Problem of the Month

Problem 0: September 2021

Problem

Some friends are playing a game involving ten cards numbered 1 through 10. In part (a), Adina, Budi, and Dewei are the players. In parts (b) and (c), Adina, Budi, Charlie, and Dewei are the players. To play the game, each player other than Dewei chooses a card and shows it to all other players, but no player looks at their own card. The game consists of a dialogue with the goal being for all players holding a card to deduce the integer on their own card. In each part of this question, the dialogue is given in the order the statements/questions occurred. **No player is allowed ask a question to which they already know the answer**.

(a) Given the dialogue below, determine the integers on Adina's and Budi's cards.

1. (Adina) Is the integer on my card larger than the integer on Budi's card?
2. (Dewei) No.
3. (Budi) I know the integer on my card.
4. (Adina) I know the integer on my card.

(b) After the dialogue below, Adina, Budi, and Charlie each know the integer on their own card. Determine all possibilities for the integers on their cards.

1. (Adina) Is the sum of the integers on the cards a perfect square?
2. (Dewei) Yes.

(c) Given the dialogue below, determine all possibilities for the integers on the cards.

1. (Adina) Are the integers on any of the cards prime?
2. (Dewei) No.
3. (Budi) Is the sum of the integers on the cards prime?
4. (Dewei) Yes.
5. The three statements below occur simultaneously.
   - (Adina) I do not know what integer is on my card.
   - (Budi) I do not know what integer is on my card.
   - (Charlie) I know what integer is on my card.
6. The two statements below occur simultaneously
   - (Adina) I still do not know what integer is on my card.
   - (Budi) I now know what integer is on my card.
7. (Adina) I now know what integer is on my card.

Hint

(a) Would this dialogue be possible if Adina saw a 3 on Budi's card? Make sure you keep in mind that no player will ever ask a question to which they already know the answer!

(b) To narrow the search for the answer, determine all possible sets of three distinct integers between 1 and 10 inclusive that have a sum equal to a perfect square. Going from there, you might want to explore what would happen for some particular configurations of the cards. For instance, if the integers are 1, 2, and 6, would it be possible for all players to determine the integer on their card from the dialogue given?

(c) The general strategy for this problem is similar to that in (b), but you will need to carefully examine how the players would be able to eliminate possibilities based on what the other players are able to infer. Experimenting with various configurations of the cards will likely be helpful.

Problem of the Month

Solution to Problem 0: September 2021

(a) Denote the integer on Adina's card by a and the integer on Budi's card by b.

If $b = 1$ or $b = 10$, then Adina would know the answer to the question in the first line and hence would not have asked it. Therefore, $b \neq 1$ and $b \neq 10$. Budi deduces this information from the fact that Adina asked the question in the first line.

After Dewei answers "No" in the second line, Budi knows the following information: the value of a, that $b \neq 1$, that $b \neq 10$, and that $a < b$. According to the third line of the dialogue, Budi is able to determine the value of b from this information.

Since $a < b$ and $b < 10$, it must be that $a \leq 8$. If $a < 8$, then there is no way for Budi to deduce the value of b from the information in the previous paragraph. Therefore, $a = 8$ and $b = 9$.

Although we have already deduced the integers on the two cards, it is worth pointing out that the final line of the dialogue makes sense. Indeed, the fact that Budi determined the value of b immediately after Adina's question was answered tells Adina that $a = 8$. Otherwise, as discussed in the previous paragraph, there is no way that Budi could have determined the value of b after line 2 in the dialogue.

(b) The smallest possible sum of three different integers from 1 to 10 inclusive is $1+2+3 = 6$ and the largest is $8 + 9 + 10 = 27$. Therefore, 9, 16, and 25 are the only possible perfect squares that can be equal to the sum of the integers on the cards. It is not difficult to deduce that there are exactly 15 sets of three distinct integers from 1 to 10 having a sum equal to a perfect square. One way to approach this is to examine possibilities by the largest integer in the set. For instance, if 10 is the largest integer, then we seek distinct positive integers x and y such that $x + y + 10$ is a perfect square. Since $10 > 9$, $x + y + 10 > 9$, so 16 and 25 are the only possibilities for this sum. This means either $x + y = 6$ or $x + y = 15$. In the former case, x and y could be 1 and 5 or 2 and 4 in either order (x and y need to be different, so $x = y = 3$ is not possible). In the latter case, x and y equal to 6 and 9 or 7 and 8. Therefore, four of the sets of three distinct positive integers are $\{1, 5, 10\}$, $\{2, 4, 10\}$, $\{6, 9, 10\}$, and $\{7, 8, 10\}$. Moreover, these four are the only such sets that contain 10. All fifteen sets are listed below.

