h)^2 + k$, with vertex $(h,k) = (7,9)$, the equation of the parabola is $y = a(x-7)^2 + 9$.

Since the point $A(9,8)$ is on the parabola, we can substitute $(x,y) = (9,8)$ into the equation $y = a(x-7)^2 + 9$ to find the value of a.

$$\begin{aligned} 8 &= a(9-7)^2 + 9 \\ 8 &= a(4) + 9 \\ -1 &= 4a \\ -\frac{1}{4} &= a \end{aligned}$$

The equation of the parabola is therefore $y = -\frac{1}{4}(x-7)^2 + 9$.

To find the y-coordinate of $B(3,b)$, we substitute $(x,y) = (3,b)$ into the equation of the parabola.

$$\begin{aligned} b &= -\frac{1}{4}(3-7)^2 + 9 \\ &= -\frac{1}{4}(16) + 9 \\ &= -4 + 9 \\ &= 5 \end{aligned}$$

Therefore, the coordinates of B are $(3,5)$.

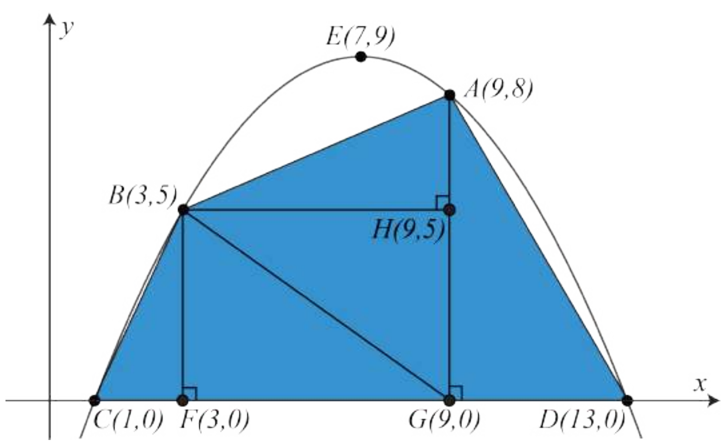
To find the x-intercepts of the parabola, we substitute $y = 0$ into the equation of the parabola.

$$\begin{aligned} 0 &= -\frac{1}{4}(x-7)^2 + 9 \\ -9 &= -\frac{1}{4}(x-7)^2 \\ 36 &= (x-7)^2 \\ \pm 6 &= x - 7 \end{aligned}$$

It follows that $x - 7 = -6$ or $x - 7 = 6$. Then the x-intercepts of the parabola are 1 and 13. Therefore, the coordinates of C and D are $C(1, 0)$ and $D(13, 0)$.

Now that we know the coordinates of A, B, C, and D, we can calculate the area of quadrilateral $ABCD$. There are many ways to do this. We will proceed as follows.

From $B(3, 5)$ and $A(9, 8)$, drop perpendiculars, intersecting the x-axis at $F(3, 0)$ and $G(9, 0)$, respectively. From $B(3, 5)$ draw a line perpendicular to AG, intersecting AG at $H(9, 5)$. Draw line segment BG.



Note that line segments BG and AG divide the quadrilateral into three regions: $\triangle CGB$, $\triangle AGD$, and $\triangle AGB$.

We will use the coordinates of the points to find the lengths of several horizontal and vertical line segments that will be required for the area calculation.

$BH = 9 - 3 = 6,\ CG = 9 - 1 = 8,\ GD = 13 - 9 = 4,\ BF = 5 - 0 = 5,$ and $AG = 8 - 0 = 8.$

To determine the area of $ABCD$, we will find the sum of the areas of $\triangle CGB$, $\triangle AGD$ and $\triangle AGB$.

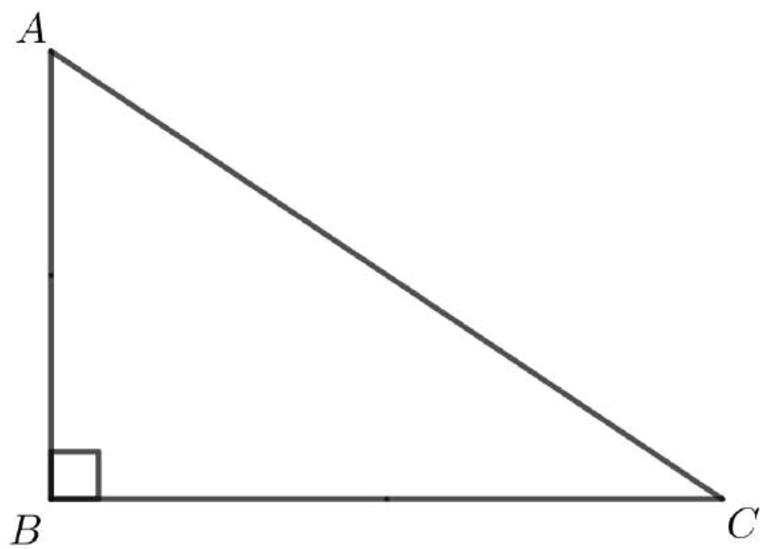
$$\begin{aligned} \text{Area } ABCD &= \text{Area } \triangle CGB + \text{Area } \triangle AGD + \text{Area } \triangle AGB \\ &= \frac{CG \times BF}{2} + \frac{AG \times GD}{2} + \frac{AG \times BH}{2} \\ &= \frac{8 \times 5}{2} + \frac{8 \times 4}{2} + \frac{8 \times 6}{2} \\ &= 20 + 16 + 24 \\ &= 60 \text{ units}^2 \end{aligned}$$

Therefore, the area of quadrilateral $ABCD$ is 60 units^2.

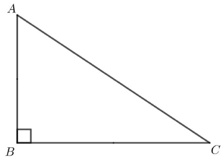
Problem of the Week
Problem E
Medians

In $\triangle ABC$, $\angle ABC = 90°$. A median is drawn from A to side BC, meeting BC at M such that $AM = 5$. A second median is drawn from C to side AB, meeting AB at N such that $CN = 2\sqrt{10}$.

Determine the length of the longest side of $\triangle ABC$.



NOTE: In a triangle, a *median* is a line segment drawn from a vertex of the triangle to the midpoint of the opposite side.



Problem of the Week
Problem E and Solution
Medians

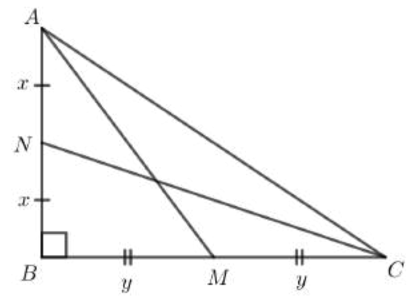
Problem

In $\triangle ABC$, $\angle ABC = 90^\circ$. A median is drawn from A to side BC, meeting BC at M such that $AM = 5$. A second median is drawn from C to side AB, meeting AB at N such that $CN = 2\sqrt{10}$.

Determine the length of the longest side of $\triangle ABC$.

Note: In a triangle, a *median* is a line segment drawn from a vertex of the triangle to the midpoint of the opposite side.

Solution



Since AM is a median, M is the midpoint of BC. Then $BM = MC$. Let $BM = MC = y$.
Since CN is a median, N is the midpoint of AB. Then $AN = NB$. Let $AN = NB = x$.

Since $\angle B = 90^\circ$, $\triangle NBC$ is a right-angled triangle. Using the Pythagorean Theorem,

$$\begin{aligned} NB^2 + BC^2 &= CN^2 \\ x^2 + (2y)^2 &= (2\sqrt{10})^2 \\ x^2 + 4y^2 &= 40 \qquad (1) \end{aligned}$$

Since $\angle B = 90^\circ$, $\triangle ABM$ is a right-angled triangle. Using the Pythagorean Theorem,

$$\begin{aligned} AB^2 + BM^2 &= AM^2 \\ (2x)^2 + y^2 &= 5^2 \\ 4x^2 + y^2 &= 25 \qquad (2) \end{aligned}$$

Adding equations (1) and (2), we get $5x^2 + 5y^2 = 65$ or $x^2 + y^2 = 13$.

The longest side of $\triangle ABC$ is the hypotenuse AC. Using the Pythagorean Theorem,

$$\begin{aligned} AC^2 &= AB^2 + BC^2 \\ &= (2x)^2 + (2y)^2 \\ &= 4x^2 + 4y^2 \\ &= 4(x^2 + y^2) \end{aligned}$$

Since $x^2 + y^2 = 13$, we have $AC^2 = 4(13)$. And since $AC > 0$, $AC = 2\sqrt{13}$ follows.

Therefore, the length of the longest side of $\triangle ABC$ is $2\sqrt{13}$.

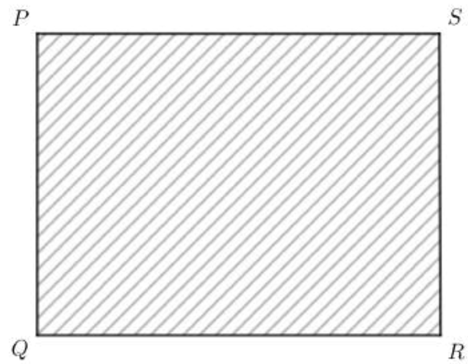
Note: The solver could have instead solved a system of equations to find $x = 2$ and $y = 3$, and then proceed to solve for the longest side. The above approach was provided to expose the solver to alternate way to think about the solution of this problem.

Algebra (A)

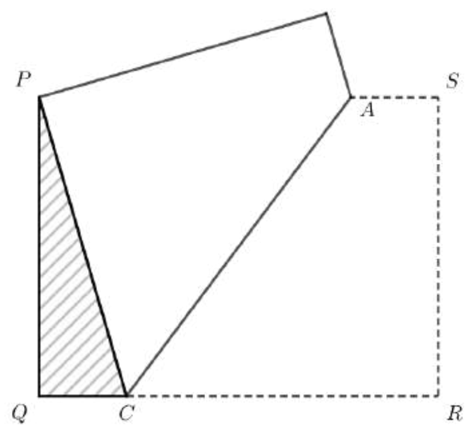
TAKE ME TO THE COVER

Problem of the Week
Problem E
Fold Once

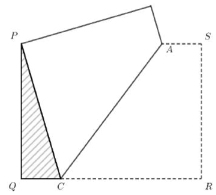
A rectangular piece of paper, $PQRS$, has $PQ = 30$ cm and $PS = 40$ cm. The paper has grey lines on one side and is plain white on the other.



The paper is folded so that the two diagonally opposite corners P and R coincide. This creates a crease along line segment AC, with A on PS and C on QR.



Determine the length of AC.



Problem of the Week

Problem E and Solution

Fold Once

Problem

A rectangular piece of paper, $PQRS$, has $PQ = 30$ cm and $PS = 40$ cm. The paper has grey lines on one side and is plain white on the other. The paper is folded so that the two diagonally opposite corners P and R coincide. This creates a crease along line segment AC, with A on PS and C on QR. Determine the length of AC.

Solution

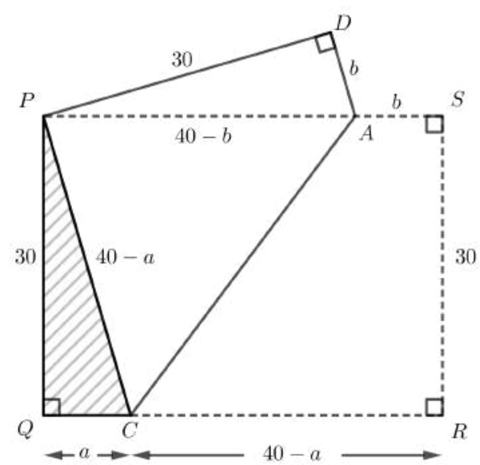
Since $PQRS$ is a rectangle, all angles inside $PQRS$ are $90°$. After the fold, R coincides with P. Label the point that S folds to as D. The angle at D is the same as the angle at S. Since $PQRS$ is a rectangle, $\angle PSR = 90°$ and $SR = 30$, and it follows that $\angle PDA = 90°$ and $PD = 30$.

Let a represent the length of QC and b represent the length of AS.
Then $CR = QR - QC = 40 - a$ and $PA = PS - AS = 40 - b$.

Since S folds to D, it follows that $AD = AS = b$.

Since R folds to P, it follows that $PC = CR = 40 - a$.

All of the information is recorded on the following diagram.



Since $\triangle PQC$ is a right-angled triangle, we can use the Pythagorean Theorem to find a.

