The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING

Problem of the Week
Problems and Solutions
2021 - 2022

Problem D (Grade 9/10)

Themes

(Click on a theme name to jump to that section.)

Number Sense (N)

Geometry & Measurement (G)

Algebra (A)

Data Management (D)

Computational Thinking (C)

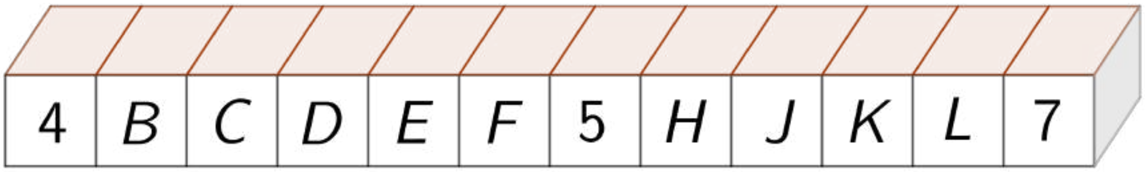
The problems in this booklet are organized into themes.
A problem often appears in multiple themes.

Number Sense (N)

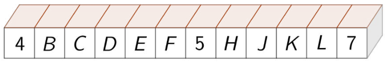
TAKE ME TO THE COVER

Problem of the Week
Problem D
Blocked Numbers

Twelve blocks are arranged as illustrated in the diagram.



Each letter shown on the front of a block represents a number. The sum of the numbers on any four consecutive blocks is 25. Determine the value of $B + F + K$.



Problem of the Week

Problem D and Solution

Blocked Numbers

Problem

Twelve blocks are arranged as illustrated in the diagram. Each letter shown on the front of a block represents a number. The sum of the numbers on any four consecutive blocks is 25. Determine the value of $B + F + K$.

Solution

Since the sum of the numbers on any four consecutive blocks is the same, looking at the first five blocks, we have

$$4 + B + C + D = B + C + D + E$$

Subtracting B, C, and D from both sides gives $E = 4$. Similarly, looking at the fifth through ninth blocks, we can show $J = 4$.

Again, since the sum of the numbers on any four consecutive blocks is the same, looking at the third through seventh blocks, we have

$$C + D + E + F = D + E + F + 5$$

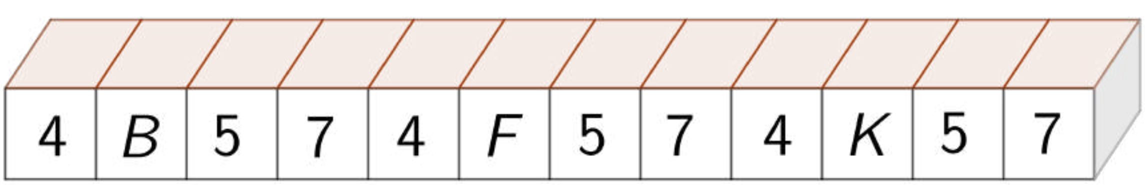
Subtracting D, E, and F from both sides gives $C = 5$. Similarly, looking at the seventh through eleventh blocks, we can show $L = 5$.

Once more, since the sum of the numbers on any four consecutive blocks is the same, looking at the eighth through twelfth blocks, we have

$$H + J + K + L = J + K + L + 7$$

Subtracting J, K, and L from both sides, gives $H = 7$. Similarly, looking at the fourth through eighth blocks, we can show $D = 7$.

Filling in the above information, the blocks now look like:



We will present two different solutions from this point.

Solution 1:
Since the sum of any four consecutive numbers is 25, using the first 4 blocks

$$\begin{aligned} 4 + B + 5 + 7 &= 25 \\ B + 16 &= 25 \\ B &= 9 \end{aligned}$$

Similarly, we can show $F = 9$ and $K = 9$.

Therefore, $B + F + K = 27$.

Solution 2:

We note that the twelve blocks are three sets of four consecutive blocks. Each of these three sets have a total of 25, so the total sum of the blocks is $3 \times 25 = 75$.

The sum is also

$$4 + B + 5 + 7 + 4 + F + 5 + 7 + 4 + K + 5 + 7 = 48 + B + F + K$$

This means

$$48 + B + F + K = 75$$

or

$$B + F + K = 27$$

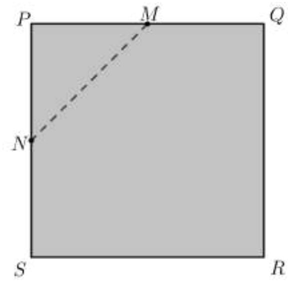
Therefore, $B + F + K = 27$.

Problem of the Week

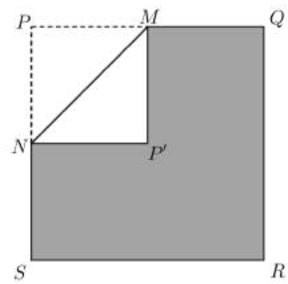
Problem D

From Square to Hexagon

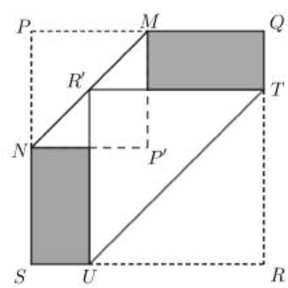
A square piece of paper, $PQRS$, has side length 40 cm. The page is grey on one side and white on the other side. Point M is the midpoint of side PQ and point N is the midpoint of side PS.



The paper is folded along MN so that P touches the paper at the point P'.



Point T lies on QR and point U lies on SR such that TU is parallel to MN, and when the paper is folded along TU, the point R touches the paper at the point R' on MN.



What is the area of hexagon $NMQTUS$?

Here are some known properties of the diagonals of a square that may be useful:

- the diagonals are equal in length; and
- the diagonals right bisect each other; and
- the diagonals bisect the corner angles.

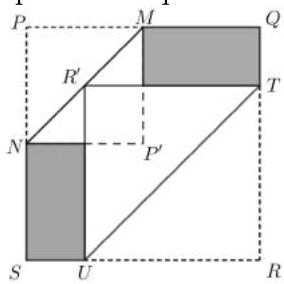
Problem of the Week

Problem D and Solution

From Square to Hexagon

Problem

A square piece of paper, $PQRS$, has side length 40 cm. The page is grey on one side and white on the other side. Point M is the midpoint of side PQ and point N is the midpoint of side PS. The paper is folded along MN so that P touches the paper at the point P'. Point T lies on QR and point U lies on SR such that TU is parallel to MN, and when the paper is folded along TU, the point R touches the paper at the point R' on MN.



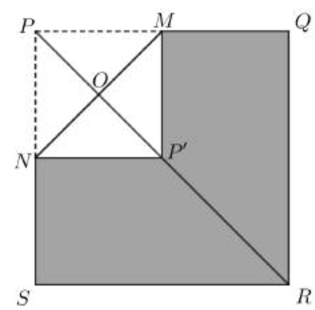
What is the area of hexagon $NMQTUS$?

Solution

To determine the area of hexagon $NMQTUS$, we will subtract the area of $\triangle PMN$ and the area of $\triangle TRU$ from the area of square $PQRS$.

Since M and N are the midpoints of PQ and PS, respectively, we know $PM = \frac{1}{2}(PQ) = 20$ cm and $PN = \frac{1}{2}(PS) = 20$ cm. Therefore, $PM = PN = 20$ and $\triangle PMN$ is an isosceles right-angled triangle. It follows that $\angle PNM = \angle PMN = 45°$.

After the first fold, P touches the paper at P'. $\triangle P'MN$ is a reflection of $\triangle PMN$ in the line segment MN. It follows that $\angle P'MN = \angle PMN = 45°$ and $\angle P'NM = \angle PNM = 45°$. Therefore, $\angle PMP' = \angle PNP' = 90°$. Since all four sides of $PMP'N$ are equal in length and all four corners are $90°$, $PMP'N$ is a square.
Since $\angle MPP' = \angle MPR = 45°$, the diagonal PP' of square $PMP'N$ lies along the diagonal PR of square $PQRS$. Let O be the intersection of the two diagonals of square $PMP'N$. It is also the intersection of MN and PR. (We will show later that this is in fact R', the point of contact of R with the paper after the second fold.)



The length of the diagonal of square $PMP'N$ can be found using the Pythagorean Theorem.

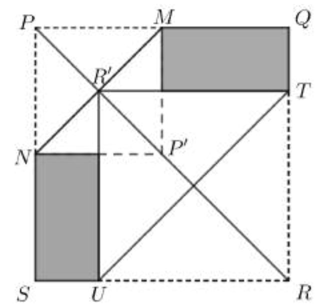
$$PP' = \sqrt{(PM)^2 + (MP')^2} = \sqrt{20^2 + 20^2} = \sqrt{800} = \sqrt{400}\sqrt{2} = 20\sqrt{2}$$

Thus, $PO = \frac{1}{2}(PP') = \frac{1}{2}(20\sqrt{2}) = 10\sqrt{2}$ cm.

In the last two steps of calculating PP', we simplified the radical. We will do this quite often in the solution. Here is the process to simplify radicals, for students who may not be familiar with this:

- Find the largest perfect square that divides into the radicand (the number under the root symbol). In this case, 400 is the largest perfect square that divides 800.
- Rewrite the radicand as the product of the perfect square and the remaining factor. In this case, we get $\sqrt{400 \times 2}$.
- Take the square root of the perfect square. In this case, we get $20\sqrt{2}$.

Since TU is parallel to MN, it follows that $\angle RTU = \angle RUT = 45°$ and $\triangle TRU$ is an isosceles right-angled triangle with $TR = RU$. When $\triangle TRU$ is reflected in the line segment TU with R' being the image of R, a square, $TRUR'$, is created. We will not present the argument here because it is very similar to the argument presented for $PMP'N$. Since $\angle TRR' = \angle TRP = 45°$, RR' lies along the diagonal PR. Also, R' lies on MN. This means that R' and O are the same point and so $PR' = PO = 10\sqrt{2}$ cm.



The length of the diagonal of square $PQRS$ can be calculated using the Pythagorean Theorem.

$$PR = \sqrt{(PQ)^2 + (QR)^2} = \sqrt{40^2 + 40^2} = \sqrt{3200} = \sqrt{1600}\sqrt{2} = 40\sqrt{2}$$

The length of RR' equals the length of PR minus the length of PR'.

$$RR' = PR - PR' = 40\sqrt{2} - 10\sqrt{2} = 30\sqrt{2}$$

But $RR' = TU$, so $TU = 30\sqrt{2}$ cm. Let $TR = RU = x$. Then, using the Pythagorean Theorem in $\triangle TRU$,

$$\begin{aligned}(TR)^2 + (RU)^2 &= (TU)^2\\ x^2 + x^2 &= (30\sqrt{2})^2\\ x^2 + x^2 &= 900 \times 2\\ 2x^2 &= 1800\\ x^2 &= 900\end{aligned}$$

And since $x > 0$, this gives $x = 30$ cm. We now have enough information to calculate the area of hexagon $NMQTUS$.

$$\begin{aligned}\text{Area } NMQTUS &= \text{Area } PQRS - \text{Area } \triangle PMN - \text{Area } \triangle TRU\\ &= PQ \times QR - \frac{PM \times PN}{2} - \frac{TR \times RU}{2}\\ &= 40 \times 40 - \frac{20 \times 20}{2} - \frac{30 \times 30}{2}\\ &= 1600 - \frac{400}{2} - \frac{900}{2}\\ &= 1600 - 200 - 450\\ &= 950\end{aligned}$$

Therefore, the area of hexagon $NMBPQD$ is 950 cm^2.

Problem of the Week
Problem D
Hundred Deck 2

Hundred Deck is a deck consisting of 100 cards numbered from 1 to 100. Each card has the same number printed on both sides. One side of the card is red and the other side of the card is yellow.

Sarai places all of the cards on a table with each card's red side facing up. She first flips over every card that has a number on it which is a multiple of 2. She then flips over every card that has a number on it which is a multiple of 3. Finally, she flips over every card that has a number on it which is a multiple of 5.

After Sarai has finished, how many cards have their red side facing up?

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Problem of the Week
Problem D and Solution
Hundred Deck 2

Problem

Hundred Deck is a deck consisting of 100 cards numbered from 1 to 100. Each card has the same number printed on both sides. One side of the card is red and the other side of the card is yellow.

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After Sarai has finished, how many cards have their red side facing up?

Solution

After flipping over all of the cards with numbers that are multiples of 2, 50 cards have their red side facing up and 50 cards have their yellow side facing up. All of the cards with their red side facing up are numbered with an odd number. All of the cards with their yellow side facing up are numbered with an even number.

Next, in the second round of flips, Sarai flips over every card that is numbered with a multiple of 3. Let's look at how many cards with their red side facing up will be flipped over to yellow and how many cards with their yellow side facing up will be flipped over to red.

There are 33 multiples of 3 from 1 to 100. They are

$$3, 6, 9, 12, 15, \ldots, 87, 90, 93, 96, 99$$

Of these numbers, 17 are odd and 16 are even. The 17 odd multiples of 3 currently have their red side facing up, and therefore are flipped over to yellow. The 16 even multiples of 3 currently have their yellow side facing up, and are therefore flipped over to red (again).

So, after the first flip there were 50 cards with their red side facing up and 50 cards with their yellow side facing up. Of the 50 red, 17 were flipped to yellow. Of the 50 yellow, 16 were flipped to red. Therefore, after the second round has finished, $50 - 17 + 16 = 49$ cards have their red side facing up and 51 cards have their yellow side facing up. The cards with their red side facing up are the cards with numbers that are odd and not a multiple of 3, or even and a multiple of 3. The cards with their yellow side facing up are the cards with numbers that are odd and a multiple of 3, or even and not a multiple of 3.

In the third round, Sarai flips all cards numbered with a multiple of 5. That is, she flips the cards numbered

$$5, 10, 15, 20, 25, 30, 35, 40, 45, 50, 55, 60, 65, 70, 75, 80, 85, 90, 95, 100$$

For these 20 numbers, let's determine their colour after the last flip by considering four cases.

- Case 1: The number is odd and not a multiple of 3.
  There are 7 cards that are numbered with a number that is a multiple of 5, odd, and not a multiple of 3. They are

  $$5, 25, 35, 55, 65, 85, 95$$

  Before the third flip, these 7 cards have their red side facing up, and are flipped to yellow in the third flip.

- Case 2: The number is even and a multiple of 3.
  There are 3 cards that are numbered with a number that is a multiple of 5, even, and a multiple of 3. They are

  $$30, 60, 90$$

  Before the third flip, these 3 cards have their red side facing up, and are flipped to yellow in the third flip.

- Case 3: The number is odd and a multiple of 3.
  There are 3 cards that are numbered with a number that is a multiple of 5, odd, and a multiple of 3. They are

  $$15, 45, 75$$

  Before the third flip, these 3 cards have their yellow side facing up, and are flipped to red in the third flip.

- Case 4: The number is even and not a multiple of 3.
  There are 7 cards that are numbered with a number that is a multiple of 5, even, and not a multiple of 3. They are

  $$10, 20, 40, 50, 70, 80, 100$$

  Before the third flip, these 7 cards have their yellow side facing up, and are flipped to red in the third flip.

Therefore, after Sarai has finished, $49 - 7 - 3 + 3 + 7 = 49$ cards have their red side facing up.

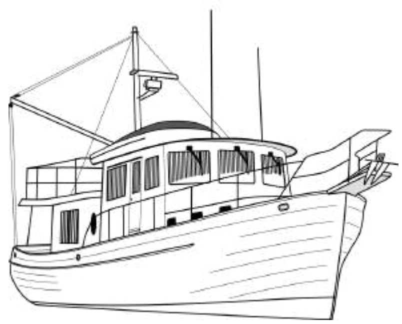
Problem of the Week

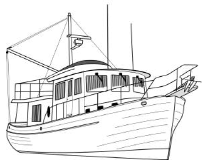
Problem D

Sale Boats

Harold, a marina manager, purchased two boats. He then sold the boats, the first at a profit of 40% and the second at a profit of 60%. The total profit on the sale of the two boats was 54% and \$88 704 was the total selling price of the two boats.

What did Harold originally pay for each of the two boats?





Problem of the Week
Problem D and Solution
Sale Boats

Problem

Harold, a marina manager, purchased two boats. He then sold the boats, the first at a profit of 40% and the second at a profit of 60%. The total profit on the sale of the two boats was 54% and \$88 704 was the total selling price of the two boats. What did Harold originally pay for each of the two boats?

Solution

Solution 1

Let a represent what Harold paid for the first boat, in dollars, and b represent what he paid for the second boat, in dollars.

The profit on the sale of the first boat was 40% or $0.4a$ dollars. Thus, the first boat sold for $a + 0.4a = 1.4a$ dollars. The profit on the sale of the second boat was 60% or $0.6b$ dollars. Thus, the second boat sold for $b + 0.6b = 1.6b$ dollars. The total selling price of the two boats was \$88 704, so we have

$$1.4a + 1.6b = 88\,704 \tag{1}$$

Harold bought both boats for a total of $(a + b)$ dollars. The profit on the sale of the two boats was 54% or $0.54(a + b)$ dollars. The two boats sold for $(a + b) + 0.54(a + b) = 1.54(a + b)$ dollars. But the total selling price was \$88 704, so

$$\begin{aligned} 1.54(a+b) &= 88\,704 \\ a+b &= 88\,704 \div 1.54 \\ a+b &= 57\,600 \\ a &= 57\,600 - b \end{aligned}$$

Substituting $a = 57\,600 - b$ into equation (1) gives

$$\begin{aligned} 1.4(57\,600 - b) + 1.6b &= 88\,704 \\ 80\,640 - 1.4b + 1.6b &= 88\,704 \\ 0.2b &= 8064 \end{aligned}$$

Dividing by 0.2, we get $b = 40\,320$. Since $b = 40\,320$ and $a + b = 57\,600$, then $a = 17\,280$ follows.

Therefore, Harold paid \$17 280 for the first boat and \$40 320 for the second boat.

Solution 2

Let a represent what Harold paid for the first boat, in dollars, and b represent what he paid for the second boat, in dollars.

The profit on the sale of the first boat was 40% or $0.4a$ dollars. The first boat sold for $a + 0.4a = 1.4a$ dollars. The profit on the sale of the second boat was 60% or $0.6b$ dollars. The second boat sold for $b + 0.6b = 1.6b$ dollars. The total selling price of the two boats was \$88 704 so we have

$$1.4a + 1.6b = 88\,704$$

Multiplying by 5, we get

$$7a + 8b = 443\,520 \tag{1}$$

Harold bought both boats for a total of $(a + b)$ dollars. The profit on the sale of the two boats was 54% or $0.54(a + b)$ dollars. The total profit is the sum of the profit from the sale of each boat, so

$$\begin{aligned} 0.54(a+b) &= 0.4a + 0.6b \\ 0.54a + 0.54b &= 0.4a + 0.6b \\ 0.14a &= 0.06b \end{aligned}$$

Multiplying by 50, we get

$$7a = 3b \tag{2}$$

Substituting $3b$ for $7a$ into equation (1), we get $3b + 8b = 443\,520$ or $11b = 443\,520$, and $b = 40\,320$ follows.

Substituting $b = 40\,320$ into equation (2), we get $7a = 120\,960$, and $a = 17\,280$ follows.

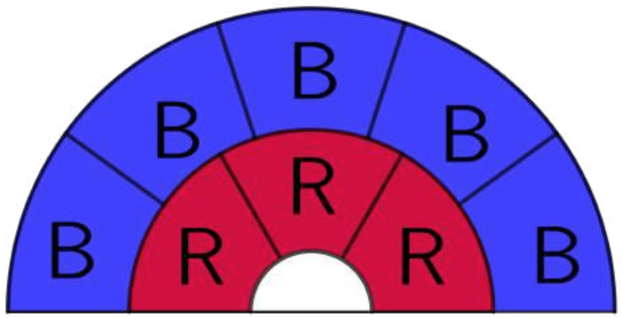
Therefore, Harold paid \$17 280 for the first boat and \$40 320 for the second boat.

Problem of the Week

Problem D

Coloured Areas

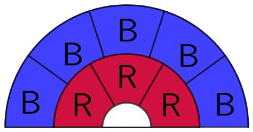
Chandra wishes to paint a sign. The sign is composed of three concentric semi-circles, creating an outer band, a middle band, and an inner semi-circle.



The outer band is divided into five regions of equal area and will be painted blue. Each of these regions is labelled with a B. The middle band is divided into three regions of equal area and will be painted red. Each of these regions is labelled with an R. The inner semi-circle will be painted white.

The diameter of the largest semi-circle is 10 m and the diameter of the middle semi-circle is 6 m.

If the ratio of the area of one region marked with an R to the area of one region marked with a B is $5 : 6$, what is the diameter of the inner semi-circle?



Problem of the Week

Problem D and Solution

Coloured Areas

Problem

Chandra wishes to paint a sign. The sign is composed of three concentric semi-circles, creating an outer band, a middle band, and an inner semi-circle. The outer band is divided into five regions of equal area and will be painted blue. Each of these regions is labelled with a B. The middle band is divided into three regions of equal area and will be painted red. Each of these regions is labelled with an R. The inner semi-circle will be painted white. The diameter of the largest semi-circle is 10 m and the diameter of the middle semi-circle is 6 m. If the ratio of the area of one region marked with an R to the area of one region marked with a B is $5:6$, what is the diameter of the inner semi-circle?

Solution

Since the area of a circle with radius r is πr^2, the area of a semi-circle with radius r is $\frac{\pi r^2}{2}$. The large semi-circle has diameter 10 m and therefore has a radius of 5 m. Thus, the area of the large semi-circle is $\frac{\pi(5)^2}{2} = \frac{25\pi}{2}$ m^2. The middle semi-circle has diameter 6 m and therefore a radius of 3 m. Thus, the area of the middle semi-circle is $\frac{\pi(3)^2}{2} = \frac{9\pi}{2}$ m^2.