$$\{1, 2, 6\}, \{1, 3, 5\}, \{1, 5, 10\}, \{1, 6, 9\}, \{1, 7, 8\}, \{2, 3, 4\}, \{2, 4, 10\}, \{2, 5, 9\},$$
$$\{2, 6, 8\}, \{3, 4, 9\}, \{3, 5, 8\}, \{3, 6, 7\}, \{4, 5, 7\}, \{6, 9, 10\}, \{7, 8, 10\}$$

After the dialogue, each player knows that the sum of the integers on the three cards is a perfect square. Each player can see two cards. If a player sees the integers 1 and 6, then the integer on their own card could be 2 or 9, but they will not be able to tell which since in either case the sum would be a perfect square. Since every player is able to deduce the integer on their card once they learn that the sum is a perfect square, it is not possible for 1 and 6 to be two of the integers on the cards. Therefore, the sets $\{1, 2, 6\}$ and $\{1, 6, 9\}$ can be eliminated as possibilities for the integers on the three cards.

$$\cancel{\{1, 2, 6\}}, \{1, 3, 5\}, \{1, 5, 10\}, \cancel{\{1, 6, 9\}}, \{1, 7, 8\}, \{2, 3, 4\}, \{2, 4, 10\}, \{2, 5, 9\},$$

$$\{2,6,8\},\{3,4,9\},\{3,5,8\},\{3,6,7\},\{4,5,7\},\{6,9,10\},\{7,8,10\}$$

If a player sees the integers 1 and 5, then the integer on their card could be 3 or 10, but they cannot determine which of the two. Therefore, the three cards cannot be $\{1,3,5\}$ or $\{1,5,10\}$. Continuing with this sort of reasoning, all possibilities except $\{2,5,9\}$, $\{3,6,7\}$, and $\{4,5,7\}$ can be eliminated from the list above.

If the integers are 2, 5, and 9, then one of the players sees the cards 2 and 5. The only way for the sum to be a perfect square is for the integer on their card to be 9 (remember, there is only one of each card so they cannot be holding a card with a 2 since they see a 2). Therefore, they know the integer on their card. The player who sees 2 and 9 knows their card must have 5 on it since there is no other integer that can be added to $2+9=11$ to get a perfect square. Finally, the player who sees 2 and 5 knows their card must have 9 on it by the same reasoning. Therefore, if the cards have 2, 5, and 9 on them (in any order), then all three players will know the integer on their card as soon as they learn that the sum is a perfect square.

By similar reasoning, if the integers are 3, 6, and 7 in any order, then each player will know the integer on their card once they learn that the sum is a square, as well as if the integers are 4, 5, and 7 in any order.

This gives a total of $3 \times 6 = 18$ configurations of the cards because there are 6 ways to distribute the three cards among the three players for each of the three possible sets of cards. However, it is possible that some of these configurations would lead to Adina knowing the answer to her question immediately from the cards she can see. To finish the argument, we will show that in any of these 18 configurations of the cards, Adina could not possibly know just from the cards that she sees that the sum is or is not a perfect square.

The possible pairs of integers that Adina can see are $\{2,5\}$, $\{2,9\}$, $\{5,9\}$, $\{3,6\}$, $\{3,7\}$, $\{6,7\}$, $\{4,5\}$, $\{4,7\}$, or $\{5,7\}$. These are the two-element subsets of the three possible sets of integers on the cards. In each of these nine cases, there is at least one possibility for the integer on her card that would make the sum a perfect square. As well, in each of these nine cases, if the integer on her card were 1, then the sum would not be a perfect square. Therefore, if Adina sees any of these nine pairs of integers, there is no way for her to know whether the sum is a perfect square. Therefore, all 18 configurations described above are possible.