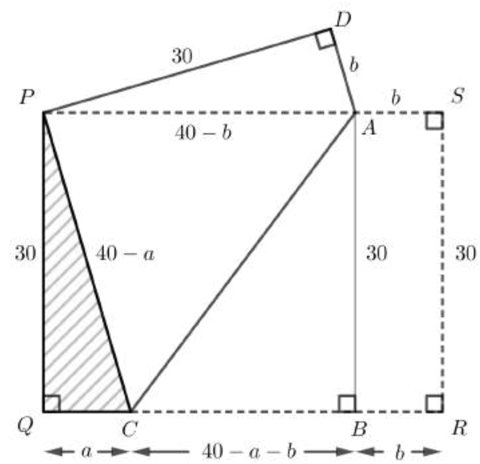
$$
\begin{aligned}
QC^2 + PQ^2 &= PC^2 \\
a^2 + 30^2 &= (40 - a)^2 \\
a^2 + 900 &= 1600 - 80a + a^2 \\
80a &= 700 \\
a &= \frac{35}{4}
\end{aligned}
$$

Since $\triangle PDA$ is a right-angled triangle, we can use the Pythagorean Theorem to find b.

$$\begin{aligned} AD^2 + PD^2 &= PA^2 \\ b^2 + 30^2 &= (40-b)^2 \\ b^2 + 900 &= 1600 - 80b + b^2 \\ 80b &= 700 \\ b &= \frac{35}{4} \end{aligned}$$

Therefore, $a = b = \dfrac{35}{4}$ cm.

We still need to find the length of the crease. From A drop a perpendicular to QR intersecting QR at B. Then $ABRS$ is a rectangle. It follows that $AB = SR = 30$ and $BR = AS = b$. Also, $CB = QR - QC - BR = 40 - a - b = 40 - \dfrac{35}{4} - \dfrac{35}{4} = \dfrac{90}{4} = \dfrac{45}{2}$ cm.



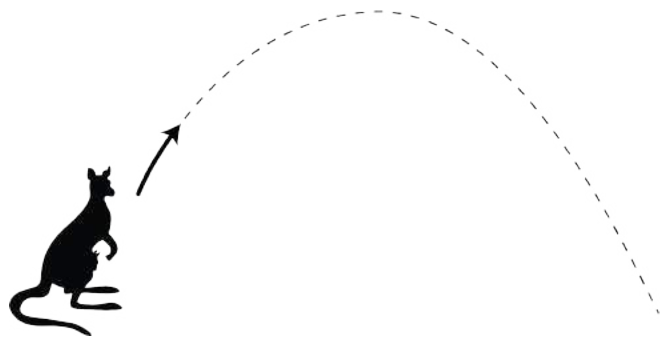
Using the Pythagorean Theorem in $\triangle CAB$,

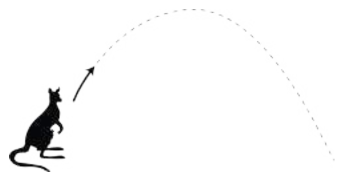
$$\begin{aligned} AC^2 &= AB^2 + CB^2 \\ &= 30^2 + \left(\frac{45}{2}\right)^2 \\ &= 900 + \frac{2025}{4} \\ &= \frac{5625}{4} \end{aligned}$$

Since $AC > 0$, it follows that $AC = \dfrac{75}{2} = 37.5$ cm. Therefore, the length of the crease is $\dfrac{75}{2}$ cm.

Problem of the Week
Problem E
Reach for the Sky

The equation $y = -5x^2 + ax + b$, where a and b are real numbers and $a \neq b$, represents a parabola. If this parabola passes through the points with coordinates (a, b) and (b, a), determine the maximum value of the parabola.





Problem of the Week

Problem E and Solution

Reach for the Sky

Problem

The equation $y = -5x^2 + ax + b$, where a and b are real numbers and $a \neq b$, represents a parabola. If this parabola passes through the points with coordinates (a, b) and (b, a), determine the maximum value of the parabola.

Solution

Since (a, b) lies on the parabola, it satisfies the equation of the parabola. We can substitute $x = a$ and $y = b$ into the equation $y = -5x^2 + ax + b$.

$$\begin{aligned} b &= -5a^2 + a^2 + b \\ b &= -4a^2 + b \\ 0 &= -4a^2 \\ 0 &= a^2 \\ 0 &= a \end{aligned}$$

The equation becomes $y = -5x^2 + 0x + b$, or simply $y = -5x^2 + b$.

Since (b, a) lies on the parabola, it satisfies the equation of the parabola. We can substitute $x = b$ and $y = a = 0$ into the equation $y = -5x^2 + b$.

$$\begin{aligned} 0 &= -5b^2 + b \\ 0 &= b(-5b + 1) \end{aligned}$$

This means that $b = 0$ or $-5b + 1 = 0$. Therefore, $b = 0$ or $b = \frac{1}{5}$.
Since $a \neq b$ and $a = 0$, then $b = 0$ is inadmissible.

Therefore, $b = \frac{1}{5}$ and the equation representing the parabola is $y = -5x^2 + \frac{1}{5}$.
The parabola opens down and the vertex of the parabola is $\left(0, \frac{1}{5}\right)$, and so the maximum value of the parabola is $\frac{1}{5}$.

Problem of the Week
Problem E
Just Sum Numbers

Kaori writes a sequence with the property that after the first two terms in the sequence, each term is equal to one more than the term before it, minus the term before that. In other words, $t_n = 1 + t_{n-1} - t_{n-2}$, for $n \geq 3$, where t_n denotes the n^{th} term in the sequence.

The first term in Kaori's sequence is x and the second term is y, where x and y are real numbers. That is, $t_1 = x$ and $t_2 = y$. Determine the sum of the first 2021 terms in her sequence, as an expression in terms of x and y.

$$t_1 + t_2 + t_3 + t_4 + t_5 + \cdots$$

$t_1 + t_2 + t_3 + t_4 + t_5 + \cdots$

Problem of the Week
Problem E and Solution
Just Sum Numbers

Problem

Kaori writes a sequence with the property that after the first two terms in the sequence, each term is equal to one more than the term before it, minus the term before that. In other words, $t_n = 1 + t_{n-1} - t_{n-2}$, for $n \geq 3$, where t_n denotes the n^{th} term in the sequence.

The first term in Kaori's sequence is x and the second term is y, where x and y are real numbers. That is, $t_1 = x$ and $t_2 = y$. Determine the sum of the first 2021 terms in her sequence, as an expression in terms of x and y.

Solution

We are given that $t_1 = x$, $t_2 = y$, and $t_n = 1 + t_{n-1} - t_{n-2}$, for $n \geq 3$.
Let's use the equation $t_n = 1 + t_{n-1} - t_{n-2}$ for $n \geq 3$.

$$\begin{aligned}
t_3 &= 1 + t_2 - t_1 = 1 + y - x \\
t_4 &= 1 + t_3 - t_2 = 1 + (1 + y - x) - y = 2 - x \\
t_5 &= 1 + t_4 - t_3 = 1 + (2 - x) - (1 + y - x) = 2 - y \\
t_6 &= 1 + t_5 - t_4 = 1 + (2 - y) - (2 - x) = 1 - y + x \\
t_7 &= 1 + t_6 - t_5 = 1 + (1 - y + x) - (2 - y) = x \\
t_8 &= 1 + t_7 - t_6 = 1 + x - (1 - y + x) = y
\end{aligned}$$

Since $t_7 = t_1$ and $t_8 = t_2$, and each term in the sequence depends only on the previous two terms, it follows that the sequence repeats every six terms.

The sum of the first six terms in the sequence is equal to
$x + y + (1 + y - x) + (2 - x) + (2 - y) + (1 - y + x) = 6$.
It follows that the sum of each successive group of six terms is also equal to 6.

We note that $2022 = 6 \times 337$, so the 2022^{nd} term of the sequence is the end of a group of six terms. Thus, the sum of the first 2022 terms in the sequence is equal to $6 \times 337 = 2022$. It also follows that $t_{2022} = t_6 = 1 - y + x$.

Since the sum of the first 2021 terms is equal to the sum of the first 2022 terms minus the 2022^{nd} term, we know that the sum of the first 2021 terms of the sequence is equal to $2022 - (1 - y + x) = 2021 + y - x$.

Problem of the Week
Problem E
Just an Average Sum

Faisal chooses four numbers. When each number is added to the mean (average) of the other three, the following sums are obtained: 25, 37, 43, and 51.

Determine the mean of the four numbers Faisal chose.

Extra Problem: Can you interpret the following picture puzzle? You may need to research the meanings of some mathematical symbols used in the puzzle.

$$B > \frac{1}{n}\sum_{i=1}^{n} x_i$$

$$B > \frac{1}{n}\sum_{i=1}^{n} x_i$$

Problem of the Week
Problem E and Solution
Just an Average Sum

Problem

Faisal chooses four numbers. When each number is added to the mean (average) of the other three, the following sums are obtained: 25, 37, 43, and 51. Determine the mean of the four numbers Faisal chose.

Extra Problem: Can you interpret the picture puzzle above? You may need to research the meanings of some mathematical symbols used in the puzzle.

Solution

Let a, b, c, and d represent the four numbers. It is possible to precisely determine the four numbers, but the problem asks for only their average, which is $\frac{a+b+c+d}{4}$.

When the first number is added to the average of the other three numbers, the result is 25. Thus,

$$a + \frac{b+c+d}{3} = 25$$

which can be rewritten as

$$3a + b + c + d = 75 \qquad (1)$$

When the second number is added to the average of the other three numbers, the result is 37. Thus,

$$b + \frac{a+c+d}{3} = 37$$

which can be rewritten as

$$a + 3b + c + d = 111 \qquad (2)$$

When the third number is added to the average of the other three numbers, the result is 43. Thus,

$$c + \frac{a+b+d}{3} = 43$$

which can be rewritten as

$$a + b + 3c + d = 129 \qquad (3)$$

When the fourth number is added to the average of the other three numbers, the result is 51. Thus,

$$d + \frac{a+b+c}{3} = 51$$

which can be rewritten as

$$a + b + c + 3d = 153 \qquad (4)$$

Adding equations (1), (2), (3), and (4), we obtain $6a + 6b + 6c + 6d = 468$. Dividing this equation by 6 gives $a + b + c + d = 78$. It follows that $\frac{a+b+c+d}{4} = 19.5$.

Therefore, the average of the four numbers is 19.5.

Although it is not required, we could solve the system of equations to determine that the numbers are: -1.5, 16.5, 25.5, and 37.5.

Extra Problem Solution:

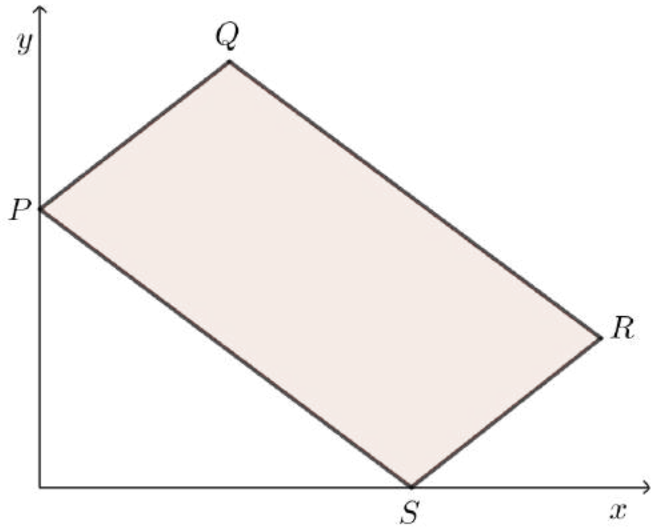
The notation $\frac{1}{n}\sum_{i=1}^{n} x_i$ is a mathematical short form which represents the average of the n numbers $x_1, x_2, \ldots, x_n$. So the picture puzzle can be interpreted as "Be greater than average".

Problem of the Week

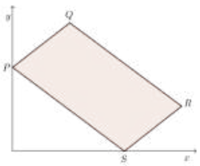
Problem E

Parallelogram Askew

Parallelogram $PQRS$ is positioned such that P lies on the positive y-axis, S lies on the positive x-axis, and Q and R lie in the first quadrant.



If vertices P, Q, and S are located at $(0, 30)$, $(k, 50)$ and $(40, 0)$, respectively, and the area of $PQRS$ is 1340 units^2, determine the coordinates of Q and R.



Problem of the Week
Problem E and Solution
Parallelogram Askew

Problem

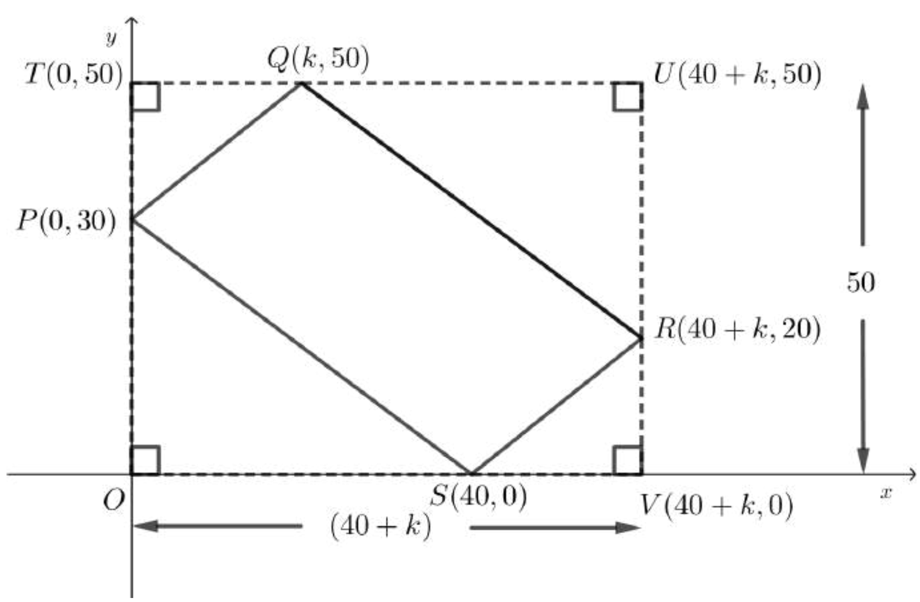
Parallelogram $PQRS$ is positioned such that P lies on the positive y-axis, S lies on the positive x-axis, and Q and R lie in the first quadrant. If vertices P, Q, and S are located at $(0, 30)$, $(k, 50)$ and $(40, 0)$, respectively, and the area of $PQRS$ is 1340 units^2, determine the coordinates of Q and R.