The area of the large semi-circle is made up of the areas of 5 regions marked with a B plus the area of the middle semi-circle. Therefore,

$$\begin{aligned} 5B + \frac{9\pi}{2} &= \frac{25\pi}{2} \\ 5B &= 8\pi \\ B &= \frac{8\pi}{5} \end{aligned}$$

Since ratio of the area of one red region marked with an R to the area of one blue region marked with a B is $5:6$, we have $\frac{R}{B} = \frac{5}{6}$. And so,

$$\begin{aligned} R &= \frac{5B}{6} \\ &= \frac{5}{6}\left(\frac{8\pi}{5}\right) \\ &= \frac{4\pi}{3} \end{aligned}$$

Let the radius of the smallest semi-circle be r.
The area of the middle semi-circle is made up of the areas of 3 regions marked with an R plus the area of the smallest semi-circle. Therefore,

$$\frac{9\pi}{2} = 3R + \frac{\pi r^2}{2}$$

Since $R = \dfrac{4\pi}{3}$, we have

$$\frac{9\pi}{2} = 4\pi + \frac{\pi r^2}{2}$$

Therefore, $\dfrac{\pi}{2} = \dfrac{\pi r^2}{2}$ or $r^2 = 1$. Thus $r = 1$, since $r > 0$.

Therefore, the diameter of the smallest semi-circle is 2 m.

Problem of the Week

Problem D

Another Average

The numbers 2124, 1984, 1742, 2344, 2074, and 1632 are each written on a card. Daniyal takes four of the cards and calculates the mean (average) of their numbers to be 2021. Determine the mean of the numbers on the remaining two cards.

Extra Problem: Can you interpret the following picture puzzle?

% % % % % % % % % %
AVERAGE

Problem of the Week
Problem D and Solution
Another Average

Problem

The numbers 2124, 1984, 1742, 2344, 2074, and 1632 are each written on a card. Daniyal takes four of the cards and calculates the mean (average) of their numbers to be 2021. Determine the mean of the numbers on the remaining two cards.

Extra Problem: Can you interpret the picture puzzle above?

Solution

At the outset, it should be noted that we could "play" with the numbers to determine which of the four numbers have an average of 2021. We could then easily determine the average of the remaining two numbers. This method works decently well on a problem with a small number of numbers. However, if we were to increase the size of the list by just a few more numbers, then the task would not be easily solved using this approach. It turns out, we can solve this problem without actually figuring out which four numbers Daniyal used.

The sum of all six numbers is

$$2124 + 1984 + 1742 + 2344 + 2074 + 1632 = 11\,900$$

Since the average of four of the numbers is 2021, then the sum of those four numbers is $4 \times 2021 = 8084$.

The sum of the two remaining numbers is $11\,900 - 8084 = 3816$. Since there are two numbers in the sum, the average of the two numbers is calculated by dividing the sum by 2. The average of the remaining two numbers is then $3816 \div 2 = 1908$.

Although not required, the two numbers that sum to 3816 are 1742 and 2074. It is then easily verified that the average of the four other numbers, 2124, 1984, 2344, and 1632, is 2021.

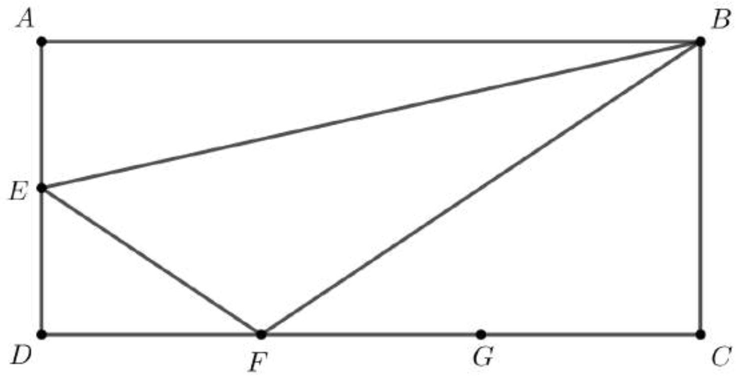
Extra Problem Answer: Ten percent above average.

Problem of the Week

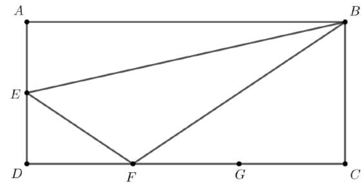
Problem D

The Whole Rectangle

In the diagram, $ABCD$ is a rectangle. Points F and G are on DC (with F closer to D) such that $DF = FG = GC$. Point E is the midpoint of AD.



If the area of $\triangle BEF$ is 30 cm^2, determine the area of rectangle $ABCD$.



Problem of the Week

Problem D and Solution

The Whole Rectangle

Problem

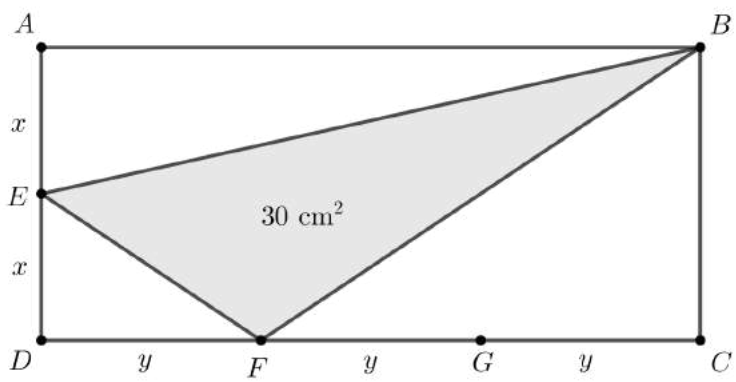
In the diagram, $ABCD$ is a rectangle. Points F and G are on DC (with F closer to D) such that $DF = FG = GC$. Point E is the midpoint of AD.

If the area of $\triangle BEF$ is 30 cm^2, determine the area of rectangle $ABCD$.

Solution

Let $DF = FG = GC = y$. Then $AB = DC = 3y$ and $FC = 2y$.

Since E is the midpoint of AD, let $AE = ED = x$. Then $AD = BC = 2x$.



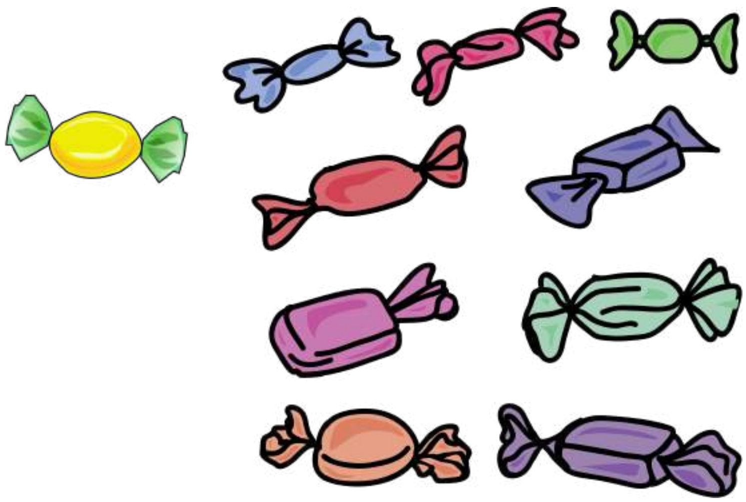
We will formulate an equation connecting the areas of the four triangles inside the rectangle to the area of the entire rectangle.

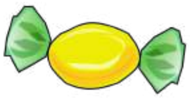
$$\begin{aligned}
\text{Area } ABCD &= \text{Area } \triangle ABE + \text{Area } \triangle BCF + \text{Area } \triangle FDE + \text{Area } \triangle BEF \\
AD \times DC &= \frac{AE \times AB}{2} + \frac{BC \times FC}{2} + \frac{DF \times ED}{2} + 30 \\
(2x)(3y) &= \frac{x \times 3y}{2} + \frac{2x \times 2y}{2} + \frac{y \times x}{2} + 30 \\
6xy &= \frac{3xy}{2} + 2xy + \frac{xy}{2} + 30 \\
12xy &= 3xy + 4xy + xy + 60 \\
4xy &= 60 \\
xy &= 15
\end{aligned}$$

Therefore, the area of rectangle $ABCD$ is $AD \times DC = (2x)(3y) = 6xy = 6(15) = 90$ cm^2.

Problem of the Week
Problem D
Sharing Sweets

Diana has 10 candies, and she wishes to give all 10 candies to her three friends, Victoria, Manuela, and Alejandra. She does not necessarily want to distribute the candy equally, but she does want each friend to receive at least one candy. In how many ways can she distribute the candies to Victoria, Manuela, and Alejandra?





Problem of the Week
Problem D and Solution
Sharing Sweets

Problem

Diana has 10 candies, and she wishes to give all 10 candies to her three friends, Victoria, Manuela, and Alejandra. She does not necessarily want to distribute the candy equally, but she does want each friend to receive at least one candy. In how many ways can she distribute the candies to Victoria, Manuela, and Alejandra?

Solution

We know that there are 10 candies and that each friend must receive at least one. We will consider the following cases:

1. Victoria receives one candy. Then Manuela and Alejandra receive a total of $10 - 1 = 9$ candies between them. This can be done in 8 possible ways:
$$(1, 8), (2, 7), (3, 6), (4, 5), (5, 4), (6, 3), (7, 2), (8, 1)$$
2. Victoria receives two candies. Then Manuela and Alejandra receive a total of $10 - 2 = 8$ candies between them. This can be done in 7 possible ways:
$$(1, 7), (2, 6), (3, 5), (4, 4), (5, 3), (6, 2), (7, 1)$$
3. Victoria receives three candies. Then Manuela and Alejandra receive a total of $10 - 3 = 7$ candies between them. This can be done in 6 possible ways:
$$(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)$$
4. Victoria receives four candies. Then Manuela and Alejandra receive a total of $10 - 4 = 6$ candies between them. This can be done in 5 possible ways:
$$(1, 5), (2, 4), (3, 3), (4, 2), (5, 1)$$
5. Victoria receives five candies. Then Manuela and Alejandra receive a total of $10 - 5 = 5$ candies between them. This can be done in 4 possible ways: $(1, 4), (2, 3), (3, 2), (4, 1)$.
6. Victoria receives six candies. Then Manuela and Alejandra receive a total of $10 - 6 = 4$ candies between them. This can be done in 3 possible ways: $(1, 3), (2, 2), (3, 1)$.
7. Victoria receives seven candies. Then Manuela and Alejandra receive a total of $10 - 7 = 3$ candies between them. This can be done in 2 possible ways: $(1, 2), (2, 1)$.
8. Victoria receives eight candies. Then Manuela and Alejandra receive a total of $10 - 8 = 2$ candies between them. This can be done in 1 way: $(1, 1)$.

Notice that Victoria cannot receive more than eight candies. If she does, then at least one of Manuela and Alejandra would have received zero candies.

Thus, the total number of ways Diana can distribute 10 candies between the three friends so that each receives at least one candy is $8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 = 36$ ways. This sum can be computed by adding the positive integers from 1 to 8. However, it is also known that the sum of the first n positive integers can be calculated using the formula $1 + 2 + \ldots + n = \dfrac{n(n + 1)}{2}$. In this case $n = 8$, so the sum is $\dfrac{8(9)}{2} = 36$.

Problem of the Week

Problem D

Ring Ring

The POTW school wants to create a phone tree system to be used in the event of an emergency school closure. Using a phone tree system, the principal phones at most three other employees, each of whom phones at most three other employees, and so on, until all of the employees have been contacted.

If the POTW school has 100 employees (including the principal) and uses a phone tree system where each employee phones 0, 1, 2, or 3 other employees, determine the maximum number of employees who do not need to make any phone calls in their phone tree system.

Problem of the Week
Problem D and Solution
Ring Ring

Problem

The POTW school wants to create a phone tree system to be used in the event of an emergency school closure. Using a phone tree system, the principal phones at most three other employees, each of whom phones at most three other employees, and so on, until all of the employees have been contacted.

If the POTW school has 100 employees (including the principal) and uses a phone tree system where each employee phones 0, 1, 2, or 3 other employees, determine the maximum number of employees who do not need to make any phone calls in their phone tree system.

Solution

It is important to note that in order to minimize the number of callers, we need to maximize the number of calls made by those who do make calls.

Once the principal makes the initial three phone calls, four people (the principal and three others) have the information. There are $100 - 4 = 96$ people left to contact.

The next three people make three calls each, for a total of 9 calls. Now 13 people have the information and 87 people still need to be contacted.

The next 9 people make 3 calls each, for a total of 27 calls. Now 40 people have the information and 60 people still need to be contacted.

From here, we will present two approaches for figuring out how many people are needed to contact the remaining 60 people.

- **Approach 1:** If the next 27 people make 3 calls each for a total of 81 calls, this is $81 - 60 = 21$ calls too many. This means $21 \div 3 = 7$ of the 27 people do not need to make any calls. Thus, only $27 - 7 = 20$ more people need to make calls.

- **Approach 2:** In order to reach the final 60 people, only $60 \div 3 = 20$ more people need to make calls because each person phones 3 people.

The total number of people required to make calls is therefore $1 + 3 + 9 + 20 = 33$.

Therefore, $100 - 33 = 67$ is the maximum number of employees who do not need to make any phone calls in the phone tree system.

A system like this is actually still very efficient at getting information to a large number of people. Close to one third of the employees need to make only 3 calls each, while about two-thirds of the employees do not need to make any calls.

Problem of the Week
Problem D
Not As It Seems

In the following table, the letters a, b, c, d, and e represent unknown numbers.

	Column 1	Column 2	Column 3
Row 1	75	b	83
Row 2	76	80	d
Row 3	a	81	85
Row 4	78	c	e

At a first glance, the numbers in the table may appear to follow a very predictable pattern. However, we need the columns and rows to follow the following rules:

1. The sum of the numbers in each of the four rows is the same.
2. The sum of the numbers in each of the three columns is the same.
3. The sum of any row does not equal the sum of any column.

Determine the values of a, b, c, d, and e.

Problem of the Week
Problem D and Solution
Not As It Seems

Problem

In the following table, the letters a, b, c, d, and e represent unknown numbers.

	Column 1	Column 2	Column 3
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1. The sum of the numbers in each of the four rows is the same.
2. The sum of the numbers in each of the three columns is the same.
3. The sum of any row does not equal the sum of any column.

Determine the values of a, b, c, d, and e.

Solution

The final answer is $a = 23$, $b = 31$, $c = 60$, $d = 33$, and $e = 51$. We will give our solution below.

Each of the first three rows has two known values and one unknown value. We also know that the sum of each row is the same.

Therefore,

$$\begin{aligned}
\text{Sum of Row 1} &= \text{Sum of Row 2} \\
75 + b + 83 &= 76 + 80 + d \\
b + 158 &= 156 + d \\
d &= b + 2
\end{aligned}$$

Also,

$$\begin{aligned} \text{Sum of Row 1} &= \text{Sum of Row 3} \\ 75 + b + 83 &= a + 81 + 85 \\ b + 158 &= a + 166 \\ a &= b - 8 \end{aligned}$$

Replacing a with $b - 8$ and d with $b + 2$, we get the following grid:

	Column 1	Column 2	Column 3
Row 1	75	b	83
Row 2	76	80	$b + 2$
Row 3	$b - 8$	81	85
Row 4	78	c	e

Now, each column has the same sum. We will use this fact to find the values of c and e.

$$\begin{aligned} \text{Sum of Column 1 } &= \text{Sum of Column 2} \\ 75 + 76 + (b - 8) + 78 &= b + 80 + 81 + c \\ b + 221 &= b + c + 161 \\ c &= 60 \end{aligned}$$

Also,

$$\begin{aligned} \text{Sum of Column 1} &= \text{Sum of Column 3} \\ 75 + 76 + (b - 8) + 78 &= 83 + (b + 2) + 85 + e \\ b + 221 &= b + e + 170 \\ e &= 51 \end{aligned}$$

Since we know $c = 60$ and $e = 51$, we can determine the row sum using the fourth row. The row sum is $78 + 60 + 51 = 189$. We can use this sum to determine the value of b.

From Row 1, $75 + b + 83 = 189$ and $b = 31$ follows.

We know that $d = b + 2$, so $d = 33$. Also, we know that $a = b - 8$, so $a = 23$.

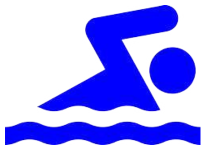
Therefore, $a = 23$, $b = 31$, $c = 60$, $d = 33$, and $e = 51$. From here, one can easily verify that each row sums to 189 and each column sums to 252.

Problem of the Week
Problem D
Let's Hit the Pool

In Wei's family, there are four children and three adults. Every weekend they all go swimming together. To use the public swimming pool, each person needs a ticket.

Wei's parents buy their tickets in bulk and keep them in a box. At the beginning of the year the ratio of adult to child tickets in the box was $11 : 14$.

Wei's family used the tickets every weekend to go swimming until they no longer had enough tickets for everyone in their family. At that point, there were no child tickets left in the box and 3 adult tickets left in the box. How many tickets were in the box at the beginning of the year?



Problem of the Week

Problem D and Solution

Let's Hit the Pool

Problem

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Wei's family used the tickets every weekend to go swimming until they no longer had enough tickets for everyone in their family. At that point, there were no child tickets left in the box and 3 adult tickets left in the box. How many tickets were in the box at the beginning of the year?

Solution

Let n represent the number of times Wei's family used the tickets to go swimming. Since they used 4 child tickets and 3 adult tickets each time, then they used $4n$ child tickets and $3n$ adult tickets in total. After they had used all the child tickets, there were 3 adult tickets left in the box. That means there were $3n + 3$ adult tickets and $4n$ child tickets in the box at the beginning of the year.

The ratio of adult to child tickets at the beginning of the year was $11 : 14$. We can use this to write and solve the following equation.

$$
\begin{aligned}
\frac{11}{14} &= \frac{3n+3}{4n} \\
(11)(4n) &= (14)(3n+3) \\
44n &= 42n + 42 \\
2n &= 42 \\
n &= 21
\end{aligned}
$$

Thus, Wei's family used the tickets to go swimming 21 times.

The total number of tickets in the box at the beginning of the year was $4n + 3n + 3 = 7n + 3$. Since $n = 21$, the total number of tickets was $7(21) + 3 = 150$.

Problem of the Week
Problem D
Five Digits

A sequence starts out with one 5, followed by two 6s, then three 7s, four 8s, five 9s, six 5s, seven 6s, eight 7s, nine 8s, ten 9s, eleven 5s, twelve 6s, and so on. (You should notice that only the five digits from 5 to 9 are used.)

The first 29 terms of the sequence appear below.

$$5, 6, 6, 7, 7, 7, 8, 8, 8, 8, 9, 9, 9, 9, 9, 5, 5, 5, 5, 5, 5, 6, 6, 6, 6, 6, 6, 6, 7, \ldots$$

Determine the 2022^{nd} digit in the sequence.

Note:
In solving the above problem, it may be helpful to use the fact that the sum of the first n positive integers is equal to $\dfrac{n(n+1)}{2}$. That is,

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

For example, $1 + 2 + 3 + 4 + 5 = 15$, and $\dfrac{5(6)}{2} = 15$.

Also, $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 = 36$, and $\dfrac{8(9)}{2} = 36$.

Problem of the Week
Problem D and Solution
Five Digits

Problem

A sequence starts out with one 5, followed by two 6s, then three 7s, four 8s, five 9s, six 5s, seven 6s, eight 7s, nine 8s, ten 9s, eleven 5s, twelve 6s, and so on. (You should notice that only the five digits from 5 to 9 are used.)

The first 29 terms of the sequence appear below.

$$5, 6, 6, 7, 7, 7, 8, 8, 8, 8, 9, 9, 9, 9, 9, 5, 5, 5, 5, 5, 5, 6, 6, 6, 6, 6, 6, 6, 7, \ldots$$

Determine the 2022^{nd} digit in the sequence.

Solution

The first group in the sequence contains one 5. The second group in the sequence contains two 6s. To the end of the second group of digits, there is a total of $1 + 2 = 3$ digits. The third group in the sequence contains three 7s. To the end of the third group of digits, there is a total of $1 + 2 + 3 = 6$ digits. The n^{th} group in the sequence contains n digits. To the end of the n^{th} group of digits, there is a total of $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$ digits.

How many groups of digits are required for there to be at least 2022 digits in the sequence?

We need to find the value of n so that $1 + 2 + 3 + \cdots + n \geq 2022$ and $1 + 2 + 3 + \cdots + (n - 1) < 2022$. At this point we will use trial and error. At the end of the solution, a more algebraic approach to finding the value of n using the quadratic formula is presented.

Suppose $n = 100$. Then $1 + 2 + 3 + \cdots + 100 = \frac{100(101)}{2} = 5050 > 2022$.

Suppose $n = 50$. Then $1 + 2 + 3 + \cdots + 50 = \frac{50(51)}{2} = 1275 < 2022$.

Suppose $n = 60$. Then $1 + 2 + 3 + \cdots + 60 = \frac{60(61)}{2} = 1830 < 2022$.

Suppose $n = 65$. Then $1 + 2 + 3 + \cdots + 65 = \frac{65(66)}{2} = 2145 > 2022$.

Suppose $n = 63$. Then $1 + 2 + 3 + \cdots + 63 = \frac{63(64)}{2} = 2016 < 2022$.

The 2022^{nd} digit is the sixth number in the next group of digits. That is, the 2022^{nd} digit is a digit in the 64^{th} group of digits.

Now, let's determine what digit is in the 64^{th} group of digits. Since we cycle through the digits and there are only five digits used, we can determine the digit by examining $\frac{64}{5} = 12\frac{4}{5}$. Thus, in the 64^{th} group of digits, the digit used is the 4^{th} digit in the sequence of digits. That is, in the 64^{th} group of digits, the digit used is an 8.

Since the 2022^{nd} digit is in the 64^{th} group of digits, it follows that the 2022^{nd} digit is an 8.

We will finish by showing how we can find the value of n algebraically.

We will first find the value of $n, n > 0$, so that

$$\begin{aligned}\frac{n(n+1)}{2} &= 2022\\ n(n+1) &= 4044\\ n^2+n-4044 &= 0\end{aligned}$$

The quadratic formula can be used to solve for n.

$$\begin{aligned}n &= \frac{-1 \pm \sqrt{1^2-4(1)(-4044)}}{2}\\ &= \frac{-1 \pm \sqrt{16\,177}}{2}\end{aligned}$$

Since $n = \frac{-1-\sqrt{16\,177}}{2} < 0$, it is inadmissible.

Then $n = \frac{-1+\sqrt{16\,177}}{2} \approx 63.09$. But n is an integer. So, interpreting our result, when $n = 63$, the sum $1+2+3+\cdots+63 < 2022$, and when $n = 64$, the sum $1+2+3+\cdots+64 > 2022$. Thus, the 2022^{nd} digit is in the 64^{th} group of digits.