(c) Since none of the cards have a prime number on them, the possibilities for the integers on the cards are 1, 4, 6, 8, 9, and 10. As well, the sum of the three integers is prime, which rules out many possibilities. By carefully checking, one finds that there are exactly seven sets of three distinct integers from the list 1, 4, 6, 8, 10 that have a prime sum. They are listed below.

$$\{1,4,6\}\ \ \{1,4,8\}\ \ \{1,6,10\}\ \ \{1,8,10\}\ \ \{4,6,9\}\ \ \{4,9,10\}\ \ \{6,8,9\}$$

The first column in the table below contains the fifteen pairs of two distinct integers selected from 1, 4, 6, 8, 9, and 10, which are the possible pairs of cards that a player can see. The right column contains the corresponding number of three-element sets from the list above of which the given two-element set is a subset. For example, the number 2 occurs to the right of $\{1,4\}$ because $\{1,4\}$ is a subset of $\{1,4,6\}$ and $\{1,4,8\}$ and none of the other sets.

$\{1,4\}$	2
$\{4,6\}$	2
$\{6,9\}$	2
$\{1,6\}$	2
$\{4,8\}$	1
$\{6,10\}$	1
$\{1,8\}$	2
$\{4,9\}$	2
$\{8,9\}$	1
$\{1,9\}$	0
$\{4,10\}$	1
$\{8,10\}$	1
$\{1,10\}$	2
$\{6,8\}$	1
$\{9,10\}$	1

If a pair in the table above has a 1 next to it, then a player who sees those two integers will know the integer on their card after the first four lines of dialogue. For instance, if a player sees 4 and 8, they will know that their card is 1 since $\{1,4,8\}$ is the only one of the seven sets above that contains 4 and 8. On the other hand, if there is a 2 next to a set in the table above, then a player who sees those two integers will not be able to determine the integer on their card after the first four lines of dialogue. For instance, if a player sees 1 and 4, then the set of integers is either $\{1,4,6\}$ or $\{1,4,8\}$, so a player who sees 1 and 4 knows that their card is 6 or 8, but cannot determine which.

We know that after the first four lines of dialogue, exactly one player is able to determine the integer on their card. Suppose the set of integers on the cards is $\{4,9,10\}$. The two-element subsets of this set are $\{4,9\}$, $\{4,10\}$, and $\{9,10\}$. Both $\{4,10\}$ and $\{9,10\}$ have a 1 next to them in the table above, so this means that if the set of integers is $\{4,9,10\}$, then two players would know the integer on their card after the first four lines of dialogue. Therefore, the set of integers on the cards is not $\{4,9,10\}$. For similar reasons, the integers cannot be $\{6,8,9\}$.

The two-element subsets of $\{1,4,6\}$ all have a 2 next to them in the table above, which means that if the set of integers is $\{1,4,6\}$, then *none* of the players would know the integer on their card after the first four lines of dialogue. Therefore, the set of integers is not $\{1,4,6\}$, and for the same reason, it is not $\{4,6,9\}$.

The three players will also deduce this information as soon as the statements in line 5 of the dialogue are spoken. Thus, after these statements are made, all three players know that the only possibilities for the set of integers on the cards are those below:

$$\{1,4,8\} \ \{1,6,10\} \ \{1,8,10\}$$

Furthermore, in each of these three cases, the integer on Charlie's card must be 1 since only the player with 1 on their card is able to deduce the integer on their card after the first 5 lines of dialogue. For instance, if the cards are 1, 4, and 8, then the players who have 4 and 8 on their card see the pairs of integers $\{1,8\}$ and $\{1,4\}$ respectively. Since these sets each have a 2 next to them in the table above, these players cannot deduce the integer on their card from the first five lines of dialogue. However, The player whose card has a 1 on it sees the set $\{4, 8\}$, which has a 1 next to it in the table above, so they can deduce the integer on their card from the first five lines. By similar reasoning, if the cards are $\{1, 6, 10\}$ or $\{1, 8, 10\}$, then the player who is able to deduce the integer on their card must be holding the card with 1 on it. Therefore, the integer on Charlie's card is 1.