Solution
Solution 1:

In this solution we will use a method known commonly as "*completing the rectangle*".

Since $PQRS$ is a parallelogram, $PQ = SR$ and PQ is parallel to SR. We can use this to find the coordinates of R. To get from P to Q, we go up 20 units and right k units. Therefore, to get from S to R we do the same. Therefore, R is located at $(40 + k, 20)$.

Enclose $PQRS$ in rectangle $OTUV$ such that OT is on the positive y-axis passing through P, TU is parallel to the positive x-axis passing through Q, UV is parallel to the positive y-axis passing through R, and OV lies along the positive x-axis passing through S. The y-coordinate of Q is the distance from the x-axis to TU and also the height, UV, of rectangle $OTUV$. It follows that $OT = UV = 50$ units. Therefore, the coordinates of T are $(0, 50)$. Similarly, the x-coordinate of R is the distance from the y-axis to UV and also the width, OV, of rectangle $OTUV$. It follows that $TU = OV = (40 + k)$ units. Therefore, the coordinates of V are $(40 + k, 0)$ and the coordinates of U are $(40 + k, 50)$.



We can now put the information together using areas to determine the value of k.

$$
\begin{aligned}
\text{Area } OTUV &= \text{Area } \triangle PTQ + \text{Area } \triangle QUR + \text{Area } \triangle RVS + \text{Area } \triangle SOP + \text{Area } PQRS \\
UV \times OV &= \frac{PT \times TQ}{2} + \frac{QU \times UR}{2} + \frac{RV \times VS}{2} + \frac{OS \times OP}{2} + 1340 \\
50 \times (40+k) &= \frac{(50-30) \times k}{2} + \frac{((40+k)-k) \times (50-20)}{2} + \frac{20 \times ((40+k)-40)}{2} + \frac{40 \times 30}{2} + 1340 \\
50 \times (40+k) &= \frac{20 \times k}{2} + \frac{40 \times 30}{2} + \frac{20 \times k}{2} + \frac{40 \times 30}{2} + 1340 \\
2000 + 50k &= 10k + 600 + 10k + 600 + 1340 \\
2000 + 50k &= 20k + 2540 \\
30k &= 540 \\
k &= 18
\end{aligned}
$$

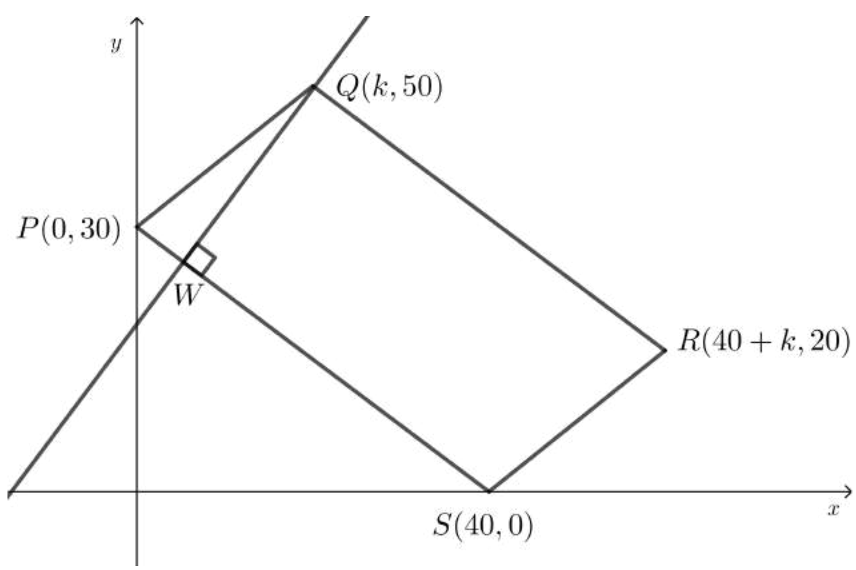
Therefore, the value of k is 18 and coordinates of Q and R are $Q(18, 50)$ and $R(58, 20)$.

Solution 2:

In this solution we will use linear equations, intersections, and lengths to find k.

Since $PQRS$ is a parallelogram, $PQ = SR$ and PQ is parallel to SR. We can use this to find the coordinates of R. To get from P to Q, we go up 20 units and right k units. Therefore, to get from S to R we do the same. Therefore, R is located at $(40 + k, 20)$.

Construct a line perpendicular to PS that passes through Q and meets PS at W.



We are going to find the coordinates of W in terms of k.

The line through PS has a slope of $-\frac{3}{4}$ and a y-intercept of 30. Therefore, the equation of this line is

$$y = -\frac{3}{4}x + 30 \tag{1}$$

The line though QW is perpendicular to PS and so has slope $\frac{4}{3}$. The equation of this line is $4x - 3y = C$. Substituting the coordinates of Q into this equation, we get $4k - 3(50) = C$ or $C = 4k - 150$. Therefore, the line through QW has equation

$$4x - 3y = 4k - 150 \tag{2}$$

W is the intersection point of the lines with equations (1) and (2).

Substituting equation (1) into equation (2), we get:

$$\begin{aligned} 4x - 3\left(-\frac{3}{4}x + 30\right) &= 4k - 150 \\ 4x + \frac{9}{4}x - 90 &= 4k - 150 \\ 16x + 9x - 360 &= 16k - 600 \\ 25x &= 16k - 240 \\ x &= 0.64k - 9.6 \end{aligned} \tag{3}$$

Substituting equation (3) into equation (1) we get:

$$\begin{aligned} y &= -\frac{3}{4}(0.64k - 9.6) + 30 \\ y &= -0.48k + 37.2 \end{aligned}$$

Therefore, the point W has coordinates $(0.64k - 9.6, -0.48k + 37.2)$.

We will now find two expressions for the length of QW.

Using the distance formula we know

$$QW = \sqrt{(0.64k - 9.6 - k)^2 + (-0.48k + 37.2 - 50)^2} \tag{4}$$

Another way to find the length QW is using the area of the parallelogram.

The length of $PS = \sqrt{(30-0)^2 + (0-40)^2} = \sqrt{2500} = 50$, since $PS > 0$.

PS is the base of the parallelogram and QW is the height. Therefore,

$$\begin{aligned} PS \times QW &= 1340 \\ 50QW &= 1340 \\ QW &= 26.8 \end{aligned} \tag{5}$$

Now equating equations (4) and (5), we can solve for k.

$$\begin{aligned} \sqrt{(0.64k - 9.6 - k)^2 + (-0.48k + 37.2 - 50)^2} &= 26.8 \\ (-0.36k - 9.6)^2 + (-0.48k - 12.8)^2 &= 718.24 \\ 0.1296k^2 + 6.912k + 92.16 + 0.2034k^2 + 12.288k + 163.84 &= 718.24 \\ 0.36k^2 + 19.2k - 462.24 &= 0 \end{aligned}$$

Using the quadratic formula, we find $k = 18$ or $k = -\frac{214}{3}$. Since $Q(k, 50)$ is in the first quadrant, we must have $k > 0$ and so $k = 18$.

Therefore, the coordinates of Q and R are $Q(18, 50)$ and $R(58, 20)$.

Problem of the Week
Problem E
Sum View

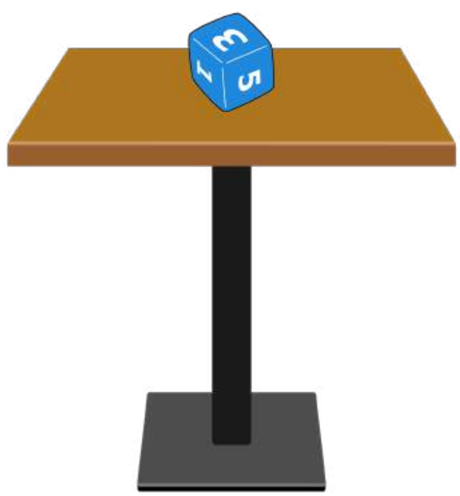
The six faces of a cube have each been marked with one of the numbers 1, 2, 3, 4, 5, and 6, with each number being used exactly once.

Three people, Paul, Lee, and Jenny, are seated around a square table.

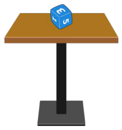
The cube is placed on the table so that from their different seat locations, each one can see the top face and two adjacent side faces. No two people see the same pair of adjacent side faces.

When Paul adds the three numbers that he can see, his total is 9. When Lee adds the three numbers that they can see, their total is 14. When Jenny adds the three numbers that she can see, her total is 15.

Determine all possibilities for the number on the bottom face of the cube.



Note that the three faces that are visible on the above cube add to 9. The picture is for illustration only. Do not assume anything from the above diagram.



Problem of the Week
Problem E and Solution
Sum View

Problem

The six faces of a cube have each been marked with one of the numbers 1, 2, 3, 4, 5, and 6, with each number being used exactly once.

Three people, Paul, Lee, and Jenny, are seated around a square table.

The cube is placed on the table so that from their different seat locations, each one can see the top face and two adjacent side faces. No two people see the same pair of adjacent side faces.

When Paul adds the three numbers that he can see, his total is 9. When Lee adds the three numbers that they can see, their total is 14. When Jenny adds the three numbers that she can see, her total is 15.

Determine all possibilities for the number on the bottom face of the cube.

Solution

We will look at the six possible cases for the number on the top face of the cube, and then decide if there are possible solutions given that top number.

Let t be the number on the top face of the cube.

- Case 1: $t = 1$

  The two side faces that Jenny sees must sum to $15 - 1 = 14$. Since the largest face value is 6, the maximum sum of the numbers on two faces is $5 + 6 = 11$. Therefore, it is not possible for the numbers on two side faces to sum to 14, and so the number on the top face cannot be 1.

- Case 2: $t = 2$

  The two side faces that Jenny sees must sum to $15 - 2 = 13$. Again, the maximum sum of the numbers on two faces is $5 + 6 = 11$. Therefore, it is not possible for the numbers on two side faces to sum to 13, and so the number on the top face cannot be 2.

- Case 3: $t = 3$

  The two side faces that Jenny sees must sum to $15 - 3 = 12$. Again, the maximum sum of the numbers on two faces is $5 + 6 = 11$. Therefore, it is not possible for the numbers on two side faces to sum to 12, and so the number on the top face cannot be 3.

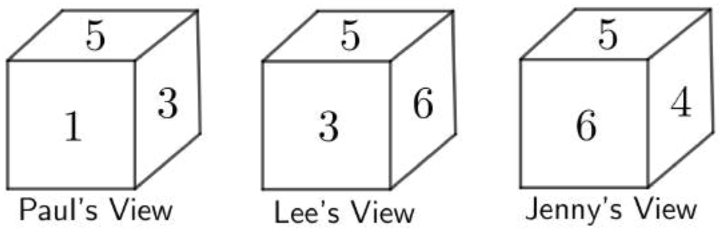
- Case 4: $t = 4$

  The two side faces that Jenny sees must sum to $15 - 4 = 11$. It is possible to get a sum of 11, and the only possible way is with the numbers on the two faces being 5 and 6. Now, the two side faces that Lee sees must sum to $14 - 4 = 10$. The only possible way to get a sum of 10 is with the numbers on the two faces being 4 and 6. But the number on the top face is also 4, so it is not possible to have a 4 on a side face.
  Therefore, the number on the top face cannot be 4.

- Case 5: $t = 5$

  The two side faces that Jenny sees must sum to $15 - 5 = 10$. The only possible way to get a sum of 10 is with the two faces being 4 and 6.
  Now, the two side faces that Lee sees must sum to $14 - 5 = 9$. It is possible to get a sum of 9, and the only possible ways are with the numbers on the two faces being 4 and 5 or 3 and 6. Since the number on the top face is 5, then the numbers on the two side faces cannot be 4 and 5. Therefore, the numbers on two side faces that Lee sees must be 3 and 6.
  Finally, the two side faces that Paul sees must sum to $9 - 5 = 4$. It is possible to get a sum of 4, and the only possible way is with the numbers on the faces being 1 and 3.

  Therefore, it is possible for the number on the top face to be 5. Then the numbers on the four side faces must be 4, 6, 3, and 1. Each person's view is shown below.

  The visible numbers are 1, 3, 4, 5, and 6. Therefore, the number on the bottom of the cube is 2.

- Case 6: $t = 6$

  The two side faces that Jenny sees must sum to $15 - 6 = 9$. It is possible to get a sum of 9, and the only possible ways are with the numbers on the two faces being 3 and 6, or 4 and 5. Since the number on the top face is 6, then the numbers on the two side faces cannot be 3 and 6. Therefore, the numbers on the two side faces that Jenny sees must be 4 and 5.