Problem of the Week

Problem D

Unusually Late

Every day, a train makes a trip from Alphatown to Betatown. Although the train is rarely late, on two different trips the train was late. On the first trip, when the train was travelling at an average speed of 56 km/h, the train was 27 minutes late. For the second trip, the train was travelling at an average speed of 54 km/h and was 42 minutes late. What is the distance between Alphatown and Betatown?

Problem of the Week
Problem D and Solution
Unusually Late

Problem

Every day, a train makes a trip from Alphatown to Betatown. Although the train is rarely late, on two different trips the train was late. On the first trip, when the train was travelling at an average speed of 56 km/h, the train was 27 minutes late. For the second trip, the train was travelling at an average speed of 54 km/h and was 42 minutes late. What is the distance between Alphatown and Betatown?

Solution

We will present three different solutions. In all three solutions, we will use the formula

$$\text{distance} = \text{speed} \times \text{time}$$

or equivalently,

$$\text{time} = \frac{\text{distance}}{\text{speed}}$$

Solution 1

Let t represent the time, in hours, taken by the train when it was 27 minutes late. Since $42 - 27 = 15$ minutes, then $t + \frac{15}{60} = t + \frac{1}{4}$ represents the time, in hours, taken by the train when it was 42 minutes late, or 15 minutes later.

For the first trip, the speed is 56 km/h and the time is t, and so the distance travelled is $56t$ km.

For the second trip, the speed is 54 km/h and the time is $t + \frac{1}{4}$, and so the distance travelled is $54\left(t + \frac{1}{4}\right)$ km.

Since the distance between Alphatown and Betatown remains constant,

$$\begin{aligned} 56t &= 54\left(t + \frac{1}{4}\right) \\ 56t &= 54t + \frac{27}{2} \\ 2t &= \frac{27}{2} \\ t &= \frac{27}{4} \end{aligned}$$

Thus, the distance between Alphatown and Betatown is $56t = 56 \times \frac{27}{4} = 378$ km.

Solution 2

Let d represent the distance, in km, between Alphatown and Betatown.

For the first trip, the speed is 56 km/h and the distance is d, and so the time for the trip is $\dfrac{d}{56}$ hours.

For the second trip, the speed is 54 km/h and the distance travelled is d, and so the time for the trip is $\dfrac{d}{54}$ hours.

Since the difference in times between the first trip and the second trip is $42 - 27 = 15$ minutes or $\frac{1}{4}$ hour,

$$\begin{aligned}
\frac{d}{54} - \frac{d}{56} &= \frac{1}{4} \\
\frac{56d - 54d}{(54)(56)} &= \frac{1}{4} \\
2d &= \frac{1}{4} \times (54)(56) \\
2d &= 756 \\
d &= 378
\end{aligned}$$

Thus, the distance between Alphatown and Betatown is 378 km.

Solution 3

This solution looks at the problem quite differently from the first two solutions.

For the first trip, if the train had first travelled for 27 minutes, it then would have completed the rest of the trip in the usual amount of time. During the 27 minutes, the train would travel $56 \times \frac{27}{60} = \frac{1512}{60} = 25.2$ km.

For the second trip, if the train had first travelled for 42 minutes, it then would have completed the rest of the trip in the usual amount of time. During the 42 minutes, the train would travel $54 \times \frac{42}{60} = \frac{2268}{60} = 37.8$ km.

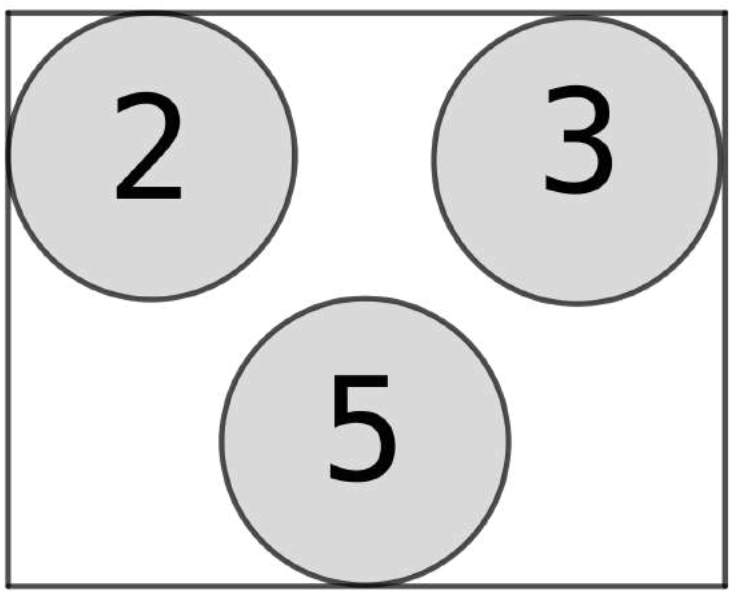
The slower train is $37.8 - 25.2 = 12.6$ km ahead of the faster train at the point when the usual time to complete the trip remains. The faster train gains 2 km/h on the slower train. Thus, it will take the faster train $\frac{12.6}{2} = 6.3$ h to catch up and thereby complete the trip. In 6.3 h, the faster train travels $56 \times 6.3 = 352.8$ km. But it had already travelled 25.2 km. Therefore, the total distance from Alphatown to Betatown is $25.2 + 352.8 = 378$ km.

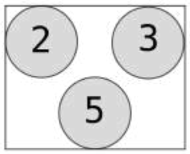
Thus, the distance between Alphatown and Betatown is 378 km.

Problem of the Week
Problem D
The Dart Game

A carnival dart game has three non-overlapping circles in a rectangle. One circle has a value of 2, another has a value of 3, and the third has a value of 5. You are allowed to throw up to 10 darts, and you start the game with a running total of 0. If a dart lands in one of the circles, you add the value of the circle to the running total. If a dart does not land in one of the circles, then you do not add anything to the running total for that throw.

Suppose you have exactly 30 points after 10 throws. Let a represent the number of throws that landed in the circle with value 5, let b represent the number of throws that landed in the circle with value 3, and let c represent the number of throws that landed in the circle with value 2. Determine all possibilities for (a, b, c).





Problem of the Week
Problem D and Solution
The Dart Game

Problem

A carnival dart game has three non-overlapping circles in a rectangle. One circle has a value of 2, another has a value of 3, and the third has a value of 5. You are allowed to throw up to 10 darts, and you start the game with a running total of 0. If a dart lands in one of the circles, you add the value of the circle to the running total. If a dart does not land in one of the circles, then you do not add anything to the running total for that throw.

Suppose you have exactly 30 points after 10 throws. Let a represent the number of throws that landed in the circle with value 5, let b represent the number of throws that landed in the circle with value 3, and let c represent the number of throws that landed in the circle with value 2. Determine all possibilities for (a, b, c).

Solution

We need to determine all possibilities for (a, b, c) with $5a + 3b + 2c = 30$ and $a + b + c \leq 10$.

We will look at cases for a. Since $6 \times 5 = 30$, then the largest value of a is 6. The smallest value is $a = 0$ since $a \geq 0$.

Let's look at the specific case where $a = 2$ to develop a process for how to determine the number of ways to get a total of 30. We will use this process for all cases, but will not show our steps in the other cases.

If $a = 2$, this will account for a total of $2 \times 5 = 10$ points. Therefore, $30 - 10 = 20$ points will be needed from landing in the circles with values of 2 and 3.

Next, we find the maximum value for b, the number of throws that landed in the circle with value 3. We want b to account for a total that is less than or equal to 20, but also give a remainder that is even, since the remaining points need to come from the circle with value 2.

If $b = 7$, then this would give $7 \times 3 = 21$ points, which exceeds 20. When $b = 6$, then this would give $6 \times 3 = 18$ points. Then $c = 1$ would make the total exactly 30. Notice here that $a + b + c = 2 + 6 + 1 = 9 \leq 10$, as required. Thus, one possibility is that $a = 2$, $b = 6$, and $c = 1$.

We then need to replace circles with a value of 3 with circles with a value 2. We note that for every two circles with a value of 3, we have a total value of 6. We can replace those two circles with three circles of value 2. This means that $b = 6 - 2 = 4$ and $c = 1 + 3 = 4$. Notice here that $a + b + c = 2 + 4 + 4 = 10 \leq 10$, as required. Thus, another possibility is that $a = 2$, $b = 4$, and $c = 4$.

We can again replace two circles with a value of 3 with three circles of value 2. This means that $b = 4 - 2 = 2$ and $c = 4 + 3 = 7$. Notice here that $a + b + c = 2 + 2 + 7 = 11 > 10$. Therefore, this is not a possibility.

We can again replace two circles with a value of 3 with three circles of value 2. This means that $b = 2 - 2 = 0$ and $c = 7 + 3 = 10$. Notice here that $a + b + c = 2 + 0 + 10 = 12 > 10$. Therefore, this is not a possibility.

We cannot again replace two circles with a value of 3 with three circles of value 2 since this would make b negative.

To summarize, when $a = 2$, there are two combinations that give a total of 30 and have $a + b + c \leq 10$.

We use this process for all the possible values of a. Our results are summarized in the table below.

a	b	c	$5a + 3b + 2c$	$a + b + c$
6	0	0	30	6
5	1	1	30	7
4	2	2	30	8
4	0	5	30	9
3	5	0	30	8
3	3	3	30	9
3	1	6	30	10
2	6	1	30	9
2	4	4	30	10
1	7	2	30	10
0	10	0	30	10

We find that there are 11 possibilities for the number of throws that have landed in each circle. The 11 possibilities for (a, b, c) are:

$(6, 0, 0)$, $(5, 1, 1)$, $(4, 2, 2)$, $(4, 0, 5)$, $(3, 5, 0)$, $(3, 3, 3)$, $(3, 1, 6)$, $(2, 6, 1)$, $(2, 4, 4)$, $(1, 7, 2)$, $(0, 10, 0)$

Problem of the Week
Problem D
Favourite Numbers

Dandan likes numbers that remind her of her name. That is, she likes six-digit numbers formed by repeating a three-digit number, such as 305 305, 417 417, and 832 832.

What is the greatest common factor of all such numbers?

Problem of the Week
Problem D and Solution
Favourite Numbers

Problem

Dandan likes numbers that remind her of her name. That is, she likes six-digit numbers formed by repeating a three-digit number, such as $305\,305$, $417\,417$, and $832\,832$.

What is the greatest common factor of all such numbers?

Solution

To get started, we look at the prime factorization of each of the given numbers.

$$\begin{aligned}305\,305 &= 5 \times 7 \times 11 \times 13 \times 61\\ 417\,417 &= 3 \times 7 \times 11 \times 13 \times 139\\ 832\,832 &= 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 7 \times 11 \times 13 \times 13\end{aligned}$$

We notice that all of these numbers are divisible by $7 \times 11 \times 13 = 1001$. These are the only factors common to all three numbers. We can pick another six-digit number formed by repeating a three-digit number and test to see if it is also divisible by 1001. The number $246\,246$, for example, is 1001×246. It would appear 1001 could be the greatest common factor of all such numbers, but we have not proven this.

Let $abc\,abc$ be any six-digit number formed by repeating the three-digit number abc.

$$\begin{aligned}abc\,abc &= abc000 + abc\\ &= 1000 \times abc + abc\\ &= 1000 \times abc + 1 \times abc\\ &= 1001 \times abc\end{aligned}$$

Since $abc\,abc = 1001 \times abc$, it is divisible by 1001. A specific number $abc\,abc$ may also have other factors, but 1001 is the largest factor common to all such numbers. In the first example $305\,305 = 1001 \times 5 \times 61$ and in the second example $417\,417 = 1001 \times 3 \times 139$. Both numbers have other factors but no other common factors greater than 1. In some cases there will be other common factors greater than 1, but not in general.

Thus, we have proven that 1001 is the greatest common factor of all six-digit numbers formed by repeating a three-digit number.

This problem is not hard if you initially "get it". The solution presented shows an approach that can be taken when you may not be certain where to begin. Try some specific examples and then attempt to generalize based on what you observe from the specific examples. Also note that discovering that 1001 worked for the three given examples and the test example is not sufficient to make a general conclusion that 1001 is the greatest common factor of all such numbers.

Geometry & Measurement (G)

TAKE ME TO THE COVER

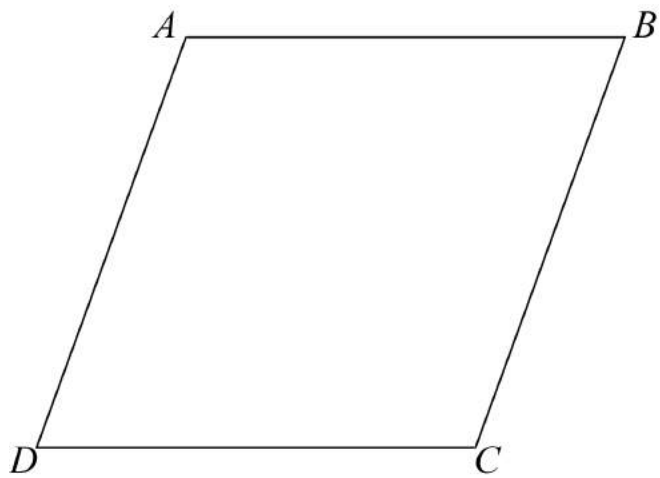
Problem of the Week

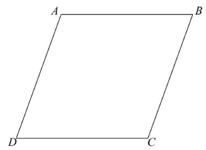
Problem D

This Angle Isn't Bad

Ewan drew rhombus $ABCD$. Recall that a rhombus is a quadrilateral with parallel opposite sides, and all four sides of equal length. In Ewan's rhombus, H is on BC in between B and C, and K is on CD in between C and D, such that $AB = AH = HK = KA$.

Determine the measure, in degrees, of $\angle BAD$.





Problem of the Week
Problem D and Solution
This Angle Isn't Bad

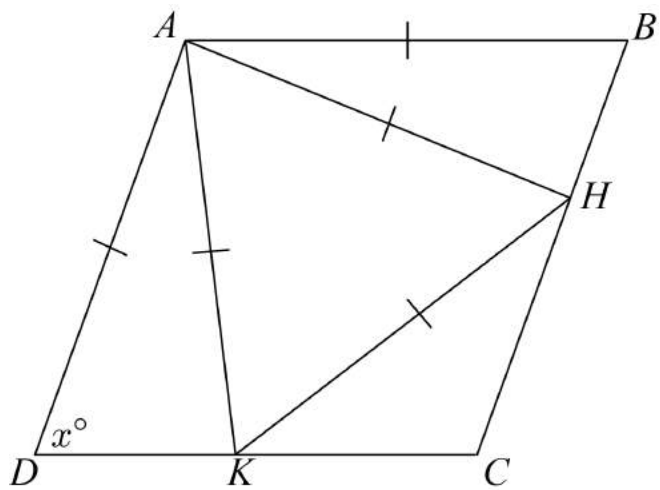
Problem

Ewan drew rhombus $ABCD$. Recall that a rhombus is a quadrilateral with parallel opposite sides, and all four sides of equal length. In Ewan's rhombus, H is on BC in between B and C, and K is on CD in between C and D, such that $AB = AH = HK = KA$.

Determine the measure, in degrees, of $\angle BAD$.

Solution

Since $ABCD$ is a rhombus, we know $AB = BC = CD = DA$. We're also given that $AB = AH = HK = KA$. Let $\angle ADK = x^\circ$.

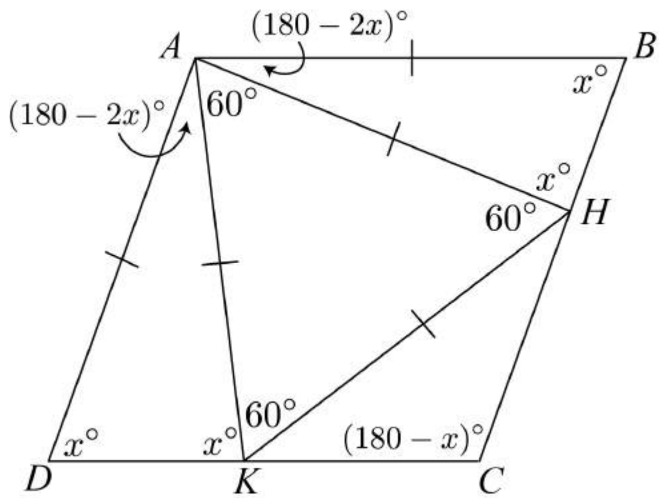


Since $AH = HK = KA$, $\triangle AHK$ is an equilateral triangle and each angle in $\triangle AHK$ is 60°. In particular, $\angle HAK = 60^\circ$.

In $\triangle ADK$, $AD = AK$ and so $\triangle ADK$ is isosceles. Therefore, $\angle AKD = \angle ADK = x^\circ$. Then $\angle DAK = (180 - 2x)^\circ$.

Since $ABCD$ is a rhombus, $AB \parallel CD$ and $\angle ADC + \angle BCD = 180^\circ$. It follows that $\angle BCD = (180 - x)^\circ$. But in the rhombus we also have $BC \parallel AD$ and $\angle BCD + \angle ABC = 180^\circ$. It follows that $\angle ABC = 180^\circ - (180 - x)^\circ = x^\circ$.

In $\triangle AHB$, $AH = AB$ and so $\triangle AHB$ is isosceles. Therefore, $\angle AHB = \angle ABH = x^\circ$. Then $\angle BAH = (180 - 2x)^\circ$.



Since $ABCD$ is a rhombus, $BC \parallel AD$, so

$$\begin{aligned} \angle BAD &= 180^\circ - \angle ABC \\ (180-2x)^\circ + 60^\circ + (180-2x)^\circ &= 180^\circ - x^\circ \\ (420-4x)^\circ &= (180-x)^\circ \\ 240^\circ &= (3x)^\circ \\ x^\circ &= 80^\circ \end{aligned}$$

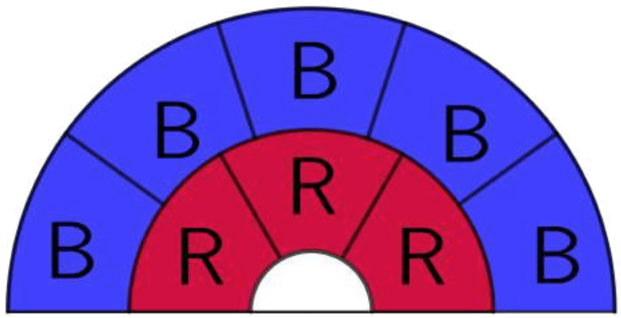
It follows that

$$\begin{aligned} \angle BAD &= (180-x)^\circ \\ &= 180^\circ - 80^\circ \\ &= 100^\circ \end{aligned}$$

Therefore, $\angle BAD = 100^\circ$.

Problem of the Week
Problem D
Coloured Areas

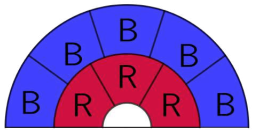
Chandra wishes to paint a sign. The sign is composed of three concentric semi-circles, creating an outer band, a middle band, and an inner semi-circle.



The outer band is divided into five regions of equal area and will be painted blue. Each of these regions is labelled with a B. The middle band is divided into three regions of equal area and will be painted red. Each of these regions is labelled with an R. The inner semi-circle will be painted white.

The diameter of the largest semi-circle is 10 m and the diameter of the middle semi-circle is 6 m.

If the ratio of the area of one region marked with an R to the area of one region marked with a B is $5:6$, what is the diameter of the inner semi-circle?



Problem of the Week
Problem D and Solution
Coloured Areas

Problem

Chandra wishes to paint a sign. The sign is composed of three concentric semi-circles, creating an outer band, a middle band, and an inner semi-circle. The outer band is divided into five regions of equal area and will be painted blue. Each of these regions is labelled with a B. The middle band is divided into three regions of equal area and will be painted red. Each of these regions is labelled with an R. The inner semi-circle will be painted white. The diameter of the largest semi-circle is 10 m and the diameter of the middle semi-circle is 6 m. If the ratio of the area of one region marked with an R to the area of one region marked with a B is $5:6$, what is the diameter of the inner semi-circle?

Solution

Since the area of a circle with radius r is πr^2, the area of a semi-circle with radius r is $\frac{\pi r^2}{2}$. The large semi-circle has diameter 10 m and therefore has a radius of 5 m. Thus, the area of the large semi-circle is $\frac{\pi(5)^2}{2} = \frac{25\pi}{2}$ m^2. The middle semi-circle has diameter 6 m and therefore a radius of 3 m. Thus, the area of the middle semi-circle is $\frac{\pi(3)^2}{2} = \frac{9\pi}{2}$ m^2.

The area of the large semi-circle is made up of the areas of 5 regions marked with a B plus the area of the middle semi-circle. Therefore,

$$\begin{aligned} 5B + \frac{9\pi}{2} &= \frac{25\pi}{2} \\ 5B &= 8\pi \\ B &= \frac{8\pi}{5} \end{aligned}$$

Since ratio of the area of one red region marked with an R to the area of one blue region marked with a B is $5:6$, we have $\frac{R}{B} = \frac{5}{6}$. And so,

$$\begin{aligned} R &= \frac{5B}{6} \\ &= \frac{5}{6}\left(\frac{8\pi}{5}\right) \\ &= \frac{4\pi}{3} \end{aligned}$$

Let the radius of the smallest semi-circle be r.
The area of the middle semi-circle is made up of the areas of 3 regions marked with an R plus the area of the smallest semi-circle. Therefore,

$$\frac{9\pi}{2} = 3R + \frac{\pi r^2}{2}$$

Since $R = \dfrac{4\pi}{3}$, we have
$$\frac{9\pi}{2} = 4\pi + \frac{\pi r^2}{2}$$

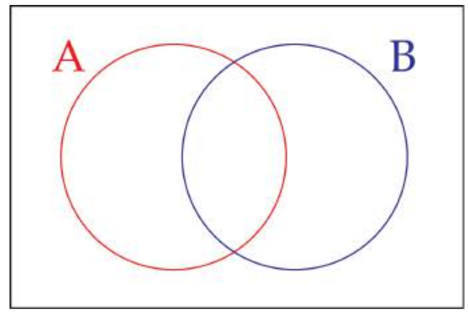
Therefore, $\dfrac{\pi}{2} = \dfrac{\pi r^2}{2}$ or $r^2 = 1$. Thus $r = 1$, since $r > 0$.

Therefore, the diameter of the smallest semi-circle is 2 m.