We have now narrowed down to 6 possibilities for how the cards are distributed:

Adina	Budi	Charlie
4	8	1
8	4	1
6	10	1
10	6	1
8	10	1
10	8	1

Since Charlie's card has 1 on it, Adina and Budi each see a 1 and one of 4, 6, 8, and 10. If Adina or Budi sees 1 and 4, then they know that the integer on their card is 8 since $\{1, 4, 8\}$ is the only possible remaining set that contains both 1 and 4. Similarly, if one of them sees 1 and 6, then they know the integer on their card is 10. If they see 1 and 8, then the integer on their card could be either 4 or 10, and if they see 1 and 10, then the integer on their card could be either 6 or 8.

In the sixth line of dialogue, we find out that after line 5, Adina does not know the integer on her card and Budi does know the integer on his card. This means Adina sees either 1 and 8 or 1 and 10, and Budi sees either 1 and 4 or 1 and 6. Of the six possibilities in the table above, the only two that satisfy both of these conditions are

Adina	Budi	Charlie
4	8	1
6	10	1

Therefore, Charlie is holding the card with 1 on it, and either Adina's card has 4 on it and Budi's has 8 on it, or Adina's card has 6 on it and Budi's has 10 on it.

In fact, both of these are possible. Denote by a the integer on Adina's card, by b the integer on Budi's card, and by c the integer on Charlie's card. We will verify that when $a = 4$, $b = 8$, and $c = 1$ the dialogue makes sense. Verifying the case when $a = 6$, $b = 10$, and $c = 1$ can be done similarly. Therefore, we assume that $a = 4$, $b = 8$, and $c = 1$.

1. Adina sees 8 and 1, neither of which is prime, so she can ask the question in line 1 since she does not know its answer.

3. Budi sees that $c = 1$ and $a = 4$ and knows that b is not prime. If $b = 8$, then $a+b+c$ is prime. If $b = 10$, then $a + b + c$ is not prime. Therefore, Budi cannot know the answer to the question in the third line before he asks it.

5.
   - Adina sees that $b = 8$ and $c = 1$, knows that $a+b+c = a+9$ is prime, and knows a is not prime. Therefore, she knows that either $a = 4$ or $a = 10$, but cannot tell which.
   - Budi sees that $a = 4$ and $c = 1$, knows that b is not prime, and knows that $a + b + c = 5 + b$ is prime. Therefore, he knows that $b = 6$ or $b = 8$, but cannot

tell which.

- Charlie sees $a = 4$ and $b = 8$, knows that c is one of 1, 6, 9, and 10, and knows that $a+b+c = 12+c$ is prime. Since $12+6 = 18$, $12+9 = 21$, and $12+10 = 22$ are all composite, Charlie knows at this point that $c = 1$.

6. – Budi knew that either $b = 6$ or $b = 8$ after he learned that $a + b + c$ was prime. He knows that Charlie knows the values of both a and b. If $b = 6$, then Charlie would not have been able to determine whether $c = 1$ or $c = 9$ since the sums $a + b + c = 4 + 6 + 1$ and $a + b + c = 4 + 6 + 9$ are both prime. Therefore, Budi concludes that $b = 8$.

   – Adina knows that $a = 4$ or $a = 10$ and she knows that Charlie knows that $b = 8$. She also knows that Charlie is able to deduce the value of c from the information revealed in the first four lines of dialogue. If $a = 10$, then Charlie could see an 8 and a 10 and would know that c was one of 1, 4, 6, and 9. The sum $8+10+1 = 19$ is prime, but $8 + 10 + 4$, $8 + 10 + 6$, and $8 + 10 + 9$ are all composite. This means Charlie would be able to deduce that $c = 1$ if $a = 10$. Similarly (as previously argued), if $a = 4$, then Charlie would be able to deduce that $c = 1$. Therefore, Charlie's ability to deduce the value of c after the first five lines does not tell Adina the value of a. As well, whether Adina is holding 4 or 10, Budi would not be able to tell whether he is holding 6 or 8 after the first five lines. This is because $4 + 1 + 6$, $4 + 1 + 8$, $10 + 1 + 6$, and $10 + 1 + 8$ are all prime. Therefore, Budi's inability to determine the value of b after the first five lines of dialogue does not tell Adina the value of a.