  Now, the two side faces that Lee sees must sum to $14 - 6 = 8$. It is possible to get a sum of 8, and the only possible ways are with the numbers on the two faces being 2 and 6 or 3 and 5. Since the number on the top face is 6, then the numbers on the two side faces cannot be 2 and 6. Therefore, the numbers on the two side faces that Lee sees must be 3 and 5.

  Finally, the two side faces that Paul sees must sum to $9 - 6 = 3$. The only possible way to get a sum of 3 is with the faces being 1 and 2.

  From Jenny's, Lee's and Paul's views, we know that the side faces must have numbers 4, 5, 3, 1, and 2. But there are only four side faces, so it is impossible for each of these five numbers to appear on a side face.

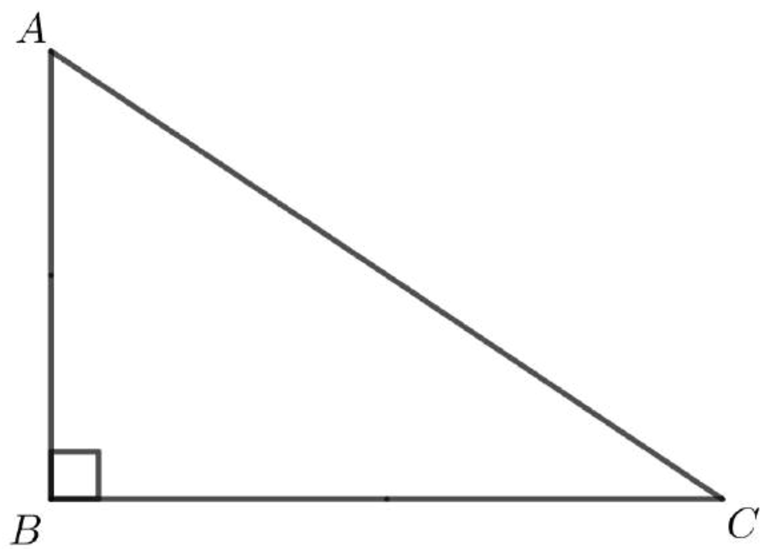
  Therefore, it is not possible for the number on the top face of the cube to be 6.

Therefore, the only possible number on the bottom face of the cube is 2.

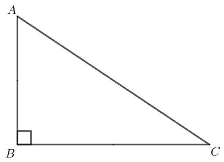
Problem of the Week
Problem E
Medians

In $\triangle ABC$, $\angle ABC = 90^\circ$. A median is drawn from A to side BC, meeting BC at M such that $AM = 5$. A second median is drawn from C to side AB, meeting AB at N such that $CN = 2\sqrt{10}$.

Determine the length of the longest side of $\triangle ABC$.



NOTE: In a triangle, a *median* is a line segment drawn from a vertex of the triangle to the midpoint of the opposite side.



Problem of the Week

Problem E and Solution

Medians

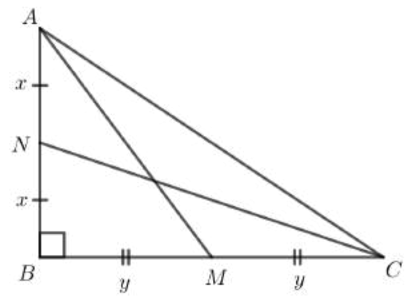
Problem

In $\triangle ABC$, $\angle ABC = 90^\circ$. A median is drawn from A to side BC, meeting BC at M such that $AM = 5$. A second median is drawn from C to side AB, meeting AB at N such that $CN = 2\sqrt{10}$.

Determine the length of the longest side of $\triangle ABC$.

NOTE: In a triangle, a *median* is a line segment drawn from a vertex of the triangle to the midpoint of the opposite side.

Solution



Since AM is a median, M is the midpoint of BC. Then $BM = MC$. Let $BM = MC = y$.
Since CN is a median, N is the midpoint of AB. Then $AN = NB$. Let $AN = NB = x$.

Since $\angle B = 90^\circ$, $\triangle NBC$ is a right-angled triangle. Using the Pythagorean Theorem,

$$\begin{aligned} NB^2 + BC^2 &= CN^2 \\ x^2 + (2y)^2 &= (2\sqrt{10})^2 \\ x^2 + 4y^2 &= 40 \qquad (1) \end{aligned}$$

Since $\angle B = 90^\circ$, $\triangle ABM$ is a right-angled triangle. Using the Pythagorean Theorem,

$$\begin{aligned} AB^2 + BM^2 &= AM^2 \\ (2x)^2 + y^2 &= 5^2 \\ 4x^2 + y^2 &= 25 \qquad (2) \end{aligned}$$

Adding equations (1) and (2), we get $5x^2 + 5y^2 = 65$ or $x^2 + y^2 = 13$.

The longest side of $\triangle ABC$ is the hypotenuse AC. Using the Pythagorean Theorem,

$$\begin{aligned} AC^2 &= AB^2 + BC^2 \\ &= (2x)^2 + (2y)^2 \\ &= 4x^2 + 4y^2 \\ &= 4(x^2 + y^2) \end{aligned}$$

Since $x^2 + y^2 = 13$, we have $AC^2 = 4(13)$. And since $AC > 0$, $AC = 2\sqrt{13}$ follows.

Therefore, the length of the longest side of $\triangle ABC$ is $2\sqrt{13}$.

NOTE: The solver could have instead solved a system of equations to find $x = 2$ and $y = 3$, and then proceed to solve for the longest side. The above approach was provided to expose the solver to alternate way to think about the solution of this problem.

Problem of the Week

Problem E

Fine Line

Suppose that $f(x) = ax + b$ and $g(x) = f^{-1}(x)$ for all values of x. That is, g is the inverse of the function f.

If $f(x) - g(x) = 2022$ for all values of x, determine all possible values for a and b.

Problem of the Week
Problem E and Solution
Fine Line

Problem

Suppose that $f(x) = ax + b$ and $g(x) = f^{-1}(x)$ for all values of x. That is, g is the inverse of the function f.

If $f(x) - g(x) = 2022$ for all values of x, determine all possible values for a and b.

Solution

Since $f(x) = ax + b$, we can determine an expression for $g(x) = f^{-1}(x)$ by letting $y = f(x)$ to obtain $y = ax + b$. We then interchange x and y to obtain $x = ay + b$, which we solve for y to obtain $ay = x - b$ or $y = \frac{x}{a} - \frac{b}{a}$.

Therefore, $f^{-1}(x) = \frac{x}{a} - \frac{b}{a}$. Note that $a \neq 0$. (This makes sense since the function $f(x) = b$ has a graph which is a horizontal line, and so cannot be invertible.)

Therefore, the equation $f(x) - g(x) = 2022$ becomes

$$(ax + b) - \left(\frac{x}{a} - \frac{b}{a}\right) = 2022$$
$$\left(a - \frac{1}{a}\right)x + \left(b + \frac{b}{a}\right) = 2022$$

This is true for all x.

From here, we will present two approaches for determining the possible values for a and b.

- **Approach 1:** Comparing coefficients

  Since the equation
  $$\left(a - \frac{1}{a}\right)x + \left(b + \frac{b}{a}\right) = 2022 = 0x + 2022$$
  is true for all x, then the coefficients of the linear expression on the left side must match the coefficients of the linear expression on the right side.

  Therefore, $a - \frac{1}{a} = 0$ and $b + \frac{b}{a} = 2022$.

  From the first of these equations, we obtain $a = \frac{1}{a}$ or $a^2 = 1$, which gives $a = 1$ or $a = -1$.

  If $a = 1$, the equation $b + \frac{b}{a} = 2022$ becomes $b + b = 2022$, which gives $b = 1011$.

  If $a = -1$ the equation $b + \frac{b}{a} = 2022$ becomes $b - b = 2022$ which is not possible.

  Therefore, we must have $a = 1$ and $b = 1011$, and so $f(x) = x + 1011$.

- **Approach 2:** Trying specific values for x

  Since the equation
  $$\left(a - \frac{1}{a}\right)x + \left(b + \frac{b}{a}\right) = 2022$$
  is true for all values of x, then it must be true for any specific values of x that we choose.

  Choosing $x = b$, we obtain
  $$\begin{aligned} \left(a - \frac{1}{a}\right)b + \left(b + \frac{b}{a}\right) &= 2022 \\ ab + b &= 2022 \qquad (1) \end{aligned}$$

  Choosing $x = 0$, we obtain
  $$\begin{aligned} 0 + b + \frac{b}{a} &= 2022 \\ b + \frac{b}{a} &= 2022 \\ \frac{ab + b}{a} &= 2022 \end{aligned}$$

  Then, substituting $ab + b = 2022$ from equation (1), we obtain
  $$\begin{aligned} \frac{ab + b}{a} &= 2022 \\ \frac{2022}{a} &= 2022 \\ a &= 1 \end{aligned}$$

  Since $a = 1$, then $ab + b = 2022$ gives $2b = 2022$ or $b = 1011$.

  Thus, $f(x) = x + 1011$.

In summary, the only possible values for a and b for which the given equation is true for all x are $a = 1$ and $b = 1011$.

Data Management (D)

TAKE ME TO THE COVER

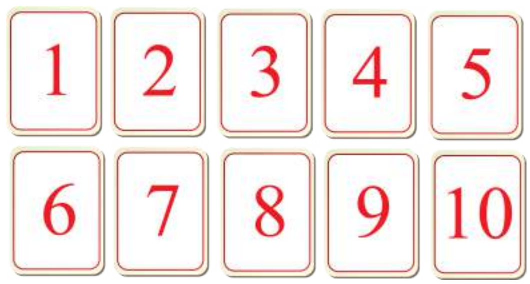
Problem of the Week

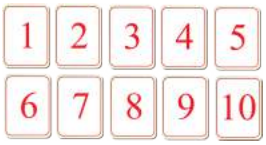
Problem E

Is it Odd?

Amna creates a game for her school carnival. She uses ten cards, each with a different integer from 1 to 10 on it, and places them face down on a table. To play the game, players randomly turn over three cards and look at the numbers. They win if the smallest number is odd and the next smallest number is even.

What is the probability that a player wins the game on their first try?





Problem of the Week
Problem E and Solution
Is it Odd?

Problem

Amna creates a game for her school carnival. She uses ten cards, each with a different integer from 1 to 10 on it, and places them face down on a table. To play the game, players randomly turn over three cards and look at the numbers. They win if the smallest number is odd and the next smallest number is even.

What is the probability that a player wins the game on their first try?

Solution

To calculate the probability we need to determine two things: the number of possible selections of three cards, and the number of these that would result in a win.

First, we will determine the number of possible selections of three cards. Since each number is distinct, then there are 10 choices for the first card, 9 choices for the second card, and 8 choices for the third card. This produces $10 \times 9 \times 8 = 720$ ordered selections. But this total includes 6 orderings for each possible selection of three numbers. For example, the three numbers 1, 2, and 3 would be included 6 times: $(1, 2, 3)$, $(1, 3, 2)$, $(2, 1, 3)$, $(2, 3, 1)$, $(3, 1, 2)$, and $(3, 2, 1)$. As we see with this example, each combination of three cards is counted six times. Therefore, there are $720 \div 6 = 120$ possible selections of three cards.

Next, we will determine the number of selections of three cards that result in a win. That is, the number of selections in which the smallest number is odd and the next smallest number is even. We can use the following table.

Smallest Number	Next Smallest Number	Possible Value(s) for Largest Number	Number of Selections of Three Cards
1	2	3, 4, 5, 6, 7, 8, 9, 10	8
	4	5, 6, 7, 8, 9, 10	6
	6	7, 8, 9, 10	4
	8	9, 10	2
3	4	5, 6, 7, 8, 9, 10	6
	6	7, 8, 9, 10	4
	8	9, 10	2
5	6	7, 8, 9, 10	4
	8	9, 10	2
7	8	9, 10	2

The total number of selections of three cards that result in a win is

$$(8 + 6 + 4 + 2) + (6 + 4 + 2) + (4 + 2) + (2) = 20 + 12 + 6 + 2 = 40.$$

Therefore, the probability that a player wins the game on their first try is $\dfrac{40}{120} = \dfrac{1}{3}$.

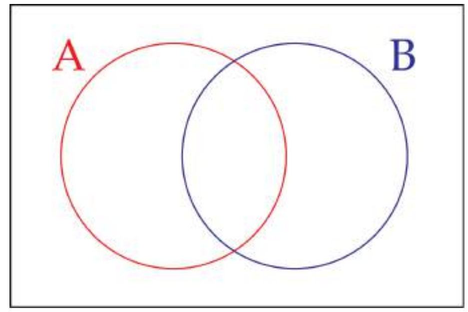
Problem of the Week

Problem E

Everything in its Place 3

(a) A Venn diagram has two circles, labelled A and B. Each circle contains functions, $f(x)$, that satisfy the following criteria.