Problem of the Week
Problem D
Everything in its Place 2

(a) A Venn diagram has two circles, labelled A and B. Each circle contains ordered pairs, (x, y), where x and y are real numbers, that satisfy the following criteria.

A: $y = -x + 1$
B: $y = 3x + 5$

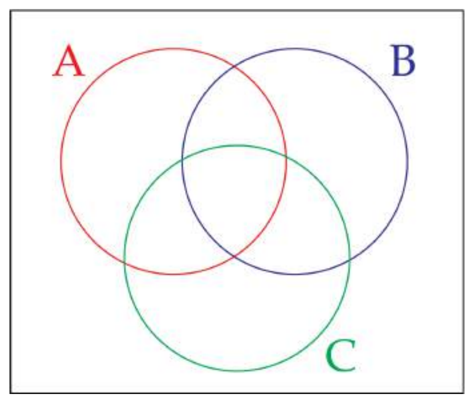


The overlapping region in the middle contains ordered pairs that are in both A and B, and the region outside both circles contains ordered pairs that are neither in A nor B.

In total this Venn diagram has four regions. Place ordered pairs in as many of the regions as you can. Is it possible to find an ordered pair for each region?

(b) A Venn diagram has three circles, labelled A, B, and C. Each circle contains integers, n, that satisfy the following criteria.

A: $3n < 20$
B: $n + 9 > 6$
C: n is even



In total this Venn diagram has eight regions. Place integers in as many of the regions as you can. Is it possible to find an integer for each region?

Problem of the Week
Problem D and Solution
Everything in its Place 2

Problem

(a) A Venn diagram has two circles, labelled A and B. Each circle contains ordered pairs, (x, y), where x and y are real numbers, that satisfy the following criteria.

A: $y = -x + 1$

B: $y = 3x + 5$

The overlapping region in the middle contains ordered pairs that are in both A and B, and the region outside both circles contains ordered pairs that are neither in A nor B. In total this Venn diagram has four regions. Place ordered pairs in as many of the regions as you can. Is it possible to find an ordered pair for each region?

(b) A Venn diagram has three circles, labelled A, B, and C. Each circle contains integers, n, that satisfy the following criteria.

A: $3n < 20$

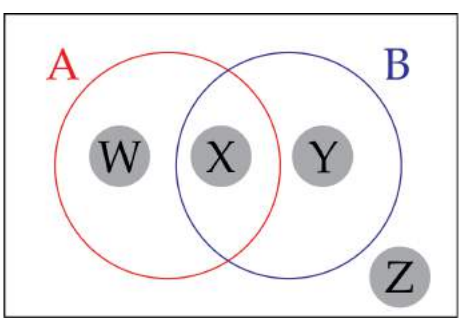
B: $n + 9 > 6$

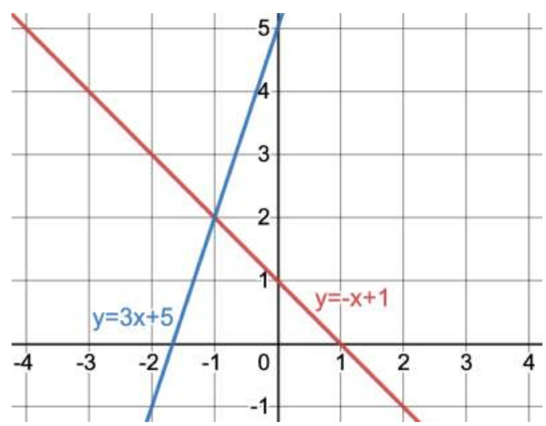
C: n is even

In total this Venn diagram has eight regions. Place integers in as many of the regions as you can. Is it possible to find an integer for each region?

Solution

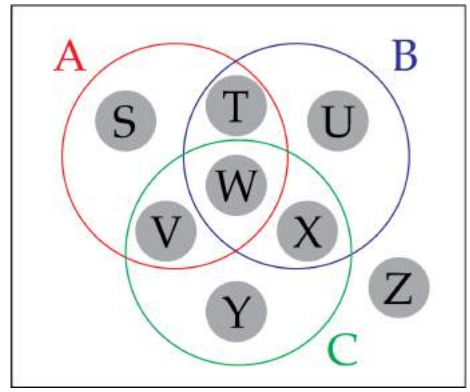
(a) We have marked the four regions W, X, Y, and Z. We plot the given equations on a grid as a reference.





- Any ordered pair, (x, y), in region W must satisfy $y = -x + 1$, but *not* $y = 3x + 5$. Any point on the line $y = -x + 1$ that is *not* on the line $y = 3x + 5$ will satisfy this. An example is $(0, 1)$.
- Any ordered pair, (x, y), in region X must satisfy both $y = -x + 1$ and $y = 3x + 5$. The only point that satisfies this is the point of intersection, $(-1, 2)$.
- Any ordered pair, (x, y), in region Y must satisfy $y = 3x + 5$, but *not* $y = -x + 1$. Any point on the line $y = 3x + 5$ that is *not* on the line $y = -x + 1$ will satisfy this. An example is $(0, 5)$.
- Any ordered pair, (x, y), in region Z must *not* satisfy $y = 3x + 5$ or $y = -x + 1$. Any point that is not on either line will satisfy this. An example is $(2, 2)$.

(b) We have marked the eight regions S, T, U, V, W, X, Y, and Z. It is helpful if we first solve the given inequalities.



For A:

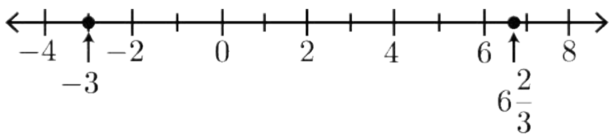
$$3n < 20$$

$$n < \frac{20}{3} = 6\frac{2}{3}$$

For B:

$$n + 9 > 6$$

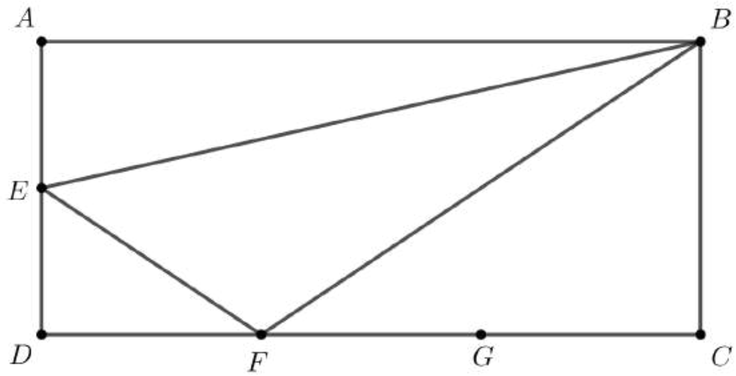
$$n > -3$$



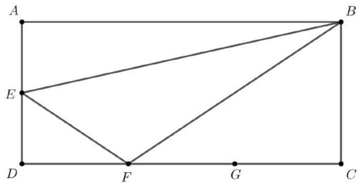
- Any integer in region S must be less than $6\frac{2}{3}$, less than or equal to -3, and an odd number. Any odd integer less than or equal to -3 will satisfy this. An example is -5.
- Any integer in region T must be less than $6\frac{2}{3}$, greater than -3, and an odd number. The only integers that satisfy this are $-1, 1, 3$, and 5.
- Any integer in region U must be greater than or equal to $6\frac{2}{3}$, greater than -3, and an odd number. Any odd integer greater than or equal to $6\frac{2}{3}$ will satisfy this. An example is 7.
- Any integer in region V must be less than $6\frac{2}{3}$, less than or equal to -3, and an even number. Any even integer less than or equal to -3 will satisfy this. An example is -4.
- Any integer in region W must be less than $6\frac{2}{3}$, greater than -3, and an even number. The only integers that satisfy this are $-2, 0, 2, 4$, and 6.
- Any integer in region X must be greater than or equal to $6\frac{2}{3}$, greater than -3, and an even number. Any even integer greater than or equal to $6\frac{2}{3}$ will satisfy this. An example is 8.
- Any integer in region Y must be greater than or equal to $6\frac{2}{3}$, less than or equal to -3, and an even number. No integer satisfies all three conditions, so this region must be left blank.
- Any integer in region Z must be greater than or equal to $6\frac{2}{3}$, less than or equal to -3, and an odd number. No integer satisfies all three conditions, so this region must also be left blank.

Problem of the Week
Problem D
The Whole Rectangle

In the diagram, $ABCD$ is a rectangle. Points F and G are on DC (with F closer to D) such that $DF = FG = GC$. Point E is the midpoint of AD.



If the area of $\triangle BEF$ is 30 cm^2, determine the area of rectangle $ABCD$.



Problem of the Week

Problem D and Solution

The Whole Rectangle

Problem

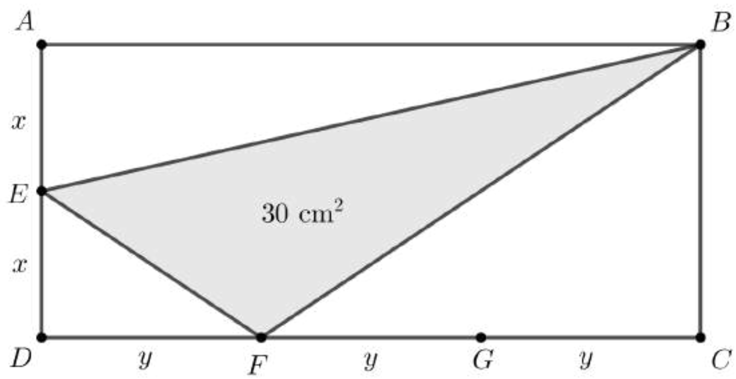
In the diagram, $ABCD$ is a rectangle. Points F and G are on DC (with F closer to D) such that $DF = FG = GC$. Point E is the midpoint of AD.

If the area of $\triangle BEF$ is 30 cm^2, determine the area of rectangle $ABCD$.

Solution

Let $DF = FG = GC = y$. Then $AB = DC = 3y$ and $FC = 2y$.

Since E is the midpoint of AD, let $AE = ED = x$. Then $AD = BC = 2x$.



We will formulate an equation connecting the areas of the four triangles inside the rectangle to the area of the entire rectangle.

$$\begin{aligned}
\text{Area } ABCD &= \text{Area } \triangle ABE + \text{Area } \triangle BCF + \text{Area } \triangle FDE + \text{Area } \triangle BEF\\
AD \times DC &= \frac{AE \times AB}{2} + \frac{BC \times FC}{2} + \frac{DF \times ED}{2} + 30\\
(2x)(3y) &= \frac{x \times 3y}{2} + \frac{2x \times 2y}{2} + \frac{y \times x}{2} + 30\\
6xy &= \frac{3xy}{2} + 2xy + \frac{xy}{2} + 30\\
12xy &= 3xy + 4xy + xy + 60\\
4xy &= 60\\
xy &= 15
\end{aligned}$$

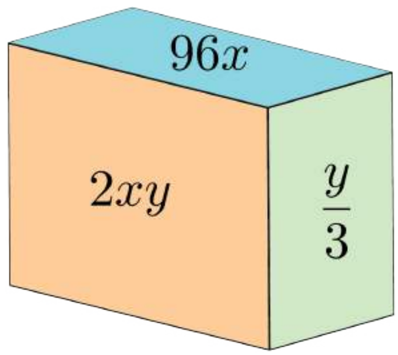
Therefore, the area of rectangle $ABCD$ is $AD \times DC = (2x)(3y) = 6xy = 6(15) = 90$ cm^2.

Problem of the Week

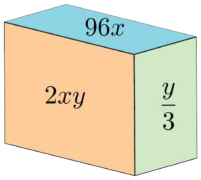
Problem D

I Want Some Volume

The areas of the front, side, and top faces of a rectangular prism are $2xy$, $\frac{y}{3}$, and $96x$ cm^2, respectively.



Calculate the volume of the rectangular prism in terms of x and y.



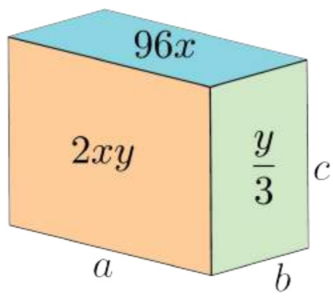
Problem of the Week
Problem D and Solution
I Want Some Volume

Problem

The areas of the front, side, and top faces of a rectangular prism are $2xy$, $\frac{y}{3}$, and $96x$ cm^2, respectively. Calculate the volume of the rectangular prism in terms of x and y.

Solution

Since $\frac{y}{3}$ and $96x$ are areas, then x and y must be positive. Let the length, width, and height of the rectangular prism be a, b, and c, respectively.



The volume is equal to the product abc.

By multiplying side lengths, we can write the following three equations using the given areas.

$$\begin{aligned} ac &= 2xy \\ bc &= \frac{y}{3} \\ ab &= 96x \end{aligned}$$

Multiplying the left sides and multiplying the right sides of each of the three equations gives us the following.

$$\begin{aligned} (ac)(bc)(ab) &= (2xy)\left(\frac{y}{3}\right)(96x) \\ a^2b^2c^2 &= 64x^2y^2 \\ (abc)^2 &= (8xy)^2 \\ \sqrt{(abc)^2} &= \pm\sqrt{(8xy)^2} \\ abc &= \pm\, 8xy \end{aligned}$$

Since all quantities are positive, we can conclude that $abc = 8xy$.

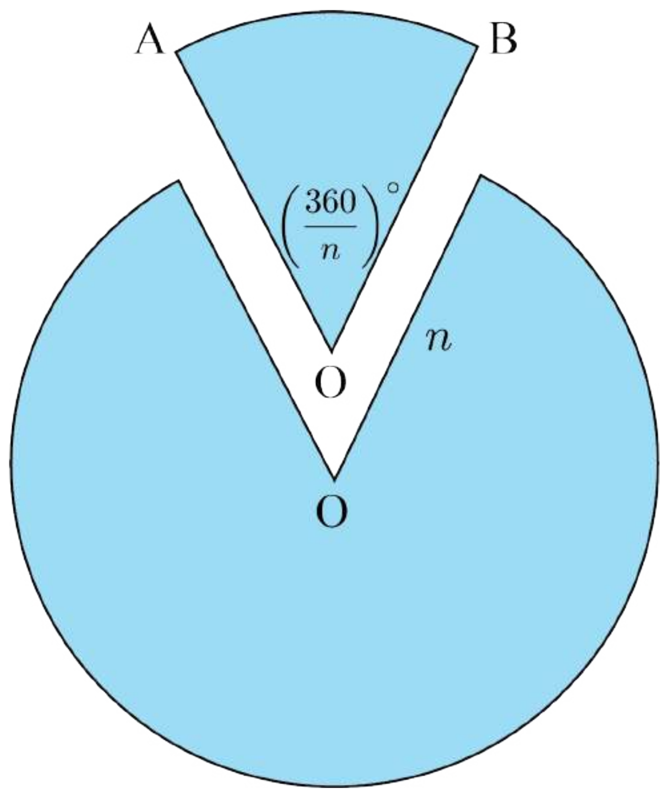
Therefore, the volume of the rectangular prism is $8xy$ cm^3.

Problem of the Week

Problem D

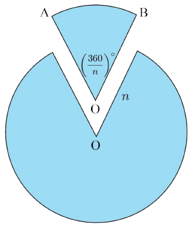
One Slice at a Time

Points A and B are on a circle with centre O and radius n so that $\angle AOB = \left(\frac{360}{n}\right)^\circ$. Sector AOB is cut out of the circle.



Determine all positive integers n for which the perimeter of sector AOB is greater than 20 and less than 30.

NOTE: You may use the fact that the ratio of the length of an arc to the circumference of the circle is the same as the ratio of the sector angle to 360°. In fact, the same ratio holds when comparing the area of a sector to the total area of the circle.



Problem of the Week

Problem D and Solution

One Slice at a Time

Problem

Points A and B are on a circle with centre O and radius n so that $\angle AOB = \left(\frac{360}{n}\right)^\circ$. Sector AOB is cut out of the circle. Determine all positive integers n for which the perimeter of sector AOB is greater than 20 and less than 30.

NOTE: You may use the fact that the ratio of the length of an arc to the circumference of the circle is the same as the ratio of the sector angle to $360°$. In fact, the same ratio holds when comparing the area of a sector to the total area of the circle.

Solution

In general, as the sector angle gets larger, so does the length of the arc, if the radius remains the same. However in this problem, as the radius n increases, the sector angle $\left(\frac{360}{n}\right)^\circ$ decreases. So it is difficult to "see" what happens to the length of the arc.

We know the ratio of the arc length to the circumference of the circle is the same as the ratio of the sector angle to $360°$. That is,

$$\frac{\text{arc length of } AB}{\text{circumference}} = \frac{\text{sector angle of } AOB}{360^\circ}$$

Rearranging, we have

$$\text{arc length of } AB = \frac{\text{sector angle of } AOB}{360^\circ} \times \text{circumference}$$

We know circumference $= \pi d = \pi \times 2n$, since $d = 2n$. Thus,

$$\text{arc length of } AB = \frac{\frac{360}{n}}{360} \times \pi \times 2n = 2\pi$$

Now we can use the arc length to calculate the perimeter of AOB.

$$\begin{aligned} \text{perimeter of } AOB &= AO + OB + \text{arc length of } AB \\ &= n + n + 2\pi \\ &= 2n + 2\pi \end{aligned}$$

If the perimeter is greater than 20, then

$$\begin{aligned} 2n + 2\pi &> 20 \\ n + \pi &> 10 \\ n &> 10 - \pi \approx 6.9 \end{aligned}$$

If the perimeter is less than 30, then

$$\begin{aligned} 2n + 2\pi &< 30 \\ n + \pi &< 15 \\ n &< 15 - \pi \approx 11.9 \end{aligned}$$

We want all integer values of n such that $n > 6.9$ and $n < 11.9$. The only integer values of n that satisfy these conditions are $n = 7$, $n = 8$, $n = 9$, $n = 10$, and $n = 11$.

Problem of the Week
Problem D
Cartesian Geocaching

Geocaching is a kind of outdoor treasure hunt where people use GPS devices to look for hidden objects, called caches. In Cartesian Geocaching, instead of using a GPS device, locations are described using Cartesian coordinates.

Hilde sets up a large field for Cartesian Geocaching, measuring the distances in kilometres so that the point $(1, 0)$ lies 1 km east of the point $(0, 0)$, for example.

Hilde starts at point $A(0, 0)$, then walks northwest in a straight line to some point B, where she hides a cache. Then, from B, she walks northeast in a straight line to point $C(0, 4)$ where she hides another cache. Finally she walks straight back to point A.

How far does Hilde walk in total?

Problem of the Week
Problem D and Solution
Cartesian Geocaching

Problem

Geocaching is a kind of outdoor treasure hunt where people use GPS devices to look for hidden objects, called caches. In Cartesian Geocaching, instead of using a GPS device, locations are described using Cartesian coordinates.

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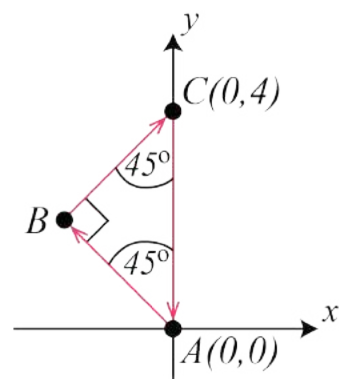
How far does Hilde walk in total?

Solution

We will show four different solutions to this problem.

Solution 1



If you travel northwest from $A(0, 0)$, the line of travel will make a 45° angle with the positive y-axis. Point B is located somewhere on this line of travel. If you travel northeast from point B to $C(0, 4)$, the line will intersect the y-axis at a 45° angle.

In $\triangle ABC$, $\angle BAC = \angle BCA = 45^\circ$. It follows that $\triangle ABC$ is isosceles. Since two of the angles in $\triangle ABC$ are 45°, then the third angle, $\angle ABC = 90^\circ$ and the triangle is right-angled.

The distance from point A to point C along the y-axis is $AC = 4$ km. Let $BC = AB = m$, for some $m > 0$. Using the Pythagorean Theorem, we can find the value of m.

$$\begin{aligned} AC^2 &= BC^2 + AB^2 \\ 4^2 &= m^2 + m^2 \\ 16 &= 2m^2 \\ 8 &= m^2 \end{aligned}$$

Then since $m > 0$, we have $m = \sqrt{8}$.

Thus, the total distance walked by Hilde is $AB + BC + AC = \sqrt{8} + \sqrt{8} + 4 = (2\sqrt{8} + 4)$ km.

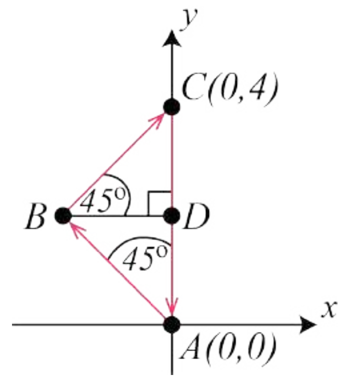
Note that the answer $\left(2\sqrt{8} + 4\right)$ is an *exact* answer. We can use a calculator to determine that this distance is approximately 9.7 km.

The exact total distance travelled can be further simplified as follows:

$$2\sqrt{8} + 4 = 2(\sqrt{4}\sqrt{2}) + 4 = 2(2\sqrt{2}) + 4 = 4\sqrt{2} + 4$$

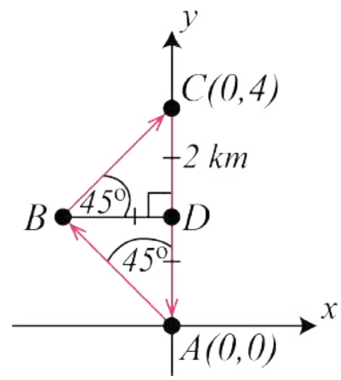
This method of simplifying radicals is developed in later mathematics courses.

Solution 2



If you travel northwest from $A(0,0)$, the line of travel will make a $45°$ angle with the positive y-axis. Point B is located somewhere on this line of travel.

From B, draw a line segment perpendicular to the y-axis, meeting the y-axis at point D. A line of travel in a northeast direction from point B to C will make $\angle DBC = 45°$.



In $\triangle ABD$, $\angle BAD = 45°$ and $\angle ADB = 90°$. It follows that $\angle ABD = 45°$, $\triangle ABD$ is isosceles and $BD = AD$.