7. If $a = 10$ and $b = 6$ or $b = 8$, then Charlie would still be able to deduce that $c = 1$ after the first four lines of dialogue. However, Budi would not be able to deduce the value of b after the first five lines of dialogue. Both of these facts follow from reasoning similar to that which is above. Therefore, once Budi announces that he has deduced the integer on his card, Adina knows that $a \neq 10$, so she knows that $a = 4$.

In conclusion, the complete set of possibilities for the integers on the cards are that Adina's card has 4, Budi's card has 8, and Charlie's card has 1, or Adina's card has 6, Budi's card has 10, and Charlie's card has 1.

Problem of the Month
Problem 1: October 2021

Problem

Suppose a, b, and c are positive integers. In this problem, a *non-negative solution* to the equation $ax + by = c$ is a pair $(x, y) = (u, v)$ of integers with $u \geq 0$ and $v \geq 0$ satisfying $au + bv = c$. For example, $(x, y) = (7, 0)$ and $(x, y) = (3, 3)$ are non-negative solutions to $3x + 4y = 21$, but $(x, y) = (-1, 6)$ is not.

(a) Determine all non-negative solutions to $5x + 8y = 120$.

(b) Determine the largest positive integer c with the property that there is no non-negative solution to $5x + 8y = c$.

In parts (c), (d), and (e), a and b are assumed to be positive integers satisfying $\gcd(a, b) = 1$.

(c) Determine the largest non-negative integer c with the property that there is no non-negative solution to $ax + by = c$. The value of c should be expressed in terms of a and b.

(d) Determine the number of non-negative integers c for which there are exactly 2021 non-negative solutions to $ax + by = c$. As with part (c), the answer should be expressed in terms of a and b.

(e) Suppose $n \geq 1$ is an integer. Determine the sum of all non-negative integers c for which there are exactly n nonnegative solutions to $ax + by = c$. The answer should be expressed in terms of a, b, and n.

Fact: You may find it useful that for integers a and b with $\gcd(a, b) = 1$, there always exist integers x and y such that $ax + by = 1$, though x and y may not be non-negative.

Hint

(a) An exhaustive search is a reasonable approach to this problem. It can be made easier if you notice that x must be a multiple of 8 and that y must be a multiple of 5.

(b) Find a positive integer c with the property that $ax+by=c$, $ax+by=c+1$, $ax+by=c+2$, $ax+by=c+3$, and $ax+by=c+4$ all have non-negative solutions.

(c), (d), (e) As always, it is good to work out a few small examples to try to guess a pattern. It might be useful to understand the set of *all* integer solutions to $ax+by=c$ for fixed a, b, and c with $\gcd(a,b)=1$. Once you do this, you might consider the integer solution $(x,y)=(u,v)$ with u negative but as close to 0 as possible.

Problem of the Month

Solution to Problem 1: October 2021

Several times throughout this solution, we will use the following fact: if $\gcd(m, n) = 1$ and km is a multiple of n, then k is a multiple of n. You might want to think about why this is true before reading the solution.

(a) Suppose x and y are integers such that $5x + 8y = 120$. Rearranging $5x + 8y = 120$, we have that $5x = 120 - 8y$, and after factoring 8 out of the right side, we get $5x = 8(15 - y)$. This means $5x$ is a multiple of 8. Using the fact given before the solution and the fact that $\gcd(5, 8) = 1$, we get that x is a multiple of 8. Similarly, $8y = 120 - 5x = 5(24 - x)$, so y is a multiple of 5.

Now suppose x and y are non-negative integers such that $5x + 8y = 120$. By the previous paragraph, there are integers $X \geq 0$ and $Y \geq 0$ such that $x = 8X$ and $y = 5Y$, which means $5(8X) + 8(5Y) = 120$. Dividing by 40, we get $X + Y = 3$. Since X and Y are non-negative integers, (X, Y) must be one of the four pairs $(0, 3)$, $(1, 2)$, $(2, 1)$, and $(3, 0)$.

Since $x = 8X$ and $y = 5Y$, this means the only possible non-negative solutions are

$x = 0$	$x = 8$	$x = 16$	$x = 24$
$y = 15$	$y = 10$	$y = 5$	$y = 0$

It is easy to check that each of these pairs is indeed a non-negative solution to $5x+8y = 120$.