A: $f(2) = -3$
B: $f(-2) = -1$

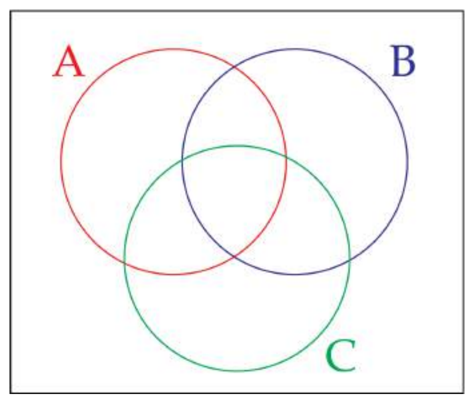


The overlapping region in the middle contains functions that are in both A and B, and the region outside both circles contains functions that are neither in A nor B.

In total this Venn diagram has four regions. Place functions in as many of the regions as you can. Is it possible to find a function for each region?

(b) A Venn diagram has three circles, labelled A, B, and C. Each circle contains ordered pairs, (x, y), where x and y are real numbers, that satisfy the following criteria.

A: $y = (x+3)^3 + 2$
B: $y = \frac{1}{2}x^2 + 1$
C: $y = |x+1|$



In total this Venn diagram has eight regions. Place ordered pairs in as many of the regions as you can. Is it possible to find an ordered pair for each region?

Problem of the Week

Problem E and Solution

Everything in its Place 3

Problem

(a) A Venn diagram has two circles, labelled A and B. Each circle contains functions, $f(x)$, that satisfy the following criteria.

A: $f(2) = -3$

B: $f(-2) = -1$

The overlapping region in the middle contains functions that are in both A and B, and the region outside both circles contains functions that are neither in A nor B. In total this Venn diagram has four regions. Place functions in as many of the regions as you can. Is it possible to find a function for each region?

(b) A Venn diagram has three circles, labelled A, B, and C. Each circle contains ordered pairs, (x, y), where x and y are real numbers, that satisfy the following criteria.

A: $y = (x + 3)^3 + 2$

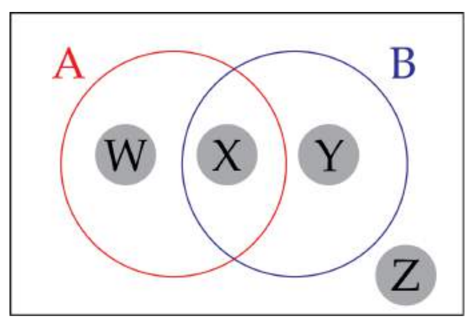
B: $y = \frac{1}{2}x^2 + 1$

C: $y = |x + 1|$

In total this Venn diagram has eight regions. Place ordered pairs in as many of the regions as you can. Is it possible to find an ordered pair for each region?

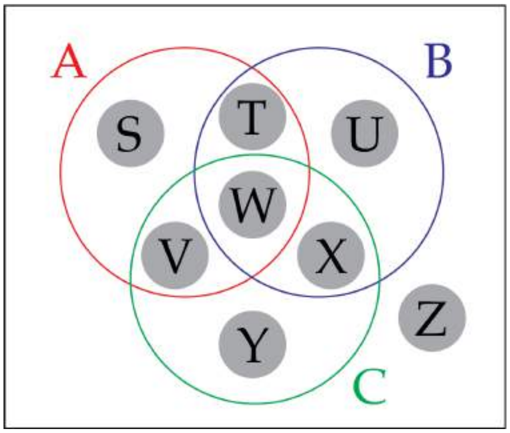
Solution

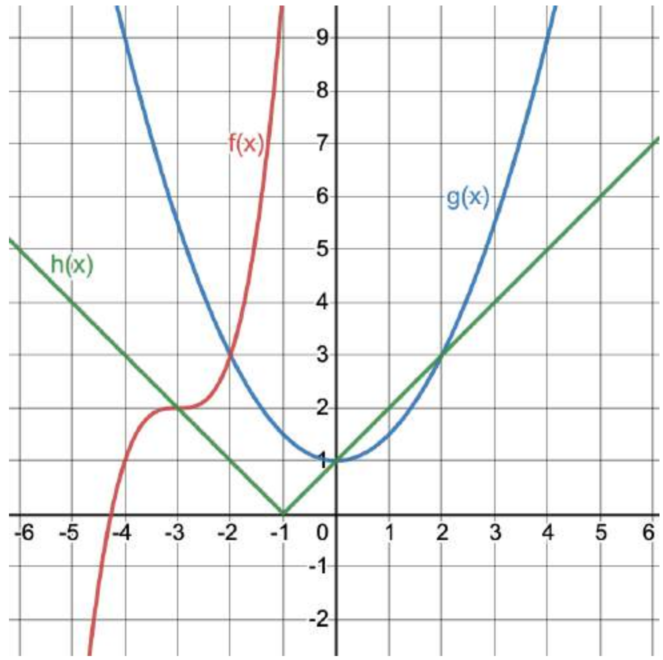
(a) We have marked the four regions W, X, Y, and Z.
When creating functions, you can think of the problem algebraically, or graphically. When thinking algebraically, a function that satisfies $f(2) = -3$ is one that evaluates to -3 when 2 is substituted for x. When thinking graphically, a function that satisfies $f(2) = -3$ is one whose graph goes through the point $(2, -3)$.



- Any function in region W must satisfy $f(2) = -3$ but *not* $f(-2) = -1$. There are infinitely many possibilities. Some examples are $f(x) = -x - 1$ and $f(x) = x^2 - 7$.
- Any function in region X must satisfy both $f(2) = -3$ and $f(-2) = -1$. In other words, the graph of the function must pass through both $(2, -3)$ and $(-2, -1)$. There are infinitely many possibilities. Some examples are $f(x) = -\frac{1}{2}x - 2$ and $f(x) = -\frac{1}{8}x^3 - 2$.
- Any function in region Y must satisfy $f(-2) = -1$ but *not* $f(2) = -3$. There are infinitely many possibilities. Some examples are $f(x) = \frac{1}{2}x$ and $f(x) = x^2 - 5$.
- Any function in region Z must satisfy neither $f(2) = -3$ nor $f(-2) = -1$. There are again infinitely many possibilities. Some examples are $f(x) = x$ and $f(x) = x^2$.

(b) We have marked the eight regions S, T, U, V, W, X, Y, and Z. We will name the functions as follows: $f(x) = (x+3)^3 + 2$, $g(x) = \frac{1}{2}x^2 + 1$, and $h(x) = |x+1|$. We have also provided a graph of the functions.





Note that the graphs of f and g intersect at $(-2, 3)$, the graphs of f and h intersect at $(-3, 2)$, and the graphs of g and h intersect at $(0, 1)$ and $(2, 3)$.

- Any ordered pair in region S must satisfy the equation $y = f(x)$, but not $y = g(x)$ or $y = h(x)$. Since the point $(-4, 1)$ is on the graph of f, but not on the graph of g or h, the ordered pair $(-4, 1)$ works. There are infinitely many others as well.
- Any ordered pair in region T must satisfy the equations $y = f(x)$ and $y = g(x)$, but not $y = h(x)$. Since the point $(-2, 3)$ is on the graph of f and on the graph of g, but not on the graph of h, the ordered pair $(-2, 3)$ works. In fact, since $(-2, 3)$ is the only point of intersection of the graphs of f and g, this is the only possible choice for region T.
- Any ordered pair in region U must satisfy the equation $y = g(x)$, but not $y = f(x)$ or $y = h(x)$. The ordered pair $(4, 9)$ works. There are infinitely many others as well.
- Any ordered pair in region V must satisfy the equations $y = f(x)$ and $y = h(x)$, but not $y = g(x)$. The ordered pair $(-3, 2)$ works. In fact, since $(-3, 2)$ is the only point of intersection of the graphs of f and h, this is the only possible choice for region V.
- Any ordered pair in region W must satisfy the equations $y = f(x)$, $y = g(x)$, and $y = h(x)$. There are no ordered pairs that satisfy this as the graphs of the three functions do not have a common point of intersection. So this region must remain empty.
- Any ordered pair in region X must satisfy the equations $y = g(x)$ and $y = h(x)$, but not $y = f(x)$. The ordered pairs $(0, 1)$ and $(2, 3)$ work. In fact, since these are the only points of intersection of the graphs of g and h, these are the only possible choices for region X.
- Any ordered pair in region Y must satisfy the equation $y = h(x)$, but not $y = f(x)$ or $y = g(x)$. The ordered pair $(-2, 1)$ works. There are infinitely many others as well.
- Any ordered pair in region Z must not satisfy the equations $y = f(x)$, $y = g(x)$, or $y = h(x)$. The ordered pair $(0, 0)$ works. There are infinitely many others as well.

Problem of the Week

Problem E

Picture This

Eight people, Alex, Braiden, Christine, Gary, Henry, Mary, Sam, and Zachary are lining up in a row for a picture. Due to the dynamics of the people involved, there are certain restrictions in the way the people will line up. Anyone with a name that ends in ‘y’ will not stand next to anyone else with a name that ends in ‘y’. (Notice that four names end in a ‘y’: Gary, Henry, Mary, and Zachary.) Also the twins, Alex and Gary, will not stand beside each other.

If the photographer randomly organizes the people, what is the probability that she arranges the people in a valid order?

The diagram below illustrates one possible valid arrangement of the people.

Problem of the Week

Problem E and Solution

Picture This

Problem

Eight people, Alex, Braiden, Christine, Gary, Henry, Mary, Sam, and Zachary are lining up in a row for a picture. Due to the dynamics of the people involved, there are certain restrictions in the way the people will line up. Anyone with a name that ends in 'y' will not stand next to anyone else with a name that ends in 'y'. (Notice that four names end in a 'y': Gary, Henry, Mary, and Zachary.) Also the twins, Alex and Gary, will not stand beside each other.

If the photographer randomly organizes the people, what is the probability that she arranges the people in a valid order?

Solution

Four of the eight people have a name that ends in 'y', and these people will not stand next to each other. The problem is further complicated by the fact that Gary and Alex cannot stand together. Since Gary also has a name that ends in 'y', we will break the problem into cases, based on Gary's position. We will number the positions from 1 to 8, starting on the left.

Let Gary's position in the line be marked with a G. Let the positions of the other people with names that end in 'y' be marked with a Y. Positions that have not yet been filled with a person will be marked with a _.

1. **Gary is in position 1.**

   We place Gary in position 1 and systematically list out the possible placements of the three other people with names that end in 'y', and see that there are 4 possible configurations:

   $$G\ \_\ Y\ \_\ Y\ \_\ Y\ \_$$
   $$G\ \_\ Y\ \_\ Y\ \_\ \_\ Y$$
   $$G\ \_\ Y\ \_\ \_\ Y\ \_\ Y$$
   $$G\ \_\ \_\ Y\ \_\ Y\ \_\ Y$$

   For each of these 4 configurations, there are 3 possible people to fill in the empty spot immediately next to Gary (these people do not have a name that ends in 'y', nor are they Alex). For each of these, there are $3 \times 2 \times 1$ ways to place the other people with a name that ends in 'y' and $3 \times 2 \times 1$ ways to place the remaining people, including Alex. Therefore, there are $4 \times 3 \times (3 \times 2 \times 1) \times (3 \times 2 \times 1) = 432$ ways to place the people properly with Gary in position 1.

2. **Gary is in position 8.**

   The possible placements of Gary and the three other people with names that end in 'y' will be the reverse of the configurations above when Gary is in position 1. Using a similar argument, we see that there are 432 ways to place the people properly with Gary in position 8.

3. **Gary is in position 2.**

   We place Gary in position 2 and systematically list out the possible placements of the three other people with names that end in 'y', and see that there is only 1 possible configuration:

   $$\_\; G \;\_\; Y \;\_\; Y \;\_\; Y$$

   For this configuration, there are 3 possible people to fill in the empty spot to the left of Gary (these people do not have a name that ends in 'y', nor are they Alex). Once that person has been placed, there are 2 possible people to put in the empty spot immediately to the right of Gary. For each of these arrangements, there are $3 \times 2 \times 1$ ways to place the other people with a name that ends in 'y' and 2×1 ways to place the remaining people, including Alex.

   Therefore, there are $1 \times (3 \times 2) \times (3 \times 2 \times 1) \times (2 \times 1) = 72$ ways to place the people properly with Gary in position 2.

4. **Gary is in position 7.**

   The possible placements of Gary and the three other people with names that end in 'y' will be the reverse of the configurations above when Gary is in position 2. Using a similar argument, we see that there are 72 ways to place the people properly with Gary in position 7.

5. **Gary is in position 3.**

   We place Gary in position 3 and systematically list out the possible placements of the three other people with names that end in 'y', and see that there are only 3 possible configurations:

   $$Y \;\_\; G \;\_\;\_\; Y \;\_\; Y$$
   $$Y \;\_\; G \;\_\; Y \;\_\; Y \;\_$$
   $$Y \;\_\; G \;\_\; Y \;\_\;\_\; Y$$

   For each of these 3 configurations, there are 3 possible people to fill in the empty spot to the left of Gary (these people do not have a name that ends in 'y', nor are they Alex). Once that person has been placed, there are 2 possible people to put in the empty spot immediately to the right of Gary. For each of these arrangements, there are $3 \times 2 \times 1$ ways to place the other people with a name that ends in 'y' and 2×1 ways to place the remaining people, including Alex.