In $\triangle CBD$, $\angle CBD = 45°$ and $\angle CDB = 90°$. It follows that $\angle BCD = 45°$, $\triangle CBD$ is isosceles and $CD = BD$.

The distance from point A to point C along the y-axis is $AC = 4$ km. Since $CD = AD$ and $AC = CD + AD$, then we know that $CD = AD = 2$ km. But $CD = BD$ so $CD = BD = AD = 2$ km.

Using the Pythagorean Theorem in right-angled $\triangle ABD$, we can calculate the length of AB.

$$\begin{aligned} AB^2 &= BD^2 + AD^2 \\ AB^2 &= 2^2 + 2^2 \\ AB^2 &= 8 \end{aligned}$$

Then since $AB > 0$, we have $AB = \sqrt{8}$.

Using the same reasoning in $\triangle CBD$, we obtain $BC = \sqrt{8}$.

Thus, the total distance walked by Hilde is $AB + BC + AC = \sqrt{8} + \sqrt{8} + 4 = (2\sqrt{8} + 4)$ km.

Note that the answer $\left(2\sqrt{8} + 4\right)$ is an *exact* answer. We can use a calculator to determine that this distance is approximately 9.7 km.

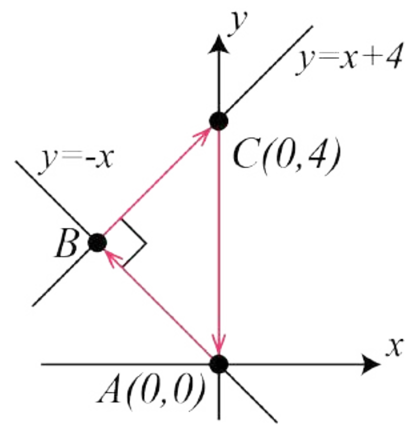
The exact total distance travelled can be further simplified as follows:

$$2\sqrt{8} + 4 = 2(\sqrt{4}\sqrt{2}) + 4 = 2(2\sqrt{2}) + 4 = 4\sqrt{2} + 4$$

This method of simplifying radicals is developed in later mathematics courses.

Solution 3

If you travel northwest from $A(0,0)$, the line of travel will make a $45°$ angle with the positive y-axis. It follows that this line has slope -1. Since this line passes through $A(0,0)$ and has slope -1, the equation of the line through A and B is $y = -x$.



Point B is located somewhere on $y = -x$. A line drawn to the northeast would be perpendicular to a line drawn to the northwest. Since a line to the northwest has slope -1, it follows that a line to the northeast would have slope 1. This second line passes through B and C, so it has slope 1 and y-intercept 4, the y-coordinate of C. The equation of the second line is $y = x + 4$.

Since point B is located on both $y = -x$ and $y = x + 4$, we can solve the system of equations to find the coordinates of B. Since $y = y$,

$$\begin{aligned} -x &= x + 4 \\ -2x &= 4 \\ x &= -2 \end{aligned}$$

Substituting $x = -2$ into $y = -x$, we obtain $y = 2$. The coordinates of B are therefore $(-2, 2)$.

Using the distance formula, we can find the lengths of AB and BC.

$$\begin{aligned} AB &= \sqrt{(-2-0)^2 + (2-0)^2} = \sqrt{4+4} = \sqrt{8} \\ BC &= \sqrt{(0-(-2))^2 + (4-2)^2} = \sqrt{4+4} = \sqrt{8} \end{aligned}$$

The distance from point A to point C along the y-axis is $AC = 4$ km. That is, $AC = 4$.

Thus, the total distance walked by Hilde is $AB + BC + AC = \sqrt{8} + \sqrt{8} + 4 = (2\sqrt{8} + 4)$ km.

Note that the answer $\left(2\sqrt{8} + 4\right)$ is an *exact* answer. We can use a calculator to determine that this distance is approximately 9.7 km.

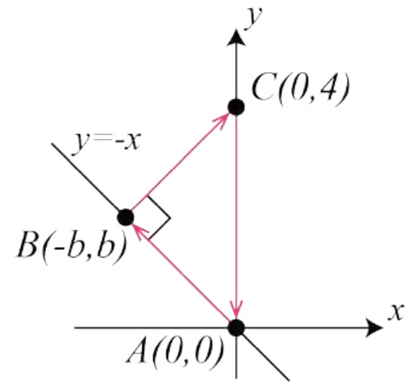
The exact total distance travelled can be further simplified as follows:

$$2\sqrt{8} + 4 = 2(\sqrt{4}\sqrt{2}) + 4 = 2(2\sqrt{2}) + 4 = 4\sqrt{2} + 4$$

This method of simplifying radicals is developed in later mathematics courses.

Solution 4

If you travel northwest from $A(0,0)$, the line of travel will make a $45°$ angle with the positive y-axis. It follows that this line has slope -1. Since this line passes through $A(0,0)$ and has slope -1, the equation of the line through A and B is $y = -x$. A line drawn to the northeast would be perpendicular to a line drawn to the northwest, so AB is perpendicular to BC, and thus $\angle ABC = 90°$.



Point B is located somewhere on $y = -x$. Let the coordinates of B be $(-b, b)$ for some $b > 0$.

Using the distance formula, we can find expressions for the lengths of AB and BC.

$$AB = \sqrt{(-b-0)^2 + (b-0)^2} = \sqrt{b^2 + b^2} = \sqrt{2b^2}$$
$$BC = \sqrt{(0-(-b))^2 + (4-b)^2} = \sqrt{b^2 + 16 - 8b + b^2} = \sqrt{2b^2 - 8b + 16}$$

The distance from point A to point C along the y-axis is $AC = 4$ km. That is, $AC = 4$.

Using the Pythagorean Theorem, we can find the value of b.

$$\begin{aligned}
AC^2 &= AB^2 + BC^2 \\
4^2 &= \left(\sqrt{2b^2}\right)^2 + \left(\sqrt{2b^2 - 8b + 16}\right)^2 \\
16 &= 2b^2 + (2b^2 - 8b + 16) \\
16 &= 4b^2 - 8b + 16 \\
0 &= 4b^2 - 8b \\
0 &= b^2 - 2b \\
0 &= b(b-2) \\
b &= 0, 2
\end{aligned}$$

Since $b > 0$, it follows that $b = 2$. We can substitute $b = 2$ into our expressions for AB and BC.

$$AB = \sqrt{2b^2} = \sqrt{2(2)^2} = \sqrt{8}$$
$$BC = \sqrt{2b^2 - 8b + 16} = \sqrt{2(2)^2 - 8(2) + 16} = \sqrt{8}$$

Thus, the total distance walked by Hilde is $AB + BC + AC = \sqrt{8} + \sqrt{8} + 4 = (2\sqrt{8} + 4)$ km.

Note that the answer $\left(2\sqrt{8} + 4\right)$ is an *exact* answer. We can use a calculator to determine that this distance is approximately 9.7 km.

The exact total distance travelled can be further simplified as follows:

$$2\sqrt{8} + 4 = 2(\sqrt{4}\sqrt{2}) + 4 = 2(2\sqrt{2}) + 4 = 4\sqrt{2} + 4$$

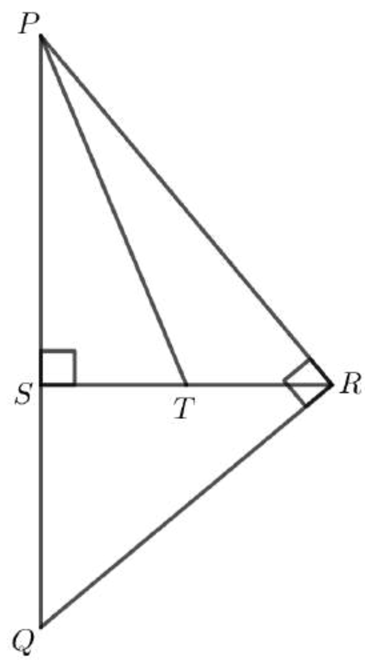
This method of simplifying radicals is developed in later mathematics courses.

Problem of the Week

Problem D

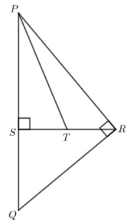
Which Term is Which?

In $\triangle PQR$, $\angle PRQ = 90^\circ$. An altitude is drawn in $\triangle PQR$ from R to PQ, intersecting PQ at S. A median is drawn in $\triangle PSR$ from P to SR, intersecting SR at T.



If the length of the median PT is 39 and the length of PS is 36, determine the length of QS.

NOTE: An *altitude* of a triangle is a line segment drawn from a vertex of the triangle perpendicular to the opposite side. A *median* is a line segment drawn from a vertex of the triangle to the midpoint of the opposite side.



Problem of the Week
Problem D and Solution
Which Term is Which?

Problem

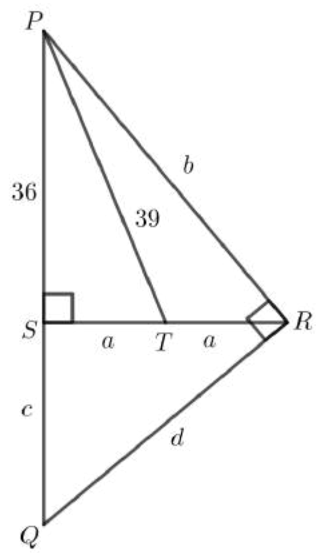
In $\triangle PQR$, $\angle PRQ = 90°$. An altitude is drawn in $\triangle PQR$ from R to PQ, intersecting PQ at S. A median is drawn in $\triangle PSR$ from P to SR, intersecting SR at T.

If the length of the median PT is 39 and the length of PS is 36, determine the length of QS.

Note: An *altitude* of a triangle is a line segment drawn from a vertex of the triangle perpendicular to the opposite side. A *median* is a line segment drawn from a vertex of the triangle to the midpoint of the opposite side.

Solution

Since T is a median in $\triangle PSR$, $ST = TR$. Let $ST = TR = a$. Let $PR = b$, $QS = c$, and $QR = d$. The variables and the given information, $PS = 36$ and $PT = 39$, are shown in the diagram.



Since $\triangle PST$ contains a right angle at S,

$$\begin{aligned} ST^2 &= PT^2 - PS^2 \\ a^2 &= 39^2 - 36^2 \\ &= 225 \end{aligned}$$

Then, since $a > 0$, $a = 15$ follows. Thus, $SR = 2a = 30$.

Since $\triangle PSR$ contains a right angle at S,

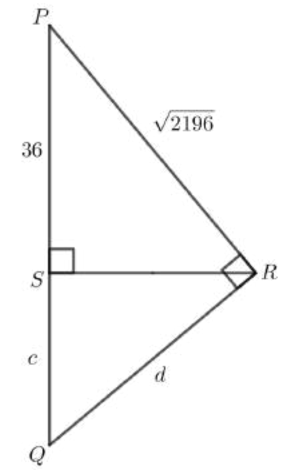
$$\begin{aligned} PR^2 &= PS^2 + SR^2 \\ b^2 &= 36^2 + 30^2 \\ &= 2196 \end{aligned}$$

Then, since $b > 0$, $b = \sqrt{2196}$ follows.

We will now use $a = 15$ and $b = \sqrt{2196}$ in the three solutions that follow.

Solution 1

In $\triangle PSR$ and $\triangle PRQ$, $\angle PSR = \angle PRQ = 90^\circ$ and $\angle SPR = \angle QPR$, a common angle. So $\triangle PSR$ is similar to $\triangle PRQ$. It follows that



$$\begin{aligned} \frac{PS}{PR} &= \frac{PR}{PQ} \\ \frac{36}{\sqrt{2196}} &= \frac{\sqrt{2196}}{36+c} \\ 1296 + 36c &= 2196 \\ 36c &= 900 \\ c &= 25 \end{aligned}$$

Thus, the length of QS is 25.

Solution 2

Since $\triangle RSQ$ contains a right angle at S, $QR^2 = QS^2 + SR^2 = c^2 + 30^2 = c^2 + 900$. Therefore, $d^2 = c^2 + 900$.

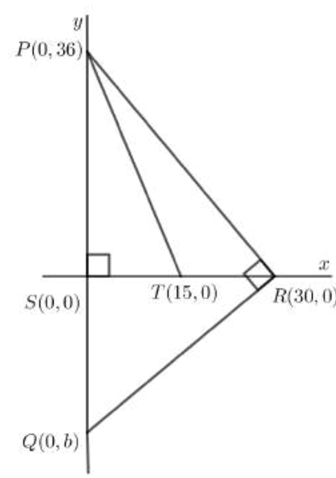
Since $\triangle PQR$ contains a right angle at R, $PQ^2 = PR^2 + QR^2$. Therefore, $(36 + c)^2 = (\sqrt{2196})^2 + d^2$, which simplifies to $1296 + 72c + c^2 = 2196 + d^2$. This further simplifies to $c^2 + 72c = 900 + d^2$.

Substituting $d^2 = c^2 + 900$, we obtain $c^2 + 72c = 900 + c^2 + 900$. Simplifying, we get $72c = 1800$ and $c = 25$ follows.

Thus, the length of QS is 25.

Solution 3

Position $\triangle PQR$ on the xy-plane so that PQ lies along the y-axis, and altitude SR lies along the positive x-axis with S at the origin. Then P has coordinates $(0, 36)$, T has coordinates $(15, 0)$, and R has coordinates $(30, 0)$.
Since Q is on the y-axis, let Q have coordinates $(0, b)$ with $b < 0$.
Notice that



$$\text{slope } PR = \frac{36-0}{0-30} = \frac{-6}{5} \text{ and slope } QR = \frac{b-0}{0-30} = \frac{b}{-30}$$

Since $\angle PRQ = 90^\circ$, $PR \perp QR$, and so their slopes are negative reciprocals of each other. That is, $\dfrac{b}{-30} = \dfrac{5}{6}$, and so $b = -25$.

It then follows that the coordinates of Q are $(0, -25)$. Thus, the length of QS is 25.

Algebra (A)

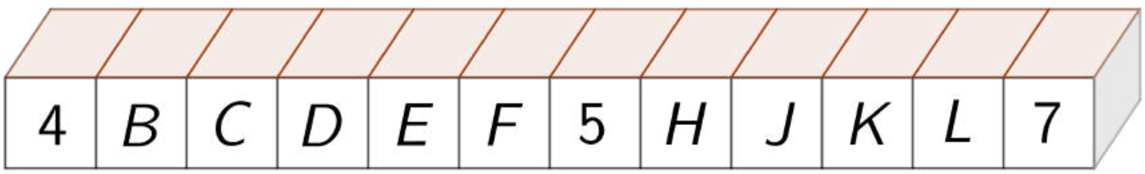
TAKE ME TO THE COVER

Problem of the Week

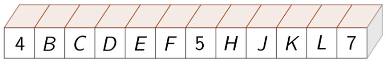
Problem D

Blocked Numbers

Twelve blocks are arranged as illustrated in the diagram.



Each letter shown on the front of a block represents a number. The sum of the numbers on any four consecutive blocks is 25. Determine the value of $B + F + K$.



Problem of the Week
Problem D and Solution
Blocked Numbers

Problem
Twelve blocks are arranged as illustrated in the diagram. Each letter shown on the front of a block represents a number. The sum of the numbers on any four consecutive blocks is 25. Determine the value of $B + F + K$.

Solution

Since the sum of the numbers on any four consecutive blocks is the same, looking at the first five blocks, we have

$$4 + B + C + D = B + C + D + E$$

Subtracting B, C, and D from both sides gives $E = 4$. Similarly, looking at the fifth through ninth blocks, we can show $J = 4$.

Again, since the sum of the numbers on any four consecutive blocks is the same, looking at the third through seventh blocks, we have

$$C + D + E + F = D + E + F + 5$$

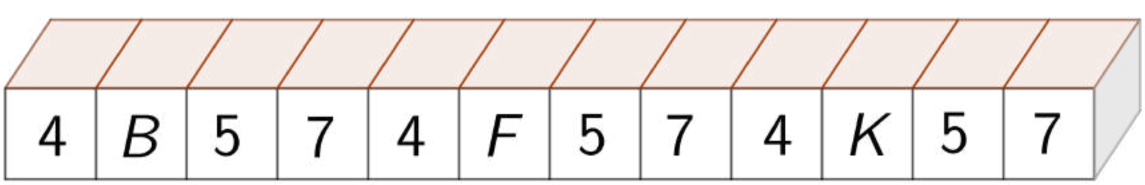
Subtracting D, E, and F from both sides gives $C = 5$. Similarly, looking at the seventh through eleventh blocks, we can show $L = 5$.

Once more, since the sum of the numbers on any four consecutive blocks is the same, looking at the eighth through twelfth blocks, we have

$$H + J + K + L = J + K + L + 7$$

Subtracting J, K, and L from both sides, gives $H = 7$. Similarly, looking at the fourth through eighth blocks, we can show $D = 7$.

Filling in the above information, the blocks now look like:



We will present two different solutions from this point.

Solution 1:
Since the sum of any four consecutive numbers is 25, using the first 4 blocks

$$\begin{aligned} 4 + B + 5 + 7 &= 25 \\ B + 16 &= 25 \\ B &= 9 \end{aligned}$$

Similarly, we can show $F = 9$ and $K = 9$.

Therefore, $B + F + K = 27$.

Solution 2:

We note that the twelve blocks are three sets of four consecutive blocks. Each of these three sets have a total of 25, so the total sum of the blocks is $3 \times 25 = 75$.

The sum is also

$$4 + B + 5 + 7 + 4 + F + 5 + 7 + 4 + K + 5 + 7 = 48 + B + F + K$$

This means

$$48 + B + F + K = 75$$

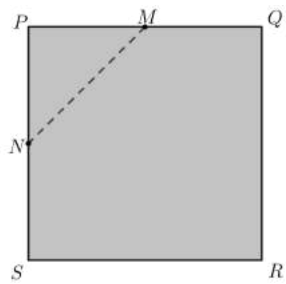
or

$$B + F + K = 27$$

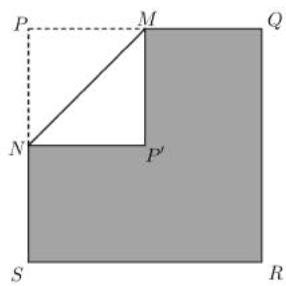
Therefore, $B + F + K = 27$.

Problem of the Week
Problem D
From Square to Hexagon

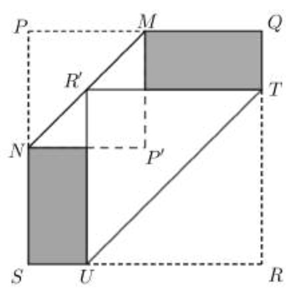
A square piece of paper, $PQRS$, has side length 40 cm. The page is grey on one side and white on the other side. Point M is the midpoint of side PQ and point N is the midpoint of side PS.



The paper is folded along MN so that P touches the paper at the point P'.



Point T lies on QR and point U lies on SR such that TU is parallel to MN, and when the paper is folded along TU, the point R touches the paper at the point R' on MN.



What is the area of hexagon $NMQTUS$?

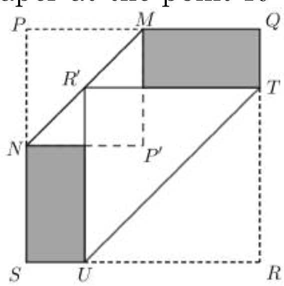
Here are some known properties of the diagonals of a square that may be useful:

- the diagonals are equal in length; and
- the diagonals right bisect each other; and
- the diagonals bisect the corner angles.

Problem of the Week
Problem D and Solution
From Square to Hexagon

Problem

A square piece of paper, $PQRS$, has side length 40 cm. The page is grey on one side and white on the other side. Point M is the midpoint of side PQ and point N is the midpoint of side PS. The paper is folded along MN so that P touches the paper at the point P'. Point T lies on QR and point U lies on SR such that TU is parallel to MN, and when the paper is folded along TU, the point R touches the paper at the point R' on MN.



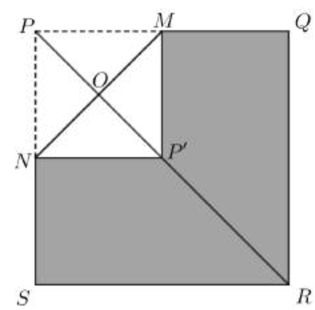
What is the area of hexagon $NMQTUS$?

Solution

To determine the area of hexagon $NMQTUS$, we will subtract the area of $\triangle PMN$ and the area of $\triangle TRU$ from the area of square $PQRS$.

Since M and N are the midpoints of PQ and PS, respectively, we know $PM = \frac{1}{2}(PQ) = 20$ cm and $PN = \frac{1}{2}(PS) = 20$ cm. Therefore, $PM = PN = 20$ and $\triangle PMN$ is an isosceles right-angled triangle. It follows that $\angle PNM = \angle PMN = 45°$.

After the first fold, P touches the paper at P'. $\triangle P'MN$ is a reflection of $\triangle PMN$ in the line segment MN. It follows that $\angle P'MN = \angle PMN = 45°$ and $\angle P'NM = \angle PNM = 45°$. Therefore, $\angle PMP' = \angle PNP' = 90°$. Since all four sides of $PMP'N$ are equal in length and all four corners are $90°$, $PMP'N$ is a square. Since $\angle MPP' = \angle MPR = 45°$, the diagonal PP' of square $PMP'N$ lies along the diagonal PR of square $PQRS$. Let O be the intersection of the two diagonals of square $PMP'N$. It is also the intersection of MN and PR. (We will show later that this is in fact R', the point of contact of R with the paper after the second fold.)



The length of the diagonal of square $PMP'N$ can be found using the Pythagorean Theorem.

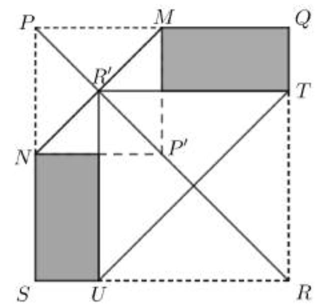
$$PP' = \sqrt{(PM)^2 + (MP')^2} = \sqrt{20^2 + 20^2} = \sqrt{800} = \sqrt{400}\sqrt{2} = 20\sqrt{2}$$

Thus, $PO = \frac{1}{2}(PP') = \frac{1}{2}(20\sqrt{2}) = 10\sqrt{2}$ cm.