(b) Observe the following:

$$\begin{aligned}
5(4) + 8(1) &= 28 \\
5(1) + 8(3) &= 29 \\
5(6) + 8(0) &= 30 \\
5(3) + 8(2) &= 31 \\
5(0) + 8(4) &= 32
\end{aligned}$$

which shows that $5x + 8y = c$ has a non-negative solution when $c = 28$, $c = 29$, $c = 30$, $c = 31$, and $c = 32$.

Next, observe that if $5x + 8y = c$ has a non-negative solution $(x, y) = (u, v)$, then

$$\begin{aligned}
5(u + 1) + 8v &= 5u + 8v + 5 \\
&= c + 5,
\end{aligned}$$

so $5x+8y = c+5$ has a non-negative solution, namely $(x, y) = (u+1, v)$. Since $5x+8y = 28$ has a non-negative solution, so does $5x + 8y = 28 + 5 = 33$. Since $5x + 8y = 29$ has a non-negative solution, so does $5x + 8y = 29 + 5 = 34$. Continuing in this way, we get that $5x + 8y = c$ has a non-negative solution for $c = 33$, $c = 34$, $c = 35$, $c = 36$, and $c = 37$. This process can be repeated to get that $5x + 8y = c$ has a non-negative solution for all $c \geq 28$. It was important that we started with five consecutive values of c for which $5x + 8y = c$ has a non-negative solution.

To finish the solution to this part, we will argue that $5x + 8y = 27$ has no non-negative solution. Together with the fact that $5x + 8y = c$ has a non-negative solution for every $c \geq 28$, this will show that the answer to the question is $c = 27$.

Suppose $5x+8y = 27$ for non-negative integers x and y. Rearranging, we have $8y = 27-5x$. Since x is a non-negative integer, $27-5x$ has a units digit of either 7 or 2. However, $27-5x$ must be a non-negative multiple of 8 since it is equal to $8y$. There are no multiples of 8 with a units digit of 7, and the smallest nonnegative multiple of 8 with a units digit of 2 is 32. Therefore, $27 - 5x$ cannot be a non-negative multiple of 8 if x is a non-negative integer, so there are no non-negative solutions to $5x + 8y = 27$.

Before moving on to the solutions to parts (c), (d), and (e), we will state two facts that will come up in their solutions. The proofs of these facts can be found at the end of this document.

Fact 1: Suppose a and b are positive integers with $\gcd(a, b) = 1$. For every integer c, the equation $ax + by = c$ has an integer solution.

Fact 2: Suppose a and b are positive integers with $\gcd(a, b) = 1$, that c is an integer, and that $(x, y) = (u, v)$ is an integer solution to $ax + by = c$ (which must exist by Fact 1). For every integer k, the pair $(u + bk, v - ak)$ is a solution to $ax + by = c$. In addition, this gives every integer solution to $ax + by = c$.

Fact 2 says that finding all integer solutions to $ax + by = c$ comes down to finding one integer solution.

(c) In part (b), we saw that when $a = 5$ and $b = 8$, the answer is $c = 27$. It may take some experimentation to guess a pattern. For example, if $a = 4$ and $b = 3$, you will find that $c = 5$ is the smallest positive integer for which $ax + by = c$ has no non-negative solution. For another example, if $a = 6$ and $b = 7$, then $c = 29$ is the largest positive integer for which $ax + by = c$ has no non-negative solution. Even now, it might be tricky to notice a pattern. If 1 is added to each of these largest values of c, one gets 28 for $a = 5$ and $b = 8$, 6 for $a = 4$ and $b = 3$, and 30 for $a = 6$ and $b = 7$. These integers factor as $28 = 4 \times 7$, $6 = 3 \times 2$, and $30 = 5 \times 6$. With such an observation, you might guess that the largest integer c for which there are no non-negative solutions to $ax + by = c$ is $(a - 1)(b - 1) - 1 = ab - a - b$. This would be a correct guess, and we will now prove it!