   Therefore, there are $3 \times (3 \times 2) \times (3 \times 2 \times 1) \times (2 \times 1) = 216$ ways to place the people properly with Gary in position 3.

6. **Gary is in position 6.**

   The possible placements of Gary and the three other people with names that end in 'y' will be the reverse of the configurations above when Gary is in position 3. Using a similar argument, we see that there are 216 ways to place the people properly with Gary in position 6.

7. **Gary is in position 4.**

   We place Gary in position 4 and systematically list out the possible placements of the three other people with names that end in 'y', and see that there are only 2 possible configurations:

   $$\_\,Y\,\_\,G\,\_\,Y\,\_\,Y$$
   $$Y\,\_\,\_\,G\,\_\,Y\,\_\,Y$$

   Using an analysis similar to that in previous cases, we see that there are $2 \times (3 \times 2) \times (3 \times 2 \times 1) \times (2 \times 1) = 144$ ways to place the people properly with Gary in position 4.

8. **Gary is in position 5.**

   The possible placements of Gary and the three other people with names that end in 'y' will be the reverse of the configurations above when Gary is in position 4. Therefore, there are 144 ways to place the people properly with Gary in position 5.

The cases have no overlapping possibilities and we have considered all of the possible placements of Gary. Therefore, there are

$$432 + 432 + 72 + 72 + 216 + 216 + 144 + 144 = 1728$$

ways for the people to line up correctly.

If the people could stand in any position in the line, the number of possible ways to line up is

$$8 \times 7 \times 6 \times \times 5 \times 4 \times 3 \times 2 \times 1 = 40\,320$$

Therefore, the probability that the photographer randomly lines up the people in a valid order is

$$\frac{1728}{40\,320} = \frac{3}{70} \approx 0.043$$

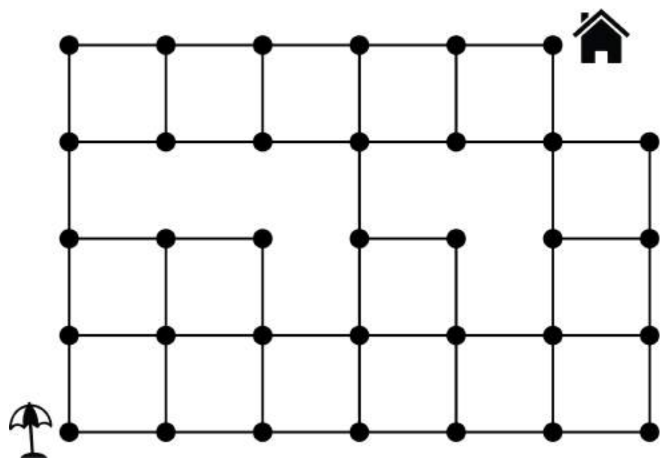
Therefore, there is approximately a 4.3% chance of the photographer arranges the people in a valid order.

Computational Thinking (C)

TAKE ME TO THE COVER

Problem of the Week
Problem E
The Long Way Home

Emilia is playing a board game where each player needs to move their game piece along the black roads from the beach in the bottom-left corner to the house in the top-right corner. The black dots represent intersections where roads meet.



In each turn a player is allowed to move their piece along a road until it reaches an intersection. Then it's another player's turn.

After Emilia finished the game, she realized that at every intersection she had turned either left or right; she never continued straight along the direction she came from in her previous turn. As well, she never went along the same part of a road more than once.

What is the fewest number of turns Emilia could have had during the game?

This problem was inspired by a past Beaver Computing Challenge (BCC) problem.

Problem of the Week
Problem E and Solution
The Long Way Home

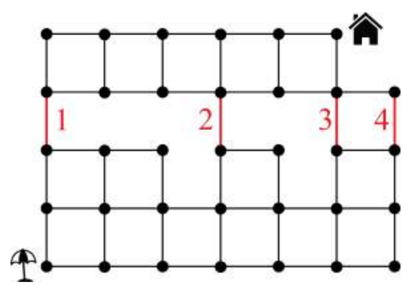
Problem

Emilia is playing a board game where each player needs to move their game piece along the black roads from the beach in the bottom-left corner to the house in the top-right corner. The black dots represent intersections where roads meet.

In each turn a player is allowed to move their piece along a road until it reaches an intersection. Then it's another player's turn.
After Emilia finished the game, she realized that at every intersection she had turned either left or right; she never continued straight along the direction she came from in her previous turn. As well, she never went along the same part of a road more than once.
What is the fewest number of turns Emilia could have had during the game?
This problem was inspired by a past Beaver Computing Challenge (BCC) problem.

Solution

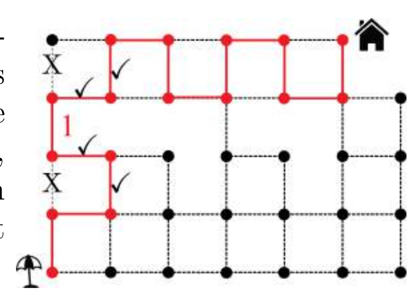
To get from the beach in the bottom-left corner to the house in the top-right corner, Emilia must have passed through at least one of the four numbered roads shown.
We will find the length of the shortest route through each of the four roads from the beach to the house, and then determine the shortest route overall. The length of this route will then be equal to the fewest number of turns Emilia could have had during the game, because in each turn she moved along the road until she reached the next intersection.



Some details, for the sake of brevity, will be omitted from the solution and left for the solver to consider further. In the solution, turns will be described in terms of north, south, east, and west, where north points to the top of the page.

- **Shortest Route Through Road 1:**

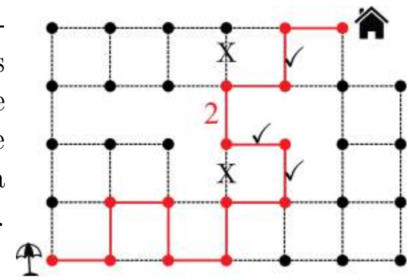
  If Emilia traveled through road 1, then for the fewest number of turns, she could not have traveled on the roads marked with an X in the diagram shown, because she never went straight through an intersection. Similarly, the roads that she must have traveled on are marked with a ✓. The route shown on the diagram is the shortest route. This route has a length of 15.



  In the part of the route between the beach and road 1, the shortest route has a length of 4 and goes north - east - north - west. It would also be possible to extend this route by going north - east - south - east - north - west - north - west, but this route is clearly longer. There is no shorter route starting from the beach that goes east first and passes through road 1. In traveling from road 1 to the house, the shortest route is shown and has a length of 10. At some points along this route, alternate choices can be made but these choices lead to invalid situations or lengthen the route. The solver may wish to confirm this.

- **Shortest Route Through Road 2:**

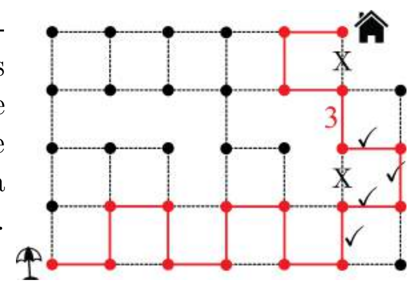
  If Emilia traveled through road 2, then for the fewest number of turns, she could not have traveled on the roads marked with an X in the diagram shown, because she never went straight through an intersection. Similarly the roads that she must have traveled on are marked with a ✓. The route shown on the diagram is the shortest route. This route has a length of 13.

  In the part of the route from the beach to road 2, the shortest route has a length of 9. In traveling from road 2 to the house, the shortest route has a length of 3. At some points along the route shown through road 2, alternate choices can be made but these choices either lead to invalid situations or lengthen the route. The solver may wish to confirm this.

- **Shortest Route Through Road 3:**

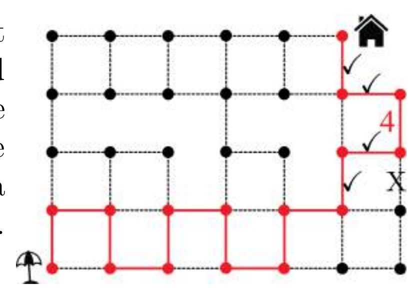
  If Emilia traveled through road 3, then for the fewest number of turns, she could not have traveled on the roads marked with an X in the diagram shown, because she never went straight through an intersection. Similarly the roads that she must have traveled on are marked with a ✓. The route shown on the diagram is the shortest route. This route has a length of 17.

  In the part of the route from the beach to road 3, the shortest route has a length of 13. In traveling from road 3 to the house, the shortest route has a length of 3. At some points along the route from the beach to road 3 and also from road 3 to the house, alternate choices can be made but these choices either lead to invalid situations or lengthen the route. The solver may wish to confirm this.

- **Shortest Route Through Road 4:**

  If Emilia traveled through road 4, then for the fewest number of turns, she could not have traveled on the road marked with an X in the diagram shown, because she never went straight through an intersection. Similarly the roads that she must have traveled on are marked with a ✓. The route shown on the diagram is the shortest route. This route has a length of 15.

  In the part of the route from the beach to road 4, the shortest route has a length of 12. In traveling from road 4 to the house, the shortest route has a length of 2. At some points along the route from the beach to the house through road 4, alternate choices can be made but these choices either lead to invalid situations, lengthen the route, or have the same length. The solver may wish to confirm this.

After examining the four possible cases, the shortest route has a length of 13 and passes through road 2. Thus, the fewest number of turns Emilia could have had during the game is 13.

Problem of the Week
Problem E
To Tell the Truth?

Four people, Nathaniel, Elizabeth, Libby, and Joel, each said two statements such that:

- one person lied in both statements;
- one person told the truth in both statements; and
- two people told the truth in one statement and a lie in the other statement.

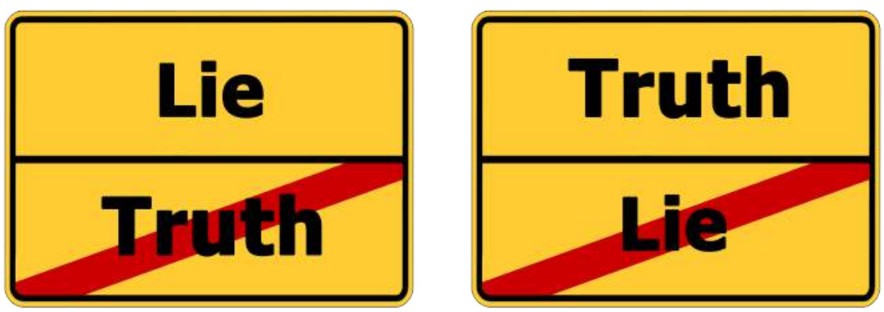
Nathaniel said “Elizabeth lied exactly once” and “Joel lied twice”.

Elizabeth said “I never lie” and “Nathaniel never lied”.

Libby said “Joel lied twice” and “Elizabeth never lied”.

Joel said “Nathaniel lied twice” and “I never lie”.

Who lied twice? Who never lied? Who lied exactly once?



Problem of the Week
Problem E and Solution
To Tell the Truth?

Problem

Four people, Nathaniel, Elizabeth, Libby, and Joel, each said two statements such that:

- one person lied in both statements;
- one person told the truth in both statements; and
- two people told the truth in one statement and a lie in the other statement.

Nathaniel said "Elizabeth lied exactly once" and "Joel lied twice".

Elizabeth said "I never lie" and "Nathaniel never lied".

Libby said "Joel lied twice" and "Elizabeth never lied".

Joel said "Nathaniel lied twice" and "I never lie".

Who lied twice? Who never lied? Who lied exactly once?

Solution

We will solve this problem by first figuring out who must have lied twice, and then figuring out who must have told the truth twice.

1. Who lied twice?

   (a) Assume that Nathaniel lied twice. If so, "Elizabeth lied once" is a lie. Therefore, "Elizabeth never lied" or "Elizabeth lied twice." But "Elizabeth never lied" cannot be true because Elizabeth says, "Nathaniel never lied." This contradicts our assumption that Nathaniel lied twice. Also, "Elizabeth lied twice" cannot be true since that means Elizabeth and Nathaniel both lied twice and this contradicts the fact that only one person lied twice. Therefore, our assumption that Nathaniel lied twice is false.

   (b) Assume that Elizabeth lied twice. If so, "Nathaniel never lied" is a lie. Then, Nathaniel lied twice or Nathaniel lied once. Nathaniel cannot have lied twice since both he and Elizabeth would have lied twice and this contradicts the fact that only one person lied twice. But "Nathaniel lied once" is also false since "Elizabeth lied exactly once" is a lie (we assumed she lied twice) and "Joel lied twice" is a lie because it contradicts the assumption that Elizabeth lied twice (and only one person can lie twice). Therefore, our assumption that Elizabeth lied twice is false.

   (c) Assume that Libby lied twice. If so, "Elizabeth never lied" is a lie and "Joel lied twice" is a lie. Since "Elizabeth never lied" is a lie, then she lied twice or she lied once. But if Elizabeth lied twice our assumption that Libby lied twice cannot be true since only one person lied twice. If Elizabeth lied exactly once, then "I never lie" must be the lie and "Nathaniel never lied" must be true. But if Nathaniel never lied, then "Joel lied twice" must be true and this contradicts the fact that only one person can lie twice. Therefore, our assumption that Libby lied twice is false.