In the last two steps of calculating PP', we simplified the radical. We will do this quite often in the solution. Here is the process to simplify radicals, for students who may not be familiar with this:

- Find the largest perfect square that divides into the radicand (the number under the root symbol). In this case, 400 is the largest perfect square that divides 800.
- Rewrite the radicand as the product of the perfect square and the remaining factor. In this case, we get $\sqrt{400 \times 2}$.
- Take the square root of the perfect square. In this case, we get $20\sqrt{2}$.

Since TU is parallel to MN, it follows that $\angle RTU = \angle RUT = 45°$ and $\triangle TRU$ is an isosceles right-angled triangle with $TR = RU$. When $\triangle TRU$ is reflected in the line segment TU with R' being the image of R, a square, $TRUR'$, is created. We will not present the argument here because it is very similar to the argument presented for $PMP'N$. Since $\angle TRR' = \angle TRP = 45°$, RR' lies along the diagonal PR. Also, R' lies on MN. This means that R' and O are the same point and so $PR' = PO = 10\sqrt{2}$ cm.



The length of the diagonal of square $PQRS$ can be calculated using the Pythagorean Theorem.

$$PR = \sqrt{(PQ)^2 + (QR)^2} = \sqrt{40^2 + 40^2} = \sqrt{3200} = \sqrt{1600}\sqrt{2} = 40\sqrt{2}$$

The length of RR' equals the length of PR minus the length of PR'.

$$RR' = PR - PR' = 40\sqrt{2} - 10\sqrt{2} = 30\sqrt{2}$$

But $RR' = TU$, so $TU = 30\sqrt{2}$ cm. Let $TR = RU = x$. Then, using the Pythagorean Theorem in $\triangle TRU$,

$$\begin{aligned}(TR)^2 + (RU)^2 &= (TU)^2\\ x^2 + x^2 &= (30\sqrt{2})^2\\ x^2 + x^2 &= 900 \times 2\\ 2x^2 &= 1800\\ x^2 &= 900\end{aligned}$$

And since $x > 0$, this gives $x = 30$ cm. We now have enough information to calculate the area of hexagon $NMQTUS$.

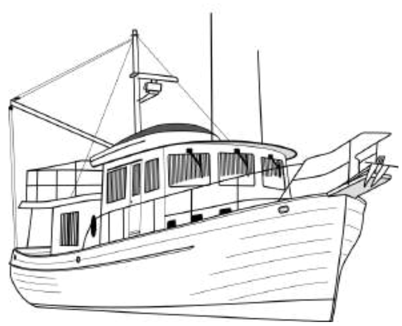
$$\begin{aligned}\text{Area } NMQTUS &= \text{Area } PQRS - \text{Area } \triangle PMN - \text{Area } \triangle TRU\\ &= PQ \times QR - \frac{PM \times PN}{2} - \frac{TR \times RU}{2}\\ &= 40 \times 40 - \frac{20 \times 20}{2} - \frac{30 \times 30}{2}\\ &= 1600 - \frac{400}{2} - \frac{900}{2}\\ &= 1600 - 200 - 450\\ &= 950\end{aligned}$$

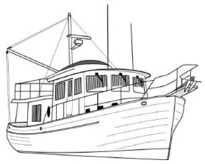
Therefore, the area of hexagon $NMBPQD$ is 950 cm^2.

Problem of the Week
Problem D
Sale Boats

Harold, a marina manager, purchased two boats. He then sold the boats, the first at a profit of 40% and the second at a profit of 60%. The total profit on the sale of the two boats was 54% and \$88 704 was the total selling price of the two boats.

What did Harold originally pay for each of the two boats?





Problem of the Week
Problem D and Solution
Sale Boats

Problem

Harold, a marina manager, purchased two boats. He then sold the boats, the first at a profit of 40% and the second at a profit of 60%. The total profit on the sale of the two boats was 54% and \$88 704 was the total selling price of the two boats. What did Harold originally pay for each of the two boats?

Solution

Solution 1

Let a represent what Harold paid for the first boat, in dollars, and b represent what he paid for the second boat, in dollars.

The profit on the sale of the first boat was 40% or $0.4a$ dollars. Thus, the first boat sold for $a + 0.4a = 1.4a$ dollars. The profit on the sale of the second boat was 60% or $0.6b$ dollars. Thus, the second boat sold for $b + 0.6b = 1.6b$ dollars. The total selling price of the two boats was \$88 704, so we have

$$1.4a + 1.6b = 88\,704 \qquad (1)$$

Harold bought both boats for a total of $(a + b)$ dollars. The profit on the sale of the two boats was 54% or $0.54(a + b)$ dollars. The two boats sold for $(a + b) + 0.54(a + b) = 1.54(a + b)$ dollars. But the total selling price was \$88 704, so

$$\begin{aligned} 1.54(a+b) &= 88\,704 \\ a+b &= 88\,704 \div 1.54 \\ a+b &= 57\,600 \\ a &= 57\,600 - b \end{aligned}$$

Substituting $a = 57\,600 - b$ into equation (1) gives

$$\begin{aligned} 1.4(57\,600 - b) + 1.6b &= 88\,704 \\ 80\,640 - 1.4b + 1.6b &= 88\,704 \\ 0.2b &= 8064 \end{aligned}$$

Dividing by 0.2, we get $b = 40\,320$. Since $b = 40\,320$ and $a + b = 57\,600$, then $a = 17\,280$ follows.

Therefore, Harold paid \$17 280 for the first boat and \$40 320 for the second boat.

Solution 2

Let a represent what Harold paid for the first boat, in dollars, and b represent what he paid for the second boat, in dollars.

The profit on the sale of the first boat was 40% or $0.4a$ dollars. The first boat sold for $a + 0.4a = 1.4a$ dollars. The profit on the sale of the second boat was 60% or $0.6b$ dollars. The second boat sold for $b + 0.6b = 1.6b$ dollars. The total selling price of the two boats was \$88 704 so we have

$$1.4a + 1.6b = 88\,704$$

Multiplying by 5, we get

$$7a + 8b = 443\,520 \tag{1}$$

Harold bought both boats for a total of $(a + b)$ dollars. The profit on the sale of the two boats was 54% or $0.54(a + b)$ dollars. The total profit is the sum of the profit from the sale of each boat, so

$$\begin{aligned} 0.54(a+b) &= 0.4a + 0.6b \\ 0.54a + 0.54b &= 0.4a + 0.6b \\ 0.14a &= 0.06b \end{aligned}$$

Multiplying by 50, we get

$$7a = 3b \tag{2}$$

Substituting $3b$ for $7a$ into equation (1), we get $3b + 8b = 443\,520$ or $11b = 443\,520$, and $b = 40\,320$ follows.

Substituting $b = 40\,320$ into equation (2), we get $7a = 120\,960$, and $a = 17\,280$ follows.

Therefore, Harold paid \$17 280 for the first boat and \$40 320 for the second boat.

Problem of the Week

Problem D

Two Equations and Two Variables

If $2x = 3y + 11$ and $2^x = 2^{4(y+1)}$, determine the value of $x + y$.

Problem of the Week
Problem D and Solution
Two Equations and Two Variables

Problem

If $2x = 3y + 11$ and $2^x = 2^{4(y+1)}$, determine the value of $x + y$.

Solution

Solution 1

Since $2^x = 2^{4(y+1)}$, it follows that $x = 4(y + 1)$, or $x = 4y + 4$. We now have the following two equations.

$$2x = 3y + 11 \qquad (1)$$
$$x = 4y + 4 \qquad (2)$$

We can substitute equation (2) into equation (1) for x.

$$\begin{aligned} 2x &= 3y + 11 \\ 2(4y + 4) &= 3y + 11 \\ 8y + 8 &= 3y + 11 \\ 5y &= 3 \\ y &= \frac{3}{5} \end{aligned}$$

Now, we can substitute $y = \frac{3}{5}$ into equation (2) to solve for x.

$$\begin{aligned} x &= 4y + 4 \\ &= 4\left(\frac{3}{5}\right) + 4 \\ &= \frac{12}{5} + \frac{20}{5} \\ &= \frac{32}{5} \end{aligned}$$

Now that we have the values of x and y, we can determine the value of $x + y$.

$$x + y = \frac{32}{5} + \frac{3}{5} = \frac{35}{5} = 7$$

Therefore, the value of $x + y$ is 7.

Solution 2

We can solve this problem in a faster way without finding the values of x and y. Since $2^x = 2^{4(y+1)}$, it follows that $x = 4(y + 1)$, or $x = 4y + 4$. We now have the following two equations.

$$2x = 3y + 11 \qquad (1)$$
$$x = 4y + 4 \qquad (2)$$

We can subtract equation (2) from equation (1), and obtain the equation $x = -y + 7$.

Rearranging this equation gives $x + y = 7$. Therefore, the value of $x + y$ is 7.

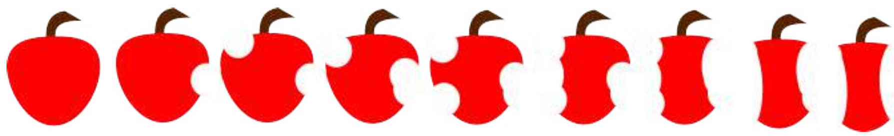
Problem of the Week

Problem D

Follow the Sequence

The first four terms of an arithmetic sequence are x, $2x$, y, and $x - y - 6$, for some integers x and y. What is the value of the 50th term in this sequence?

NOTE: An *arithmetic sequence* is a sequence in which each term after the first is obtained from the previous term by adding a constant. For example, $3, 5, 7, 9$ are the first four terms of an arithmetic sequence.



Problem of the Week
Problem D and Solution
Follow the Sequence

Problem

The first four terms of an arithmetic sequence are x, $2x$, y, and $x - y - 6$, for some integers x and y. What is the value of the 50th term in this sequence?

NOTE: An *arithmetic sequence* is a sequence in which each term after the first is obtained from the previous term by adding a constant. For example, $3, 5, 7, 9$ are the first four terms of an arithmetic sequence.

Solution

Since each term is obtained by adding the same number to the previous term, then the differences between pairs of consecutive terms are equal. Thus, from the first three terms we can conclude

$$\begin{aligned} 2x - x &= y - 2x \\ x &= y - 2x \\ 3x &= y \end{aligned}$$

Now we can substitute $y = 3x$ into the fourth term to write it in terms of x.

$$\begin{aligned} x - y - 6 &= x - 3x - 6 \\ &= -2x - 6 \end{aligned}$$

Therefore, in terms of x, the first four terms are x, $2x$, $3x$, and $-2x - 6$.
However since $2x - x = x$, the common difference is x, so we can also write the fourth term as $4x$. Thus,

$$\begin{aligned} 4x &= -2x - 6 \\ 6x &= -6 \\ x &= -1 \end{aligned}$$

Thus, the first four terms of the sequence are -1, -2, -3, and -4.

To get the 50th term, we must add the common difference 49 times to the first term, to get $-1 + 49(-1) = -50$.

Therefore, the 50th term of the sequence is -50.

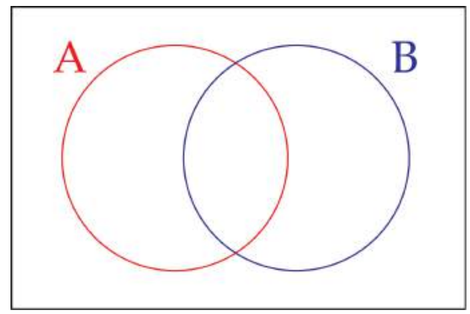
Problem of the Week

Problem D

Everything in its Place 2

(a) A Venn diagram has two circles, labelled A and B. Each circle contains ordered pairs, (x, y), where x and y are real numbers, that satisfy the following criteria.

A: $y = -x + 1$
B: $y = 3x + 5$

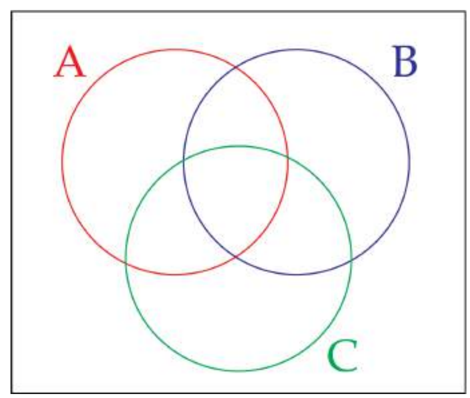


The overlapping region in the middle contains ordered pairs that are in both A and B, and the region outside both circles contains ordered pairs that are neither in A nor B.

In total this Venn diagram has four regions. Place ordered pairs in as many of the regions as you can. Is it possible to find an ordered pair for each region?

(b) A Venn diagram has three circles, labelled A, B, and C. Each circle contains integers, n, that satisfy the following criteria.

A: $3n < 20$
B: $n + 9 > 6$
C: n is even



In total this Venn diagram has eight regions. Place integers in as many of the regions as you can. Is it possible to find an integer for each region?

Problem of the Week
Problem D and Solution
Everything in its Place 2

Problem

(a) A Venn diagram has two circles, labelled A and B. Each circle contains ordered pairs, (x, y), where x and y are real numbers, that satisfy the following criteria.

A: $y = -x + 1$

B: $y = 3x + 5$

The overlapping region in the middle contains ordered pairs that are in both A and B, and the region outside both circles contains ordered pairs that are neither in A nor B. In total this Venn diagram has four regions. Place ordered pairs in as many of the regions as you can. Is it possible to find an ordered pair for each region?

(b) A Venn diagram has three circles, labelled A, B, and C. Each circle contains integers, n, that satisfy the following criteria.

A: $3n < 20$

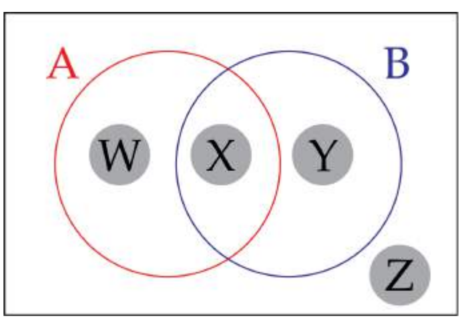
B: $n + 9 > 6$

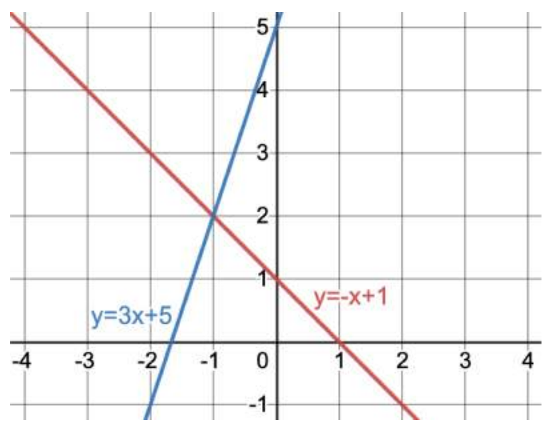
C: n is even

In total this Venn diagram has eight regions. Place integers in as many of the regions as you can. Is it possible to find an integer for each region?

Solution

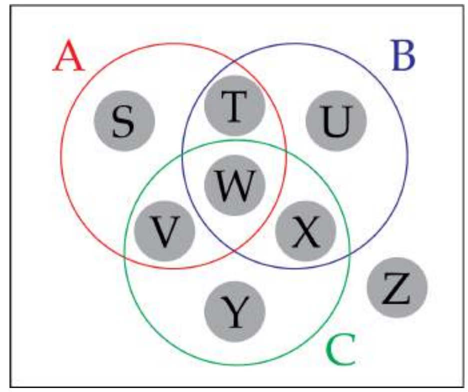
(a) We have marked the four regions W, X, Y, and Z. We plot the given equations on a grid as a reference.





- Any ordered pair, (x, y), in region W must satisfy $y = -x + 1$, but *not* $y = 3x + 5$. Any point on the line $y = -x + 1$ that is *not* on the line $y = 3x + 5$ will satisfy this. An example is $(0, 1)$.
- Any ordered pair, (x, y), in region X must satisfy both $y = -x + 1$ and $y = 3x + 5$. The only point that satisfies this is the point of intersection, $(-1, 2)$.
- Any ordered pair, (x, y), in region Y must satisfy $y = 3x + 5$, but *not* $y = -x + 1$. Any point on the line $y = 3x + 5$ that is *not* on the line $y = -x + 1$ will satisfy this. An example is $(0, 5)$.
- Any ordered pair, (x, y), in region Z must *not* satisfy $y = 3x + 5$ or $y = -x + 1$. Any point that is not on either line will satisfy this. An example is $(2, 2)$.

(b) We have marked the eight regions S, T, U, V, W, X, Y, and Z. It is helpful if we first solve the given inequalities.



For A:

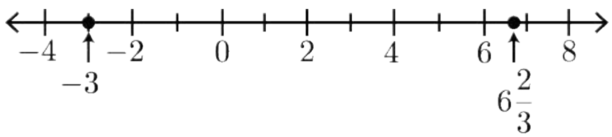
$$3n < 20$$

$$n < \frac{20}{3} = 6\frac{2}{3}$$

For B:

$$n + 9 > 6$$

$$n > -3$$



- Any integer in region S must be less than $6\frac{2}{3}$, less than or equal to -3, and an odd number. Any odd integer less than or equal to -3 will satisfy this. An example is -5.
- Any integer in region T must be less than $6\frac{2}{3}$, greater than -3, and an odd number. The only integers that satisfy this are $-1, 1, 3$, and 5.
- Any integer in region U must be greater than or equal to $6\frac{2}{3}$, greater than -3, and an odd number. Any odd integer greater than or equal to $6\frac{2}{3}$ will satisfy this. An example is 7.
- Any integer in region V must be less than $6\frac{2}{3}$, less than or equal to -3, and an even number. Any even integer less than or equal to -3 will satisfy this. An example is -4.
- Any integer in region W must be less than $6\frac{2}{3}$, greater than -3, and an even number. The only integers that satisfy this are $-2, 0, 2, 4$, and 6.
- Any integer in region X must be greater than or equal to $6\frac{2}{3}$, greater than -3, and an even number. Any even integer greater than or equal to $6\frac{2}{3}$ will satisfy this. An example is 8.
- Any integer in region Y must be greater than or equal to $6\frac{2}{3}$, less than or equal to -3, and an even number. No integer satisfies all three conditions, so this region must be left blank.
- Any integer in region Z must be greater than or equal to $6\frac{2}{3}$, less than or equal to -3, and an odd number. No integer satisfies all three conditions, so this region must also be left blank.

Problem of the Week

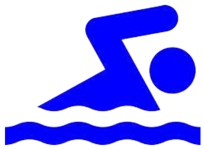
Problem D

Let's Hit the Pool

In Wei's family, there are four children and three adults. Every weekend they all go swimming together. To use the public swimming pool, each person needs a ticket.

Wei's parents buy their tickets in bulk and keep them in a box. At the beginning of the year the ratio of adult to child tickets in the box was $11 : 14$.

Wei's family used the tickets every weekend to go swimming until they no longer had enough tickets for everyone in their family. At that point, there were no child tickets left in the box and 3 adult tickets left in the box. How many tickets were in the box at the beginning of the year?



Problem of the Week

Problem D and Solution

Let's Hit the Pool

Problem

In Wei's family, there are four children and three adults. Every weekend they all go swimming together. To use the public swimming pool, each person needs a ticket.

Wei's parents buy their tickets in bulk and keep them in a box. At the beginning of the year the ratio of adult to child tickets in the box was $11 : 14$.

Wei's family used the tickets every weekend to go swimming until they no longer had enough tickets for everyone in their family. At that point, there were no child tickets left in the box and 3 adult tickets left in the box. How many tickets were in the box at the beginning of the year?

Solution

Let n represent the number of times Wei's family used the tickets to go swimming. Since they used 4 child tickets and 3 adult tickets each time, then they used $4n$ child tickets and $3n$ adult tickets in total. After they had used all the child tickets, there were 3 adult tickets left in the box. That means there were $3n + 3$ adult tickets and $4n$ child tickets in the box at the beginning of the year.

The ratio of adult to child tickets at the beginning of the year was $11 : 14$. We can use this to write and solve the following equation.

$$\begin{aligned}
\frac{11}{14} &= \frac{3n+3}{4n} \\
(11)(4n) &= (14)(3n+3) \\
44n &= 42n + 42 \\
2n &= 42 \\
n &= 21
\end{aligned}$$

Thus, Wei's family used the tickets to go swimming 21 times.

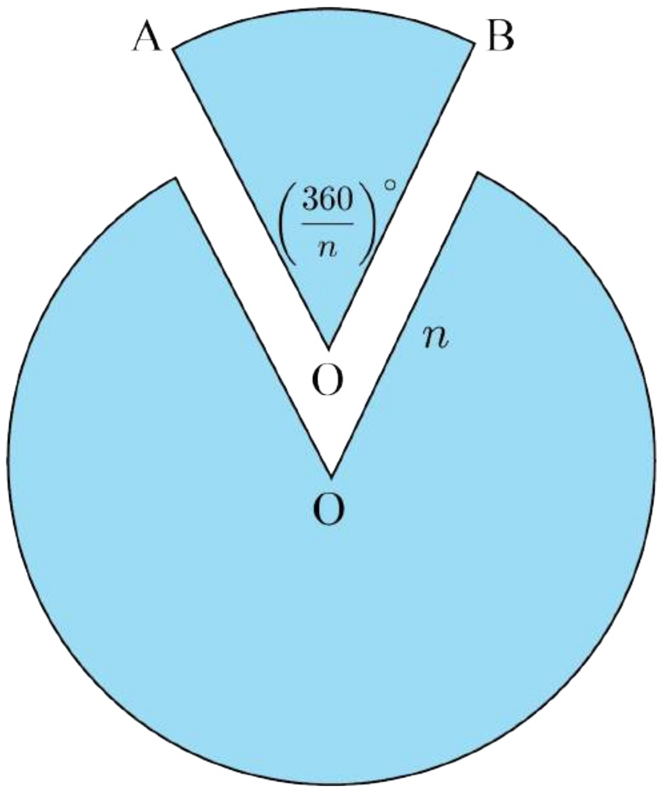
The total number of tickets in the box at the beginning of the year was $4n + 3n + 3 = 7n + 3$. Since $n = 21$, the total number of tickets was $7(21) + 3 = 150$.

Problem of the Week

Problem D

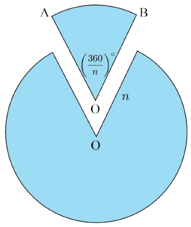
One Slice at a Time

Points A and B are on a circle with centre O and radius n so that $\angle AOB = \left(\frac{360}{n}\right)^\circ$. Sector AOB is cut out of the circle.