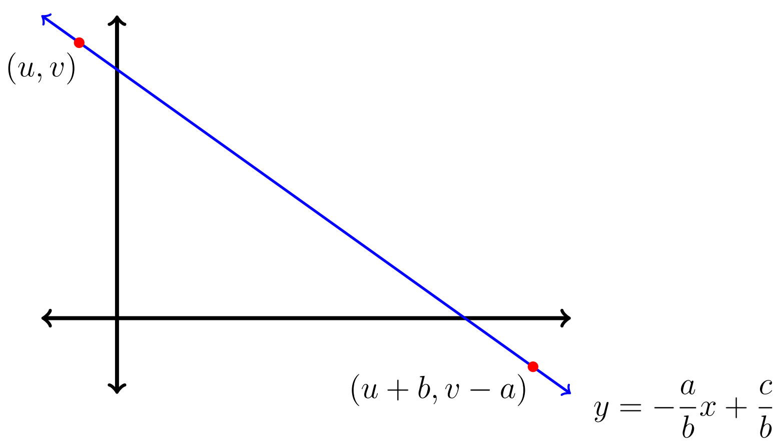
We will prove two statements.

- If a and b are positive integers with $\gcd(a, b) = 1$ and $ax+by = c$ has no non-negative solution, then $c \leq ab - a - b$.
- If a and b are positive integers with $\gcd(a, b) = 1$, then $ax + by = ab - a - b$ has no non-negative solution.

The first bullet point implies that if $c > ab - a - b$, then $ax + by = c$ *does* have a non-negative solution. Therefore, the two statements above combine to imply that the answer to the question is $c = ab - a - b$.

Assume that c is a positive integer such that $ax+by=c$ has no non-negative solution. We can rearrange $ax+by=c$ to $y=-\frac{a}{b}x+\frac{c}{b}$. This is the equation of a line with negative slope and a positive y-intercept. Furthermore, the solutions to $ax+by=c$ are exactly the *lattice points* that lie on the line [A lattice point is a point in the plane whose coordinates are both integers.]. By Fact 2, the integer solutions to $ax+by=c$, which are the lattice points on the line, are exactly the ordered pairs of the form $(u+bk, v-ak)$ where $(x,y)=(u,v)$ is any fixed integer solution and k takes every integer value. This means there are infinitely many lattice points on the line and that their x-coordinates occur at $x=u$ and every integer multiple of b to the right and left of u. Likewise, their y-coordinates occur at v and every integer multiple of a above and below v.

Thus, there must be a solution $(x,y)=(u,v)$ with the property that $u<0$ but $u+b\geq 0$. We will fix the solution $(x,y)=(u,v)$ to be the lattice point on the line $y=-\frac{a}{b}x+\frac{c}{b}$ that is closest to the y-axis among those with a negative x-coordinate. Since $u<0$ and u is an integer, it must be that $u\leq -1$. The diagram below depicts the line $y=-\frac{a}{b}x+\frac{c}{b}$ as well as the lattice point (u,v), and the next lattice point on the line moving from (u,v) to the right. We are assuming there are no non-negative solutions, which means the next lattice point cannot be in the first quadrant. However, it has a positive x-coordinate by the assumption on (u,v), so it must appear below the x-axis in order to fail to be a non-negative solution.



The next lattice point on the line moving to the right from (u,v) is $(u+b, v-a)$. As mentioned above, it must be in the fourth quadrant, which means $v-a<0$. Since v and a are both integers, so is $v-a$, which means $v-a\leq -1$ which can be rearranged to $v\leq a-1$.

We now have that $au+bv=c$ as well as $u\leq -1$ and $v\leq a-1$. Therefore,

$$\begin{aligned} c &= au+bv \\ &\leq a(-1)+b(a-1) \\ &= ab-a-b. \end{aligned}$$

Therefore, if $ax+by=c$ has no non-negative solutions, then $c\leq ab-a-b$, as claimed.

For the second statement, suppose $ax+by=ab-a-b$ for integers x and y. Rearranging and factoring, we get $a(x+1)+b(y+1)=ab$. Since both $a(x+1)$ and ab are multiples of a, it must also be the case that $b(y+1)$ is a multiple of a. We are assuming that $\gcd(a,b)=1$, so this means $y+1$ is a multiple of a. Therefore, there is some integer Y so that $y+1=aY$. By similar reasoning, there is an integer X such that $x+1=bX$.