We are told that one person lied twice and none of Nathaniel, Elizabeth or Libby lied twice. Therefore, by elimination, Joel is the one who lied twice.

2. Who never lied?

   (a) Assume that Elizabeth never lied. Then her statement that "Nathaniel never lied" must be true. Then, there are then two people who never lied. This contradicts the fact that only one person never lied. Therefore, our assumption that Elizabeth never lied is false.

   (b) Assume that Libby never lied. Then her statement that "Elizabeth never lied" must be true. Then, there are then two people who never lied. This contradicts the fact that only one person never lied. Therefore, our assumption that Libby never lied is false.

   Joel lied twice. Elizabeth and Libby lied. Therefore, by elimination, Nathaniel is the one who never lied. It then follows that Elizabeth and Libby each make one true statement and tell one lie.

We can now check our results.

Nathaniel never lied. Then his statements are both true. Elizabeth lied exactly once is true and Joel lied twice is true.

Elizabeth lied once. Then one of her statements is true and the other is a lie. Her statement that she never lies is a lie and her statement that Nathaniel never lies is true.

Libby lied once. Then one of her statements is true and the other is a lie. Her statement that Joel lied twice is true and her statement that Elizabeth never lied is a lie.

Joel lied twice. Then both of his statements are lies. Nathaniel lied twice is a lie. And his statement the he never lies is a lie.

Problem of the Week

Problem E

Ducks in a Row

There are 10 rubber ducks arranged in a row on a desk. On the bottom of each duck there is one of the following letters: A, B, C, D, E, F, G, H, I, or J. Each letter occurs exactly once.

Amina has written the following algorithm to find a particular letter.

Pick up the leftmost duck and look on the bottom. If it's the letter you're looking for, put the duck back down and stop. Otherwise pick up the next duck and look on the bottom. If it's the letter you're looking for, put the duck back down, swap it with the duck to its left, and stop. Repeat this until you find the letter you're looking for.

> For example, suppose the ducks were in the following order.
>
> G, B, F, J, A, I, E, H, C, D
>
> Amina uses her algorithm to find the letter E. She looks at the first 6 ducks and returns each of them to the desk. When she looks at the seventh duck and sees that it is the E, she swaps ducks E and I. So to locate the letter E, Amina looked at 7 ducks, and the ducks would now be in the following order.
>
> G, B, F, J, A, E, I, H, C, D
>
> Suppose Amina now wants to find the letter J. She looks at the G, B, and F and puts them back down. She then looks at the fourth duck, sees the J, and swaps ducks F and J. The ducks would now be in the following order.
>
> G, B, J, F, A, E, I, H, C, D
>
> After searching for the E and the J, Amina has looked at at a total of $7 + 4 = 11$ ducks.

If the 10 ducks begin in some unknown order and Amina uses her algorithm to search for each of the ten letters exactly once, what is the maximum possible number of ducks that Amina picks up to look on the bottom?

Problem of the Week
Problem E and Solution
Ducks in a Row

Problem

There are 10 rubber ducks arranged in a row on a desk. On the bottom of each duck there is one of the following letters: A, B, C, D, E, F, G, H, I, or J. Each letter occurs exactly once. Amina has written the following algorithm to find a particular letter.

Pick up the leftmost duck and look on the bottom. If it's the letter you're looking for, put the duck back down and stop. Otherwise pick up the next duck and look on the bottom. If it's the letter you're looking for, put the duck back down, swap it with the duck to its left, and stop. Repeat this until you find the letter you're looking for.

If the 10 ducks begin in some unknown order and Amina uses her algorithm to search for each of the ten letters exactly once, what is the maximum possible number of ducks that Amina picks up to look on the bottom?

Solution

If no swaps were required as a result of finding a duck, how many ducks would Amina have to look at in total?

Let's number the positions 1 through 10, starting with the leftmost position as number 1. At some point Amina is looking for the duck in position 1. She would have to look at 1 duck to find it. At some point she is looking for the duck in position 2. She would have to look at 2 ducks to find it. At some point she is looking for the duck in position 3. She would have to look at 3 ducks to find it. This continues until at some point she is looking for the duck in position 10. She would have to look at 10 ducks to find it. To locate all 10 ducks, Amina would have to look at $1 + 2 + 3 + \cdots + 10 = 55$ ducks.

Since Amina looks for each letter exactly once, swapping the position of one letter with the position of another letter can only have the effect of increasing the number of ducks looked at for the letter on the preceding duck by one. The number of ducks looked at to find other letters would not be affected. Therefore, swapping can only increase the number of ducks looked at (by one) for all but the first search. This means swapping can increase the number of ducks looked at by at most 9 in total making the maximum total number of ducks looked at equal to $55 + 9 = 64$.

On the next page, an illustration of how this maximum can be achieved is shown.

Is 64 an achievable maximum?

First we will put the ducks in order, left to right, from A to J.

A, B, C, D, E, F, G, H, I, J,

Now we will search for each letter in order from B to J and search for A last.

Since B is in the second position, we must look at 2 ducks to find it. We then swap A and B.

B, A, C, D, E, F, G, H, I, J

Since C is in the third position, we must look at 3 ducks to find it. We then swap A and C.

B, C, A, D, E, F, G, H, I, J

Since D is in the fourth position, we must look at 4 ducks to find it. We then swap A and D.

B, C, D, A, E, F, G, H, I, J

Since E is in the fifth position, we must look at 5 ducks to find it. We then swap A and E.

B, C, D, E, A, F, G, H, I, J

Since F is in the sixth position, we must look at 6 ducks to find it. We then swap A and F.

B, C, D, E, F, A, G, H, I, J

Since G is in the seventh position, we must look at 7 ducks to find it. We then swap A and G.

B, C, D, E, F, G, A, H, I, J

Since H is in the eighth position, we must look at 8 ducks to find it. We then swap A and H.

B, C, D, E, F, G, H, A, I, J

Since I is in the ninth position, we must look at 9 ducks to find it. We then swap A and I.

B, C, D, E, F, G, H, I, A, J

Since J is in the tenth position, we must look at 10 ducks to find it. We then swap A and J.

B, C, D, E, F, G, H, I, J, A

Finally, since A is in the tenth position, we must look at 10 ducks to find it. We then swap A and J (again).

B, C, D, E, F, G, H, I, A, J

We have looked at a total of $2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 10 = 64$ ducks to find each of the letters.

On the next page, we look at a possible extension and a connection to Computer Science.

Extension:

Suppose you have n ducks, each with something different on them. You lay the ducks out in a similar manner to how we handled the 10 different ducks. You search for each of the different ducks, one at a time. What is the maximum number of ducks you must look at in order to locate all of the ducks using the search described in the problem?

Connection to Computer Science:

This problem was inspired by a problem from the Beaver Computer Challenge in 2012. You can see the problem and others like it at http://www.cemc.uwaterloo.ca/contests/bcc.html.

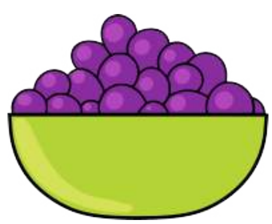
One of the fundamental problems in computer science is how to organize data in order to search within it quickly. There are many ways to do this: using binary trees, splay trees, skip lists, sorted arrays, etc. The technique outlined in this problem is the idea of moving found items closer to the "front," with the assumption that if we search for something once, it is quite likely that the same item will be searched for again. The transpose (swap) heuristic used by Amina in this problem is one technique for doing this. Other heuristics include move-to-front, which moves a found element to the very front of the list. Moreover, this problem highlights the process of performing worst-case analysis for an algorithm. Computer scientists care about "what is the worst possible input for this algorithm, and how long will it take to execute on that input?" In this question, we are asking about the worst-case performance of the transpose heuristic on a list of size 10.

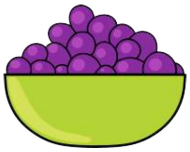
Problem of the Week
Problem E
Taking Grapes

Quinn and Birgitta are playing a game using a bowl of grapes. The rules of the game are as follows:

- Write the numbers 1, 2, 3, 4, and 5 on a whiteboard.
- Players take turns choosing a number from the whiteboard, removing that number of grapes from the bowl, and then erasing that number from the whiteboard.
- The game continues until all the numbers are erased or a player is not able to take any of the remaining numbers of grapes from the bowl.
- The last player who erased a number from the whiteboard is the winner.

The game starts with 8 grapes and Quinn goes first. Quinn has developed a winning strategy so that she is guaranteed to win this game, regardless of how many grapes Birgitta takes on her turns. Find all the possible first moves in Quinn's winning strategy. Justify your answer.





Problem of the Week
Problem E and Solution
Taking Grapes

Problem

Quinn and Birgitta are playing a game using a bowl of grapes. The rules of the game are as follows:

- Write the numbers $1,\ 2,\ 3,\ 4,$ and 5 on a whiteboard.
- Players take turns choosing a number from the whiteboard, removing that number of grapes from the bowl, and then erasing that number from the whiteboard.
- The game continues until all the numbers are erased or a player is not able to take any of the remaining numbers of grapes from the bowl.
- The last player who erased a number from the whiteboard is the winner.

The game starts with 8 grapes and Quinn goes first. Quinn has developed a winning strategy so that she is guaranteed to win this game, regardless of how many grapes Birgitta takes on her turns. Find all the possible first moves in Quinn's winning strategy. Justify your answer.

Solution

Quinn has five options to begin with. She can remove $1,\ 2,\ 3,\ 4,$ or 5 grapes. Let's look at each case.

- *Case 1:* Quinn removes 5 grapes on her first turn.

  If Quinn removes 5 grapes, then Birgitta can remove the remaining 3 grapes and win the game. Therefore, Quinn could lose if she starts by removing 5 grapes.

- *Case 2:* Quinn removes 4 grapes on her first turn.

  If Quinn removes 4 grapes, then there will be 4 grapes remaining. Birgitta can then remove $1,\ 2,$ or 3 grapes since Quinn will have erased the number 4 from the whiteboard.

  If Birgitta removes 1 grape, then there will be 3 grapes remaining. Quinn can then remove the remaining 3 grapes and win the game.

  If Birgitta removes 2 grapes, then there will be 2 grapes remaining. Quinn's only option will be to remove 1 grape, since the number 2 will have already been erased from the whiteboard. Then there will be 1 grape remaining, but Birgitta will not be able to remove it, since the number 1 will have been erased from the whiteboard on the previous turn. Therefore, Quinn will win the game.

  If Birgitta removes 3 grapes, then there will be 1 grape remaining. Quinn can then remove the remaining grape and win the game.

  Therefore, no matter what Birgitta does on her turn, Quinn can win if she starts the game by removing 4 grapes.

- *Case 3:* Quinn removes 3 grapes on her first turn.

  If Quinn removes 3 grapes, then Birgitta can remove the remaining 5 grapes and win the game. Therefore, Quinn could lose if she starts by removing 3 grapes.

- *Case 4:* Quinn removes 2 grapes on her first turn.

  If Quinn removes 2 grapes, then there will be 6 grapes remaining. Birgitta can then remove 1, 3, 4, or 5 grapes since Quinn will have erased the number 2 from the whiteboard.

  If Birgitta removes 1 grape, then there will be 5 grapes remaining. Quinn can then remove the 5 remaining grapes and win the game.

  If Birgitta removes 3 grapes, then there will be 3 grapes remaining. Quinn's only option will be to remove 1 grape, since the numbers 2 and 3 will have already been erased from the whiteboard. Then there will be 2 grapes remaining, but Birgitta will not be able to remove any of them, since the numbers 1 and 2 will have already been erased from the whiteboard. Therefore, Quinn will win the game.

  If Birgitta removes 4 grapes, then there will be 2 grapes remaining. Quinn's only option will be to remove 1 grape, since the number 2 will have already been erased from the whiteboard. Then there will be 1 grape remaining, but Birgitta will not be able to remove it since the number 1 will have been erased from the whiteboard on the previous turn. Therefore, Quinn will win the game.

  If Birgitta removes 5 grapes, then there will be 1 grape remaining. Quinn can then remove the remaining grape and win the game.

  Therefore, no matter what Birgitta does on her turn, Quinn can win if she starts the game by removing 2 grapes.

- *Case 5:* Quinn removes 1 grape on her first turn.

  If Quinn removes 1 grape, then there will be 7 grapes remaining. Birgitta can then remove 2, 3, 4, or 5 grapes.

  If Birgitta removes 2 grapes, then there will be 5 grapes remaining. Quinn can remove the remaining 5 grapes and win the game.

  If Birgitta removes 3 grapes, then there will be 4 grapes remaining. Quinn can remove the remaining 4 grapes and win the game.

  If Birgitta removes 4 grapes, then there will be 3 grapes remaining. Quinn can remove the remaining 3 grapes and win the game.

  If Birgitta removes 5 grapes, then there will be 2 grapes remaining. Quinn can remove the remaining 2 grapes and win the game.

  Therefore, no matter what Birgitta does on her turn, Quinn can win if she starts the game by removing 1 grape.