Determine all positive integers n for which the perimeter of sector AOB is greater than 20 and less than 30.

NOTE: You may use the fact that the ratio of the length of an arc to the circumference of the circle is the same as the ratio of the sector angle to 360°. In fact, the same ratio holds when comparing the area of a sector to the total area of the circle.



Problem of the Week

Problem D and Solution

One Slice at a Time

Problem

Points A and B are on a circle with centre O and radius n so that $\angle AOB = \left(\frac{360}{n}\right)^\circ$. Sector AOB is cut out of the circle. Determine all positive integers n for which the perimeter of sector AOB is greater than 20 and less than 30.

Note: You may use the fact that the ratio of the length of an arc to the circumference of the circle is the same as the ratio of the sector angle to 360°. In fact, the same ratio holds when comparing the area of a sector to the total area of the circle.

Solution

In general, as the sector angle gets larger, so does the length of the arc, if the radius remains the same. However in this problem, as the radius n increases, the sector angle $\left(\frac{360}{n}\right)^\circ$ decreases. So it is difficult to "see" what happens to the length of the arc.

We know the ratio of the arc length to the circumference of the circle is the same as the ratio of the sector angle to 360°. That is,

$$\frac{\text{arc length of } AB}{\text{circumference}} = \frac{\text{sector angle of } AOB}{360^\circ}$$

Rearranging, we have

$$\text{arc length of } AB = \frac{\text{sector angle of } AOB}{360^\circ} \times \text{circumference}$$

We know circumference $= \pi d = \pi \times 2n$, since $d = 2n$. Thus,

$$\text{arc length of } AB = \frac{\frac{360}{n}}{360} \times \pi \times 2n = 2\pi$$

Now we can use the arc length to calculate the perimeter of AOB.

$$\begin{aligned} \text{perimeter of } AOB &= AO + OB + \text{arc length of } AB \\ &= n + n + 2\pi \\ &= 2n + 2\pi \end{aligned}$$

If the perimeter is greater than 20, then

$$\begin{aligned} 2n + 2\pi &> 20 \\ n + \pi &> 10 \\ n &> 10 - \pi \approx 6.9 \end{aligned}$$

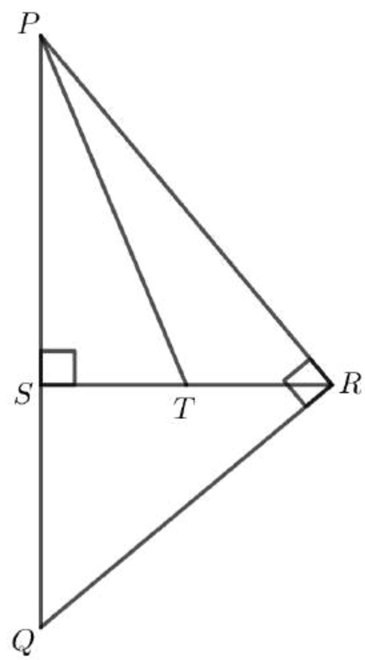
If the perimeter is less than 30, then

$$\begin{aligned} 2n + 2\pi &< 30 \\ n + \pi &< 15 \\ n &< 15 - \pi \approx 11.9 \end{aligned}$$

We want all integer values of n such that $n > 6.9$ and $n < 11.9$. The only integer values of n that satisfy these conditions are $n = 7$, $n = 8$, $n = 9$, $n = 10$, and $n = 11$.

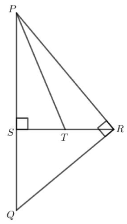
Problem of the Week
Problem D
Which Term is Which?

In $\triangle PQR$, $\angle PRQ = 90^\circ$. An altitude is drawn in $\triangle PQR$ from R to PQ, intersecting PQ at S. A median is drawn in $\triangle PSR$ from P to SR, intersecting SR at T.



If the length of the median PT is 39 and the length of PS is 36, determine the length of QS.

Note: An *altitude* of a triangle is a line segment drawn from a vertex of the triangle perpendicular to the opposite side. A *median* is a line segment drawn from a vertex of the triangle to the midpoint of the opposite side.



Problem of the Week

Problem D and Solution

Which Term is Which?

Problem

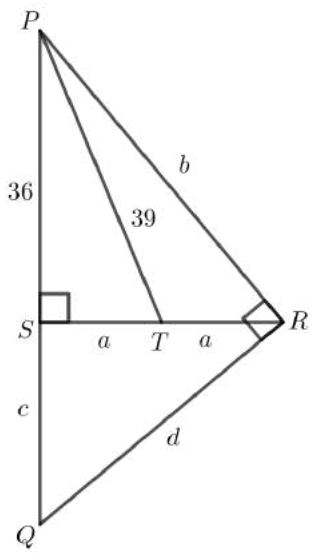
In $\triangle PQR$, $\angle PRQ = 90^\circ$. An altitude is drawn in $\triangle PQR$ from R to PQ, intersecting PQ at S. A median is drawn in $\triangle PSR$ from P to SR, intersecting SR at T.

If the length of the median PT is 39 and the length of PS is 36, determine the length of QS.

NOTE: An *altitude* of a triangle is a line segment drawn from a vertex of the triangle perpendicular to the opposite side. A *median* is a line segment drawn from a vertex of the triangle to the midpoint of the opposite side.

Solution

Since T is a median in $\triangle PSR$, $ST = TR$. Let $ST = TR = a$. Let $PR = b$, $QS = c$, and $QR = d$. The variables and the given information, $PS = 36$ and $PT = 39$, are shown in the diagram.



Since $\triangle PST$ contains a right angle at S,

$$\begin{aligned} ST^2 &= PT^2 - PS^2 \\ a^2 &= 39^2 - 36^2 \\ &= 225 \end{aligned}$$

Then, since $a > 0$, $a = 15$ follows. Thus, $SR = 2a = 30$.

Since $\triangle PSR$ contains a right angle at S,

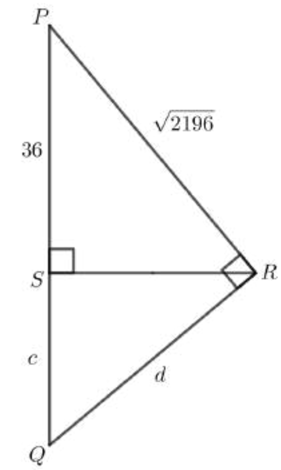
$$\begin{aligned} PR^2 &= PS^2 + SR^2 \\ b^2 &= 36^2 + 30^2 \\ &= 2196 \end{aligned}$$

Then, since $b > 0$, $b = \sqrt{2196}$ follows.

We will now use $a = 15$ and $b = \sqrt{2196}$ in the three solutions that follow.

Solution 1

In $\triangle PSR$ and $\triangle PRQ$, $\angle PSR = \angle PRQ = 90^\circ$ and $\angle SPR = \angle QPR$, a common angle. So $\triangle PSR$ is similar to $\triangle PRQ$. It follows that



$$\begin{aligned}\frac{PS}{PR} &= \frac{PR}{PQ}\\ \frac{36}{\sqrt{2196}} &= \frac{\sqrt{2196}}{36+c}\\ 1296+36c &= 2196\\ 36c &= 900\\ c &= 25\end{aligned}$$

Thus, the length of QS is 25.

Solution 2

Since $\triangle RSQ$ contains a right angle at S, $QR^2 = QS^2 + SR^2 = c^2 + 30^2 = c^2 + 900$. Therefore, $d^2 = c^2 + 900$.

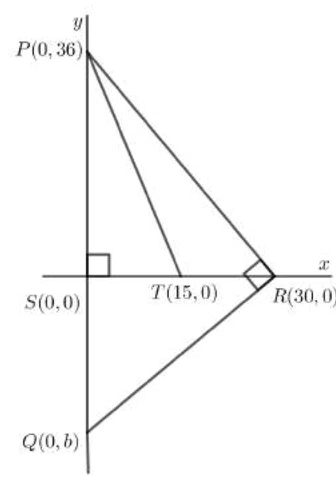
Since $\triangle PQR$ contains a right angle at R, $PQ^2 = PR^2 + QR^2$. Therefore, $(36+c)^2 = (\sqrt{2196})^2 + d^2$, which simplifies to $1296 + 72c + c^2 = 2196 + d^2$. This further simplifies to $c^2 + 72c = 900 + d^2$.

Substituting $d^2 = c^2 + 900$, we obtain $c^2 + 72c = 900 + c^2 + 900$. Simplifying, we get $72c = 1800$ and $c = 25$ follows.

Thus, the length of QS is 25.

Solution 3

Position $\triangle PQR$ on the xy-plane so that PQ lies along the y-axis, and altitude SR lies along the positive x-axis with S at the origin. Then P has coordinates $(0,36)$, T has coordinates $(15,0)$, and R has coordinates $(30,0)$.
Since Q is on the y-axis, let Q have coordinates $(0,b)$ with $b<0$.
Notice that



$$\text{slope } PR = \frac{36-0}{0-30} = \frac{-6}{5} \text{ and slope } QR = \frac{b-0}{0-30} = \frac{b}{-30}$$

Since $\angle PRQ = 90^\circ$, $PR \perp QR$, and so their slopes are negative reciprocals of each other. That is, $\dfrac{b}{-30} = \dfrac{5}{6}$, and so $b = -25$.

It then follows that the coordinates of Q are $(0,-25)$. Thus, the length of QS is 25.

Problem of the Week

Problem D

Fraction Distraction

Find all ordered pairs, (a, b), that satisfy $\dfrac{a-b}{a+b} = 9$ and $\dfrac{ab}{a+b} = -60$.

$$(a, b)$$

(a, b)

Problem of the Week
Problem D and Solution
Fraction Distraction

Problem

Find all ordered pairs, (a, b), that satisfy $\dfrac{a-b}{a+b} = 9$ and $\dfrac{ab}{a+b} = -60$.

Solution

Multiplying both sides of the first equation, $\dfrac{a-b}{a+b} = 9$, by $a+b$ gives $a - b = 9a + 9b$ and so $-8a = 10b$ or $-4a = 5b$. Thus, $a = -\dfrac{5}{4}b$.

Multiplying both sides of the second equation, $\dfrac{ab}{a+b} = -60$, by $a+b$ gives $ab = -60a - 60b$. Substituting $a = -\dfrac{5}{4}b$ into $ab = -60a - 60b$, we get

$$\begin{aligned}
ab &= -60a - 60b \\
\left(-\frac{5}{4}b\right)(b) &= -60\left(-\frac{5}{4}b\right) - 60b \\
-\frac{5}{4}b^2 &= 75b - 60b \\
-\frac{5}{4}b^2 &= 15b \\
b^2 &= -12b \\
b^2 + 12b &= 0
\end{aligned}$$

Notice that $b = 0$ satisfies this equation. Thus $b = 0$ is one possibility. When $b \neq 0$, we can divide both sides of the equation by b to get $b + 12 = 0$, or $b = -12$. Thus, $b = 0$ or $b = -12$.

If $b = 0$, then $a = -\dfrac{5}{4}(0) = 0$. But this gives us a denominator of 0 in each of the original equations. Therefore, $b \neq 0$.

If $b = -12$, then $a = -\dfrac{5}{4}(-12) = 15$.

Therefore, the only ordered pair that satisfies both equations is $(15, -12)$.

Data Management (D)

TAKE ME TO THE COVER

Problem of the Week

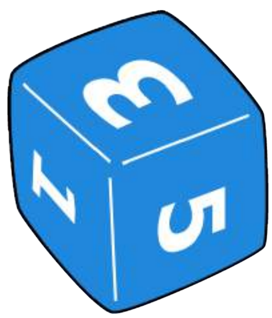
Problem D

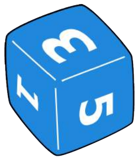
Not So Random

Kimi created a digital die that can be controlled with a program. She then programmed it as follows.

- Initially it has the numbers $1,\ 2,\ 3,\ 4,\ 6$, and 8 on its faces.
- If an odd number is rolled, all the odd numbers on the die double, but the even numbers remain the same.
- If an even number is rolled, all the even numbers on the die are halved, but the odd numbers remain the same.

Kimi rolls the die once and the numbers on the die change as described above. She then rolls the die again, but this time something goes wrong and none of the numbers change. What is the probability that she rolled a 2 on her second roll?





Problem of the Week
Problem D and Solution
Not So Random

Problem

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Kimi rolls the die once and the numbers on the die change as described above. She then rolls the die again, but this time something goes wrong and none of the numbers change. What is the probability that she rolled a 2 on her second roll?

Solution

Solution 1

In this solution, we will determine the possibilities for the first and second roll to count the total number of possible outcomes. We will then count the number of outcomes in which the second roll is a 2 and determine the probability.

- If the first roll is odd, the numbers on the die change from $1,\ 2,\ 3,\ 4,\ 6,\ 8$ to $2,\ 2,\ 6,\ 4,\ 6,\ 8$ as a result of doubling the odd numbers. If we write the possible first and second rolls as an ordered pair, then the following 12 combinations are possible.

  $$(1,2),(1,2),(1,6),(1,4),(1,6),(1,8),(3,2),(3,2),(3,6),(3,4),(3,6),(3,8)$$

- If the first roll is even, the numbers on the die change from $1,\ 2,\ 3,\ 4,\ 6,\ 8$ to $1,\ 1,\ 3,\ 2,\ 3,\ 4$ as a result of halving the even numbers. If we write the possible first and second rolls as an ordered pair, then the following 24 combinations are possible.

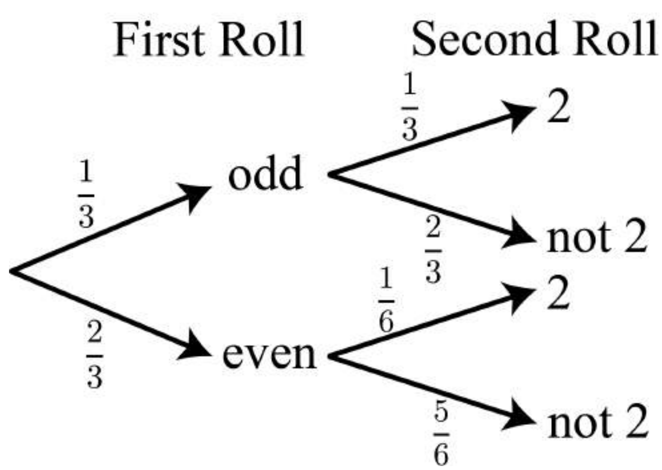
  $$(2,1),(2,1),(2,3),(2,2),(2,3),(2,4),(4,1),(4,1),(4,3),(4,2),(4,3),(4,4),$$
  $$(6,1),(6,1),(6,3),(6,2),(6,3),(6,4),(8,1),(8,1),(8,3),(8,2),(8,3),(8,4)$$

There are 36 possible outcomes in total. Of these outcomes, 8 have a second roll of 2. Therefore, the probability of rolling a 2 on the second roll is $\frac{8}{36}=\frac{2}{9}$.

Solution 2

In this solution, we will show the possibilities on a tree diagram.

- The probability of rolling an odd number on the first roll is $\frac{2}{6} = \frac{1}{3}$. The numbers on the die then change from $1,\ 2,\ 3,\ 4,\ 6,\ 8$ to $2,\ 2,\ 6,\ 4,\ 6,\ 8$ as a result of doubling the odd numbers. The probability of rolling a 2 on the second roll is now $\frac{2}{6} = \frac{1}{3}$.
- The probability of rolling an even number on the first roll is $\frac{4}{6} = \frac{2}{3}$. The numbers on the die then change from $1,\ 2,\ 3,\ 4,\ 6,\ 8$ to $1,\ 1,\ 3,\ 2,\ 3,\ 4$ as a result of halving the even numbers. The probability of rolling a 2 on the second roll is now $\frac{1}{6}$.



To calculate the probability of rolling an odd number on the first roll and then a 2 on the second roll, we multiply the probabilities of each to obtain $\frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$.

To calculate the probability of rolling an even number on the first roll and then a 2 on the second roll, we multiply the probabilities of each to obtain $\frac{2}{3} \times \frac{1}{6} = \frac{1}{9}$.

Then, to calculate the probability of rolling an odd number on the first roll and then a 2 on the second roll, or an even number on the first roll and then a 2 on the second roll, we add their probabilities to obtain $\frac{1}{9} + \frac{1}{9} = \frac{2}{9}$.

Therefore, the probability of rolling a 2 on the second roll is $\frac{2}{9}$.

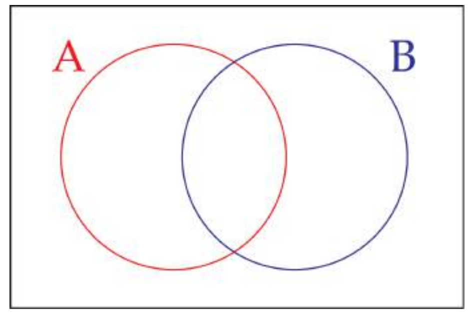
Problem of the Week

Problem D

Everything in its Place 2

(a) A Venn diagram has two circles, labelled A and B. Each circle contains ordered pairs, (x, y), where x and y are real numbers, that satisfy the following criteria.

A: $y = -x + 1$
B: $y = 3x + 5$

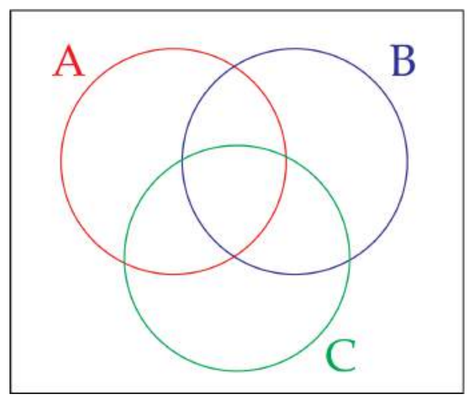


The overlapping region in the middle contains ordered pairs that are in both A and B, and the region outside both circles contains ordered pairs that are neither in A nor B.

In total this Venn diagram has four regions. Place ordered pairs in as many of the regions as you can. Is it possible to find an ordered pair for each region?

(b) A Venn diagram has three circles, labelled A, B, and C. Each circle contains integers, n, that satisfy the following criteria.

A: $3n < 20$
B: $n + 9 > 6$
C: n is even



In total this Venn diagram has eight regions. Place integers in as many of the regions as you can. Is it possible to find an integer for each region?

Problem of the Week
Problem D and Solution
Everything in its Place 2

Problem

(a) A Venn diagram has two circles, labelled A and B. Each circle contains ordered pairs, (x, y), where x and y are real numbers, that satisfy the following criteria.

A: $y = -x + 1$

B: $y = 3x + 5$

The overlapping region in the middle contains ordered pairs that are in both A and B, and the region outside both circles contains ordered pairs that are neither in A nor B. In total this Venn diagram has four regions. Place ordered pairs in as many of the regions as you can. Is it possible to find an ordered pair for each region?

(b) A Venn diagram has three circles, labelled A, B, and C. Each circle contains integers, n, that satisfy the following criteria.

A: $3n < 20$

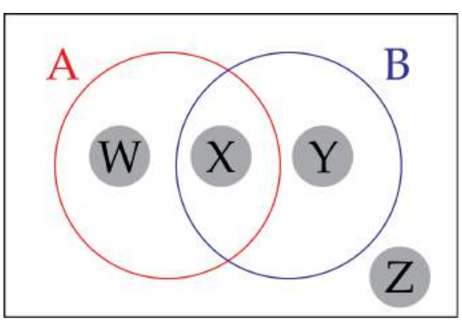
B: $n + 9 > 6$

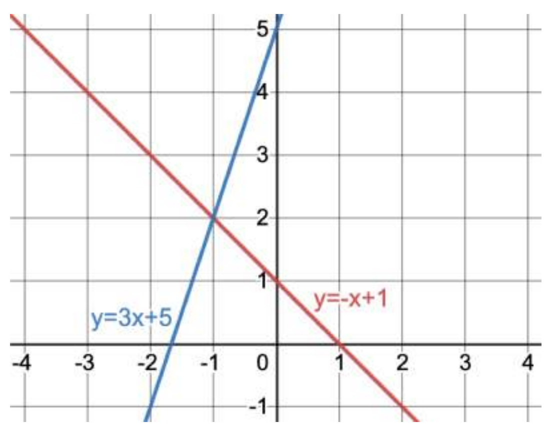
C: n is even

In total this Venn diagram has eight regions. Place integers in as many of the regions as you can. Is it possible to find an integer for each region?

Solution

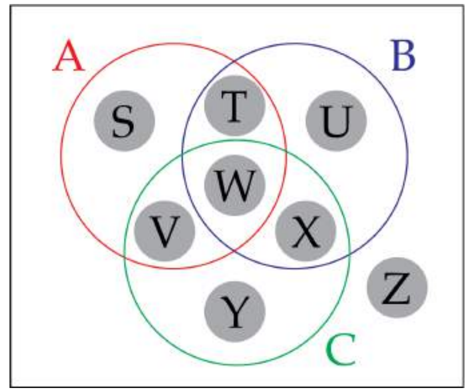
(a) We have marked the four regions W, X, Y, and Z. We plot the given equations on a grid as a reference.





- Any ordered pair, (x, y), in region W must satisfy $y = -x + 1$, but *not* $y = 3x + 5$. Any point on the line $y = -x + 1$ that is *not* on the line $y = 3x + 5$ will satisfy this. An example is $(0, 1)$.
- Any ordered pair, (x, y), in region X must satisfy both $y = -x + 1$ and $y = 3x + 5$. The only point that satisfies this is the point of intersection, $(-1, 2)$.
- Any ordered pair, (x, y), in region Y must satisfy $y = 3x + 5$, but *not* $y = -x + 1$. Any point on the line $y = 3x + 5$ that is *not* on the line $y = -x + 1$ will satisfy this. An example is $(0, 5)$.
- Any ordered pair, (x, y), in region Z must *not* satisfy $y = 3x + 5$ or $y = -x + 1$. Any point that is not on either line will satisfy this. An example is $(2, 2)$.

(b) We have marked the eight regions S, T, U, V, W, X, Y, and Z. It is helpful if we first solve the given inequalities.