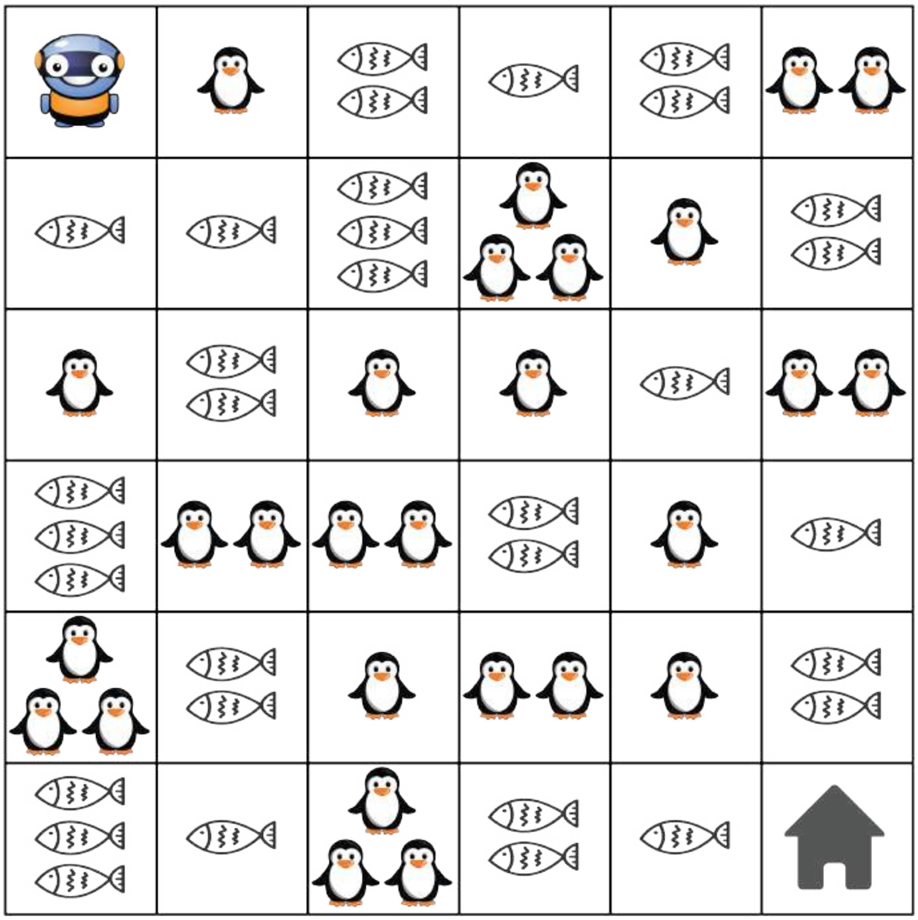
Therefore, Quinn's winning strategy starts by removing 1, 2, or 4 grapes.

Problem of the Week

Problem E

Feeding Penguins

Sabrina is playing a video game that uses a 6 by 6 grid. Her character starts in the top-left square and needs to get to the house in the bottom-right corner. All the other squares contain either fish or penguins, as shown in the following grid.



Sabrina can move only right or down through the grid. When she gets to a square with fish, she picks up all the fish. When she gets to a square with penguins, she must feed one fish to each penguin.

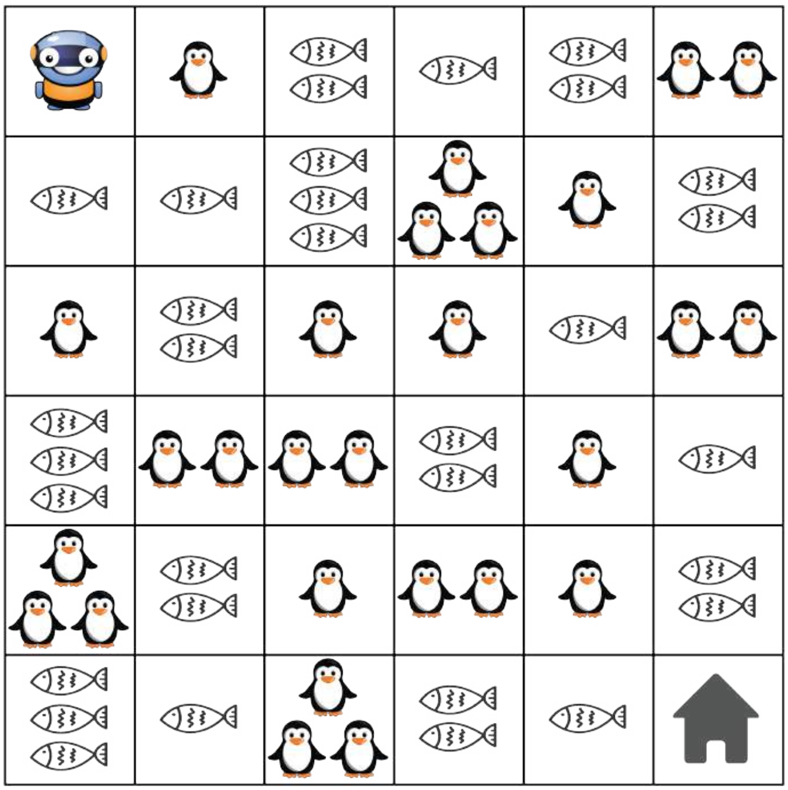
If Sabrina starts with 5 fish, what is the maximum possible number of fish she could have with her when she arrives at the house?

This problem was inspired by a past Beaver Computing Challenge (BCC) problem.

Problem of the Week
Problem E and Solution
Feeding Penguins

Problem

Sabrina is playing a video game that uses a 6 by 6 grid. Her character starts in the top-left square and needs to get to the house in the bottom-right corner. All the other squares contain either fish or penguins, as shown in the following grid.



Sabrina can move only right or down through the grid. When she gets to a square with fish, she picks up all the fish. When she gets to a square with penguins, she must feed one fish to each penguin.

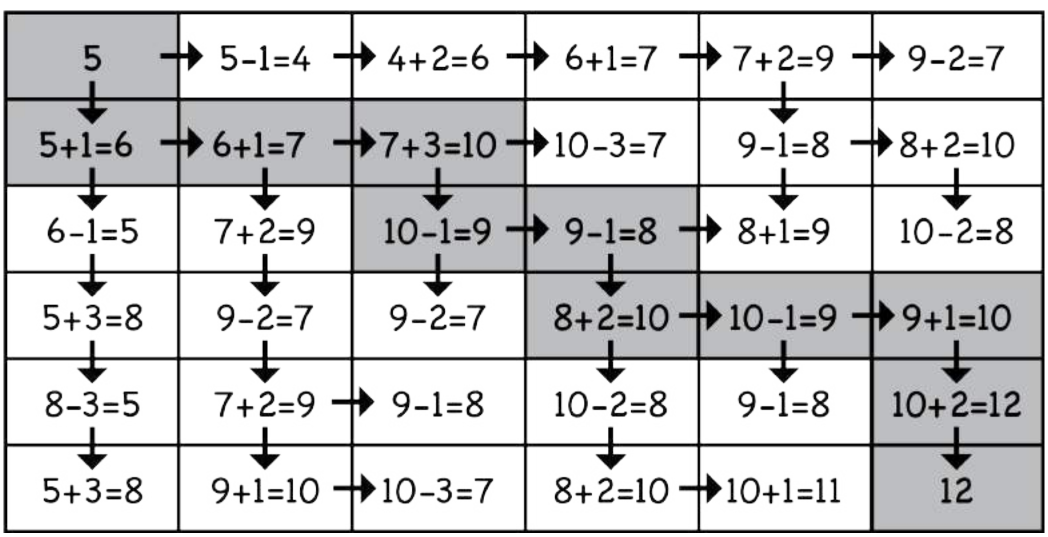
If Sabrina starts with 5 fish, what is the maximum possible number of fish she could have with her when she arrives at the house?

This problem was inspired by a past Beaver Computing Challenge (BCC) problem.

Solution

One way to solve this problem is by counting the number of fish Sabrina has as she moves through the grid, recording the maximum possible number of fish for each square. To start, Sabrina has 5 fish. If she moves to the right then she lands on a square with one penguin. After feeding it she has $5 - 1 = 4$ fish. If she instead moves down initially, then she lands on a square with one fish. After picking it up, she has $5 + 1 = 6$ fish. There are two ways to get to the square in the second row and second column. If Sabrina comes from the left, she will have 6 fish, but if she comes from above, she will have only 4 fish. Since $6 > 4$, the best way to reach the square in the second row and second column is to come from the left. We record this on our grid with an arrow. Then, since the square in the second row and second column has one fish, Sabrina will pick it up and then have $6 + 1 = 7$ fish.

In this way, for each square we calculate the total possible number of fish coming from the left or from above, and use the higher number in that square. Each arrow indicates the direction from which the higher number was calculated. The completed grid is shown. A shaded path through the squares in the grid is also indicated. Following the shaded path allows Sabrina to end up with the maximum possible number of fish when she arrives at the house.

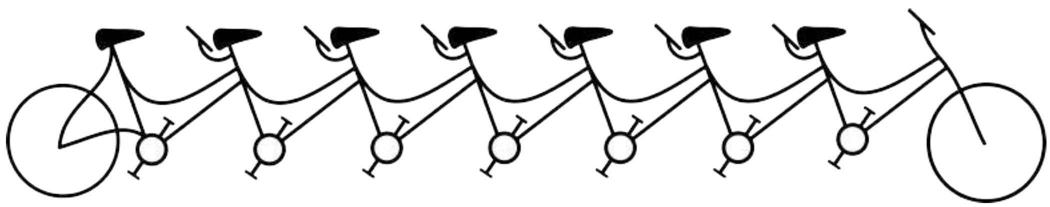


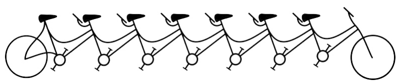
Thus, the maximum possible number of fish Sabrina could have with her when she arrives at the house is 12.

Problem of the Week
Problem E
Let's All Ride a Bike

Grayson's Groupcycles rents bikes with multiple seats for large groups of people. They rent 7-seater, 13-seater, and 25-seater bikes. A group of 14 people could fit on two 7-seater bikes, however a group of 15 people could not fit exactly on any of the bikes since no combination of bikes have exactly 15 seats.

What is the largest group size that *cannot* fit exactly on any combination of bikes from Grayson's Groupcycles?





Problem of the Week
Problem E and Solution
Let's All Ride a Bike

Problem

Grayson's Groupcycles rents bikes with multiple seats for large groups of people. They rent 7-seater, 13-seater, and 25-seater bikes. A group of 14 people could fit on two 7-seater bikes, however a group of 15 people could not fit exactly on any of the bikes since no combination of bikes have exactly 15 seats.

What is the largest group size that *cannot* fit exactly on any combination of bikes from Grayson's Groupcycles?

Solution

Any group size that is a multiple of 7 can fit on multiple 7-seater bikes. Similarly, any group size that is a multiple of 13 can fit on multiple 13-seater bikes, and any group size that is a multiple of 25 can fit on multiple 25-seater bikes. These multiples are listed below.

$$\text{multiples of } 7 : 7,\ 14,\ 21,\ 28,\ 35,\ 42,\ 49,\ \ldots$$
$$\text{multiples of } 13 : 13,\ 26,\ 39,\ 52,\ \ldots$$
$$\text{multiples of } 25 : 25,\ 50,\ 75,\ \ldots$$

Putting these numbers together, along with sums of the different multiples, gives us the following list of group sizes that can fit exactly on some combination of bikes from Grayson's Groupcycles.

$$7,\ 13,\ 14,\ 20\ (=7+13),\ 21,\ 25,\ 26,\ 27\ (=14+13),\ 28,\ 32\ (=7+25),\ 33\ (=7+26),$$
$$34\ (=21+13),\ 35,\ 38\ (=13+25),\ 39,\ 40\ (=14+26),\ 41\ (=28+13),\ 42,$$
$$45\ (=7+13+25),\ 46\ (=21+25),\ 47\ (=21+26),\ 48\ (=35+13),\ 49,\ 50,\ 51\ (=26+25)$$

The missing numbers from the above list correspond to the group sizes that cannot fit exactly on any combination of bikes. The largest of these group sizes appears to be 44, however we must justify that this is the maximum group size that cannot fit exactly on any combination of bikes. To do this, we note that group sizes of 45, 46, 47, 48, 49, 50, and 51 can all fit exactly on some combination of bikes. This corresponds to 7 consecutive group sizes. If we add 7 to each of these group sizes, the additional 7 people could fit on a 7-seater bike. It follows that group sizes of 52, 53, 54, 55, 56, 57, and 58 can also fit exactly on some combination of bikes. In this way, we can continue to add 7 to each of these group sizes to obtain the next set of 7 consecutive group sizes and determine that they can also fit exactly on some combination of bikes. It follows that every group size of 45 or more can fit exactly on some combination of bikes.

Thus, the largest group size that cannot fit exactly on any combination of bikes from Grayson's Groupcycles is 44.

Extension: It turns out there are only 26 group sizes that cannot fit exactly on any combination of bikes. If Grayson's Groupcycles added a 3-seater bike, how many group sizes wouldn't be able to fit exactly on any combination of bikes?