For A:

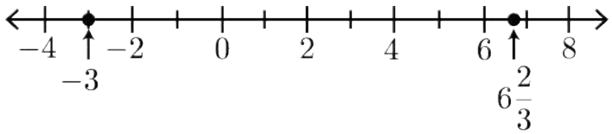
$$3n < 20$$

$$n < \frac{20}{3} = 6\frac{2}{3}$$

For B:

$$n + 9 > 6$$

$$n > -3$$



- Any integer in region S must be less than $6\frac{2}{3}$, less than or equal to -3, and an odd number. Any odd integer less than or equal to -3 will satisfy this. An example is -5.
- Any integer in region T must be less than $6\frac{2}{3}$, greater than -3, and an odd number. The only integers that satisfy this are $-1, 1, 3$, and 5.
- Any integer in region U must be greater than or equal to $6\frac{2}{3}$, greater than -3, and an odd number. Any odd integer greater than or equal to $6\frac{2}{3}$ will satisfy this. An example is 7.
- Any integer in region V must be less than $6\frac{2}{3}$, less than or equal to -3, and an even number. Any even integer less than or equal to -3 will satisfy this. An example is -4.
- Any integer in region W must be less than $6\frac{2}{3}$, greater than -3, and an even number. The only integers that satisfy this are $-2, 0, 2, 4$, and 6.
- Any integer in region X must be greater than or equal to $6\frac{2}{3}$, greater than -3, and an even number. Any even integer greater than or equal to $6\frac{2}{3}$ will satisfy this. An example is 8.
- Any integer in region Y must be greater than or equal to $6\frac{2}{3}$, less than or equal to -3, and an even number. No integer satisfies all three conditions, so this region must be left blank.
- Any integer in region Z must be greater than or equal to $6\frac{2}{3}$, less than or equal to -3, and an odd number. No integer satisfies all three conditions, so this region must also be left blank.

Computational Thinking (C)

TAKE ME TO THE COVER

Problem of the Week

Problem D

The Spy's The Limit

A group of five spies, Agent A, Agent B, Agent C, Agent D, and Agent E, meet every Friday to share all the information they have uncovered over the previous week. To avoid suspicion, a spy can never be seen with more than one other spy at a time. As well, the spies always communicate face to face to avoid a paper trail.

Every Friday the spies conduct several rounds of meetings at various locations around town. Each round consists of two simultaneous meetings, which involve four spies in total. There is always one spy sitting out of the round.

In each meeting, each spy passes along all of the information they know. This includes both the information they gathered the previous week, as well as all of the information passed on to them from other spies in earlier meetings that day.

Determine the minimum number of rounds of meetings required in order for each spy to learn all of the information gathered by each of the other spies during the previous week.

This problem was inspired by a past Beaver Computing Challenge (BCC) problem.

Problem of the Week
Problem D and Solution
The Spy's The Limit

Problem

A group of five spies, Agent A, Agent B, Agent C, Agent D, and Agent E, meet every Friday to share all the information they have uncovered over the previous week. To avoid suspicion, a spy can never be seen with more than one other spy at a time. As well, the spies always communicate face to face to avoid a paper trail.
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Determine the minimum number of rounds of meetings required in order for each spy to learn all of the information gathered by each of the other spies during the previous week.

This problem was inspired by a past Beaver Computing Challenge (BCC) problem.

Solution

In this solution, we will first show that in order for each spy to learn all of the information gathered by each of the other spies, at least four rounds are needed. Then we will show that this is possible in exactly four rounds. Thus, we will conclude that the minimum number of rounds needed is four.

In the first round, two meetings can take place, and at least one spy will sit out. Suppose Agent E sits out the first round. Since Agent E was not involved, Agent E's original information is known only by Agent E. Therefore, it is not possible for anyone to know all of the information after one round.

In the second round, Agent E could meet with another spy or Agent E could sit out again.

- Suppose Agent E meets with another spy. Then only two spies would know Agent E's original information. In the third round, these two spies could meet with at most two other spies, so after three rounds, at most four spies would know Agent E's original information. Therefore, at least one more round would be needed, and so at least four rounds in total are needed.
- Suppose Agent E sits out again on the second round. Then in the third round Agent E could meet with another spy, and so only two spies would know Agent E's original information. Using similar reasoning to that above, we can show that in this case, at least five rounds in total would be needed.

We have shown that at least four rounds are needed if Agent E meets with another spy in the second round. We will now show that this can be done in exactly four rounds.

In the first round, suppose Agent A meets with Agent B, Agent C meets with Agent D, and Agent E sits out.

We can summarize the information each spy knows after the first round in the following table.

Agent	Agents whose information is known by this Agent
A	A, B
B	A, B
C	C, D
D	C, D
E	E

In the second round, suppose Agent E meets with Agent A, Agent B meets with Agent C, and Agent D sits out. Now Agent A knows the original information from Agent B and Agent E, but not Agent C or Agent D. Agent B knows the original information from Agent A, Agent C and Agent D, but not Agent E.

We can summarize the information each spy knows after the second round in the following table.

Agent	Agents whose information is known by this Agent
A	A, B, E
B	A, B, C, D
C	A, B, C, D
D	C, D
E	A, B, E

In the third round, suppose Agent A meets with Agent C, Agent D meets with Agent E, and Agent B sits out. Then the following table summarizes the information each spy knows after the third round.

Agent	Agents whose information is known by this Agent
A	A, B, C, D, E
B	A, B, C, D
C	A, B, C, D, E
D	A, B, C, D, E
E	A, B, C, D, E

In the fourth round, Agent B can meet with any other spy to learn the original information gathered by Agent E. No other meeting needs to take place in this round, as the remaining spies all know all the information gathered by each of the other spies.

We have shown that at least four rounds are needed and we've also shown that it is possible for each spy to learn the information gathered by each of the other spies in exactly four rounds. Therefore, the minimum number of rounds required for each spy to learn the information gathered by each of the other spies is four.

Extension:

Suppose there were 6 spies instead of 5. Determine the minimum number of rounds of meetings required so that all of the information known by each spy has been shared with every other spy. You may be surprised by the result.

Problem of the Week
Problem D
Out of This World

On a far away planet, there are two types of inhabitants: Veres, who always tell the truth, and Nugators, who always lie.

Four of the inhabitants of the planet are seated around a circular table. When asked, “Are you a Vere or a Nugator?”, all four replied, “Vere”. When asked, “Is the person on your right a Vere or a Nugator?”, all four replied, “ Nugator”.

How many Veres are seated at the table? Verify that your solution is the only possible solution.

Problem of the Week
Problem D and Solution
Out of This World

Problem

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How many Veres are seated at the table? Verify that your solution is the only possible solution.

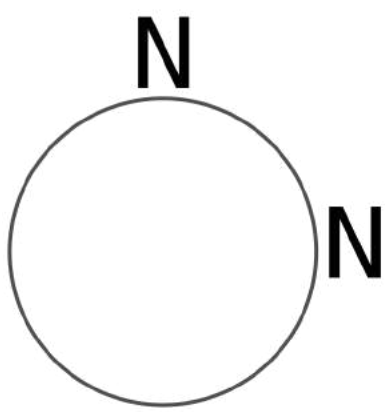
Solution

There are really five possibilities to check: there could be four Veres, there could be three Veres and one Nugator, there could be two Veres and two Nugators, there could be one Vere and three Nugators, or there could be four Nugators.

We can eliminate cases as follows:

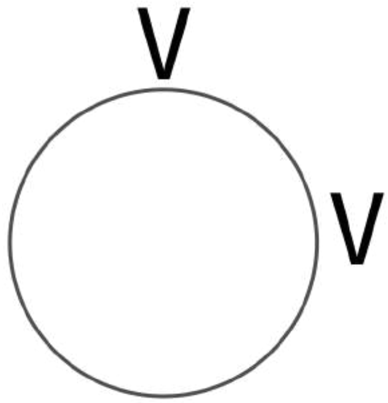
1. **Can two Nugators be seated beside each other?**

   Suppose there are two Nugators seated beside each other at the table. Since Nugators always lie, when the two Nugators answer the first question, they will both lie and say “Vere”. However, in responding to the second question, the Nugator with the other Nugator on their right would lie and say “Vere”. But everyone responded “Nugator”. This is a contradiction. Therefore, there cannot be two Nugators seated beside each other. This conclusion effectively eliminates the possibility that there are four Nugators, or one Vere and three Nugators.

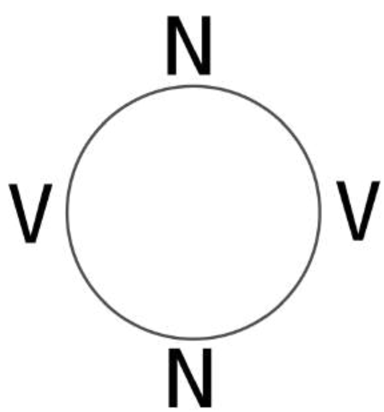


2. **Can two Veres be seated beside each other?**

   Suppose there are two Veres seated beside each other at the table. Since Veres always tell the truth, when the two Veres answer the first question, they will both tell the truth and say “Vere”. However, in responding to the second question, the Vere with the other Vere on their right would tell the truth and say “Vere”. But everyone responded “Nugator”. This is a contradiction. Therefore, there cannot be two Veres seated beside each other. This conclusion effectively eliminates the possibility that there are four Veres, or three Veres and one Nugator.



The only possibility left is that there are two Veres and two Nugators seated at the table, and the two Nugators are not sitting next to each other and the two Veres are not sitting next to each other. The diagram illustrates how they must be seated relative to each other.



We can confirm that this arrangement satisfies the problem. Since all Nugators lie and all Veres tell the truth, they will all answer the first question “Vere”. Since all Nugators lie and all Veres tell the truth, they will all answer the second question “Nugator”.

Therefore, there are two Veres and two Nugators, and when seated at a circular table they alternate Vere, Nugator, Vere, Nugator.

Problem of the Week

Problem D

Just My Three Cents

In Canada, pennies are 1 cent coins that were used up until 2012. Adeline and Bai are playing a game using three pennies and a game board consisting of a row of 8 squares. To start the game, the pennies are placed in the three leftmost squares, as shown.

The rules of the game are as follows:

- On a player's turn, the player must move one penny one or more squares to the right.
- The penny may not pass over any other penny or land on a square that is occupied by another penny.
- The game ends when the pennies are in the three rightmost squares. The last player to move a penny wins the game.

Adeline knows that if she goes first she can always win the game, regardless of where Bai moves the pennies on her turns. Describe Adeline's first move and her winning strategy.

Problem of the Week

Problem D and Solution

Just My Three Cents

Problem

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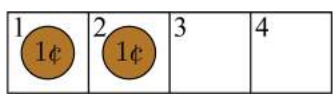
Adeline knows that if she goes first she can always win the game, regardless of where Bai moves the pennies on her turns. Describe Adeline's first move and her winning strategy.

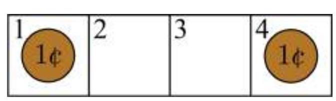
Solution

First, consider playing the game with just two pennies and four squares. We will number the squares from 1 to 4, starting on the left. The two pennies would start in squares 1 and 2.

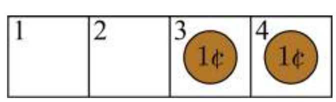


Player 1 has two options for their first turn. They can move the penny in square 2 to either square 4 or square 3.

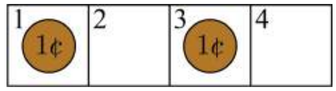
- *Option 1:* Player 1 moves the penny in square 2 to square 4. Then the pennies would be in squares 1 and 4.

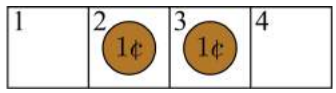
  If Player 2 moves the penny in square 1 to square 3, then they would win the game because the pennies would be in squares 3 and 4.

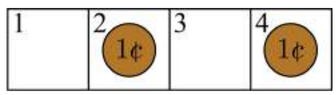
- *Option 2:* Player 1 moves the penny in square 2 to square 3. Then the pennies would be in squares 1 and 3.

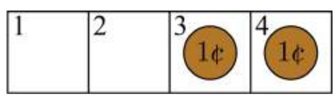
  Then Player 2 has two options for their turn. They can either move the penny in square 3 or move the penny in square 1. However, if Player 2 wants to win the game, they should not move the penny in square 3 to square 4. If they do, then the pennies would be in squares 1 and 4, and then Player 1 could move the penny in square 1 to square 3 and win the game. So, Player 2 should move the penny in square 1 to square 2. Then the pennies would be in squares 2 and 3.

  Player 1 would be forced to move the penny in square 3 to square 4. Then the pennies would be in squares 2 and 4.

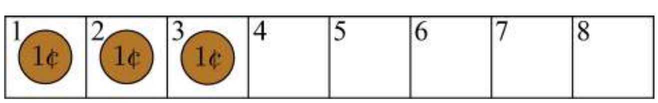
  

  Player 2 would then move the penny in square 2 to square 3, and win the game because the pennies would be in squares 3 and 4.

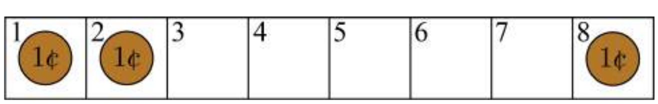
  

In the game with just two pennies and four squares, Player 2 is always able to win, regardless of what Player 1 does on their turn. If you look closely, you will see that the winning strategy for Player 2 is to copy whatever Player 1 did with the other penny. The two pennies start together. Player 1 must move the rightmost penny, creating a gap between the two pennies. On the following turn, Player 2 can move the other penny in such a way that there is no longer a gap between the two pennies. The number of squares really does not matter. Whatever Player 1 does with the penny on the right, Player 2 "mimics" with the penny on the left. Player 2 wins in this version of the game, but in our game Player 1 is supposed to win and we have an extra penny.

In our game, Adeline is Player 1 and the pennies start in squares 1, 2, and 3.



If Adeline first moves the penny in square 3 to square 8, then what is left in squares 1 to 7 is a two penny game with a total of 7 squares.



Now, whatever Bai does on her turn with the penny in square 2, Adeline should "mimic" with the penny in square 1. This will guarantee that Adeline will win the game.

Problem of the Week

Problem D

Time for Cake

Finn and Lea own a cake business. Finn does all the baking while Lea does all the decorating. One day they need to complete five cake orders. The order in which they complete the cakes doesn't matter, however all cakes need to be baked before they can be decorated. A cake can be decorated at any time after it has been baked. Also Finn and Lea can each work on only one cake at a time.

The times to bake and decorate each of the cakes are shown in the table below.

Order Number	Cake Type	Baking Time (min)	Decorating Time (min)
1	Carrot Cake	50	20
2	Vanilla Birthday Cake	30	60
3	Strawberry Cheesecake	70	40
4	Rainbow Layer Cake	100	90
5	Angel Food Cake	80	10

If Finn and Lea start working on these orders at 9:30 a.m., what is the earliest time that they can be completely finished all five cakes? Justify your answer.

Problem of the Week
Problem D and Solution
Time for Cake

Problem

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If Finn and Lea start working on these orders at 9:30 a.m., what is the earliest time that they can be completely finished all five cakes? Justify your answer.

Solution

The sum of all the baking times is 330 minutes. Similarly, the sum of all the decorating times is 220 minutes. Since $330 > 220$, we can conclude that it will take at least 330 minutes to complete all the orders. Furthermore, since the last cake to be baked still needs to be decorated after that, it is not possible to complete all the orders in 330 minutes. The shortest decorating time is 10 minutes, so the shortest possible time to complete all the orders is $330 + 10 = 340$ minutes. Now we need to see if we can arrange the orders in such a way that they can all be completed in 340 minutes.

If we put the orders with the shortest decorating times at the end, then Finn won't be waiting a long time after he has finished baking. Similarly, if we put the orders with the shortest baking times at the beginning, then Lea won't be waiting a long time before she can start decorating. We will look at the baking and decorating times from smallest to largest and place orders with shorter baking times at the beginning, and orders with shorter decorating times at the end.

The smallest time is a decorating time of 10 minutes for Order 5. So we place that order last.

Position	1^{st}	2^{nd}	3^{rd}	4^{th}	5^{th}
Order #					5

The next smallest time is a decorating time of 20 minutes for Order 1. We place that order second last.

Position	1^{st}	2^{nd}	3^{rd}	4^{th}	5^{th}
Order #				1	5

The next smallest time is a baking time of 30 minutes for Order 2. We place that order first.

Position	1st	2nd	3rd	4th	5th
Order #	2			1	5

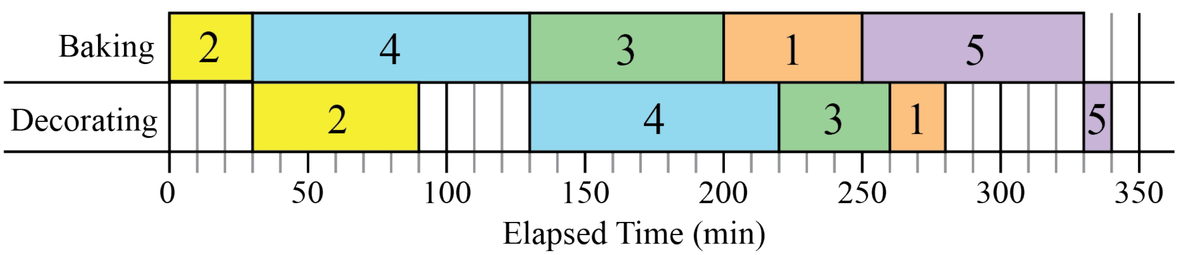
The next smallest time is a decorating time of 40 minutes for Order 3. We place that order third last.

Position	1st	2nd	3rd	4th	5th
Order #	2		3	1	5

The remaining order to be placed is Order 4. We place that order in the empty spot in second position. The positions of all of the orders are now determined.

Position	1st	2nd	3rd	4th	5th
Order #	2	4	3	1	5

Now we create a timeline to help calculate how long it takes to complete the orders in this way. The timeline shows the time slots during which each order is baked and decorated. Since baking and decorating different cakes can happen at the same time, there are two simultaneous schedules shown on the timeline, one for baking and one for decorating. Finn bakes the cakes back-to-back, and Lea starts decorating each cake after it has finished baking and once Lea is available.



Using the timeline, we can see it takes 340 minutes to complete the orders, so we have found an arrangement that allows all orders to be completed in 340 minutes.

Since 340 minutes is equal to 5 h 40 min, and Finn and Lea start working on the orders at 9:30 a.m., it follows that they will be completely finished at 3:10 p.m., at the earliest.

NOTE: It turns out that this is not the only arrangement that produces a time of 340 minutes. The complete list of arrangements of order numbers that produce a time of 340 minutes is below, where the order numbers are written in the order that they are completed.

- 1, 2, 4, 3, 5
- 2, 1, 4, 3, 5
- 2, 3, 4, 1, 5
- 2, 4, 1, 3, 5
- 2, 4, 3, 1, 5
- 3, 2, 4, 1, 5
- 4, 1, 2, 3, 5
- 4, 1, 3, 2, 5
- 4, 2, 1, 3, 5
- 4, 2, 3, 1, 5
- 4, 3, 1, 2, 5
- 4, 3, 2, 1, 5

Connections to Computer Science

To solve this problem we used a *greedy strategy*, which is a strategy for solving optimization problems. Using this strategy, we made the optimal choice at each step, in hopes of finding the optimal solution. Greedy strategies do not always produce optimal solutions, but are still useful because they are easy to describe and implement, and often give a good approximation to the optimal solution.

Problem of the Week

Problem D

Fast Bikers

The top five finishers in a bike race were Albine, Farrah, Jasna, Nuan, and Terese, in some order. They all rode a different-coloured bike (black, white, green, blue, or red), and were a different age (18, 21, 25, 26, or 29). Use the clues below to determine each person's age, bike colour, and final position in the race.

1. The five people in the race were the person who finished first, the 26-year old, Nuan, the person with the white bike, and Albine.
2. Jasna rode the white bike and finished third.
3. Nuan is 3 years younger than the person who rode the red bike.
4. The oldest person rode the white bike and finished just before the youngest person, who rode the black bike.
5. The person with the green bike finished first, and someone younger finished right after.
6. Terese finished right after Nuan.

You may find the following table useful in organizing your solution.

	1st	2nd	3rd	4th	5th	Black	White	Green	Blue	Red	18	21	25	26	29
Albine															
Farrah															
Jasna															
Nuan															
Terese															
18															
21															
25															
26															
29															
Black															
White															
Green															
Blue															
Red															

Problem of the Week

Problem D and Solution

Fast Bikers

Problem

The top five finishers in a bike race were Albine, Farrah, Jasna, Nuan, and Terese, in some order. They all rode a different-coloured bike (black, white, green, blue, or red), and were a different age (18, 21, 25, 26, or 29). Use the clues below to determine each person's age, bike colour, and final position in the race.

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Nuan															
Terese															
18															
21															
25															
26															
29															
Black															
White															
Green															
Blue															
Red															

Solution

First we will give the answer, and then an explanation as to how we arrived at the answer.

- Farrah is 25 years old, rode the green bike, and finished first.
- Albine is 21 years old, rode the red bike, and finished second.
- Jasna is 29 years old, rode the white bike, and finished third.
- Nuan is 18 years old, rode the black bike, and finished fourth.
- Terese is 26 years old, rode the blue bike, and finished fifth.

There are many different ways to arrive at the answer above. You may have used the table provided to keep track of matches that were confirmed or deemed impossible while examining and combining the different clues. Below we present an explanation in words only. It may be helpful to follow along by filling in the table as you read the explanation.

From clues 2 and 4, Jasna rode the white bike, finished third, and is 29 years old. Clue 4 also tells us that the 18-year old rode the black bike and finished fourth.

From clue 3, Nuan and the person who rode the red bike must be either 18 and 21 years old, or 26 and 29 years old. However, from clue 1, we know Nuan is not 26 years old. It follows that Nuan must be 18 years old and the person who rode the red bike must be 21 years old. That means Nuan is the person who finished fourth and rode the black bike.

From clue 5, Terese finished fifth. It follows from clue 1 that Terese is 26 years old since she did not finish first or ride the white bike.

The people who finished first and second must therefore be 25 and 21 years old, in some order. It follows from clue 5 that the 25-year old must have finished first, followed by the 21-year old, who we already determined rode the red bike. Clue 5 also tells us the person who finished first rode a green bike. It follows that Terese rode a blue bike.

From clue 1, Albine did not finish first, so that means she finished second, and Farrah finished first.