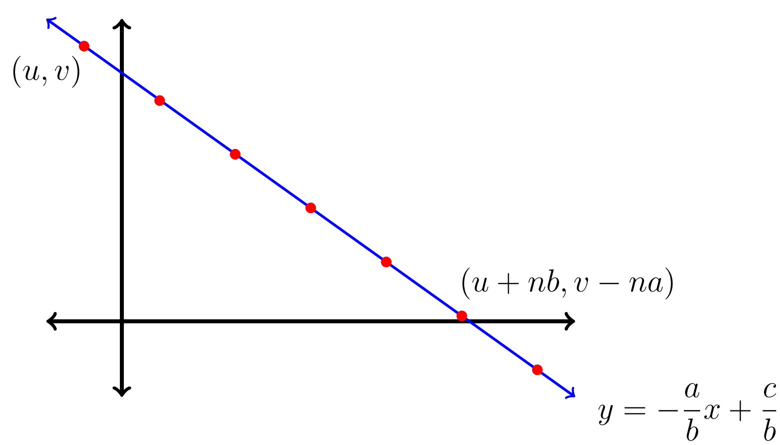
Substituting $y+1=aY$ and $x+1=bX$ into $a(x+1)+b(y+1)=ab$, we get the equation $abX+abY=ab$, and since ab must be positive, we can divide through by it to get $X+Y=1$. If the sum of two integers is 1, then one of them must be non-positive. Therefore, either $X\le 0$ or $Y\le 0$. By how X and Y are defined, this means either $\frac{x+1}{b}\le 0$ or $\frac{y+1}{a}\le 0$. Since a and b are positive, this means either $x+1\le 0$ or $y+1\le 0$, which implies that one of x and y is negative. Therefore, no integer solution to $ax+by=ab-a-b$ can be non-negative.

As discussed earlier, we have shown that $ax+by=c$ has a non-negative solution for every integer $c>ab-a-b$ and we have now shown that $ax+by=ab-a-b$ has no non-negative solution. Therefore, $c=ab-a-b$ is the largest integer with the property that $ax+by=c$ has no non-negative solution.

(d) We will show that for every positive integer n, there are exactly ab positive integers c for which there are exactly n non-negative solutions to $ax+by=c$.

Suppose c has the property that there are exactly n non-negative solutions to $ax+by=c$. Following the reasoning in the solution to (c), we are interested in lattice points on the line $y=-\frac{a}{b}x+\frac{c}{b}$. As discussed earlier, there are infinitely many such lattice points and we can choose (u,v) to be the lattice point on the line with the property that $u<0$ and $u+b\ge 0$. The next n lattice points on the line moving to the right are those of the form $(u+kb,v-ka)$ where k ranges over the integers from 1 through n inclusive.

In order for there to be exactly n non-negative solutions, the first n lattice points on the line to the right of (u,v) must be in the first quadrant. The diagram below is similar to the one in the solution to part (c), but it depicts the situation for $n=5$. The point (u,v) is in the second quadrant, the next five moving along the line to the right are in the first quadrant, and the next lattice point, $(u+(n+1)b,v-(n+1)a)$, is in the fourth quadrant.



For $(u+(n+1)b,v-(n+1)a)$ to be the first lattice point on the line to the right of (u,v) that does not correspond to a non-negative solution, it must not be in the first quadrant while $(u+nb,v-na)$ must be in the first quadrant. Since k and b are positive, the assumptions on (u,v) imply that $(u+kb,v-ka)$ has a non-negative x-coordinate for *every* positive integer k. This means $(u+nv, v-na)$ has a non-negative y-coordinate and $(u+(n+1)b, v-(n+1)a)$ has a negative y-coordinate. This leads to the two inequalities $v-na \geq 0$ and $v-(n+1)a < 0$.

From our assumption about c, we have deduced that there is a nonnegative solution (u, v) satisfying u and v satisfying $u < 0$, $u+b \geq 0$, $v-na \geq 0$, and $v-(n+1)a < 0$. Since all quantities are integers, we can replace the inequality $u < 0$ with $u \leq -1$ and replace $v-(n+1)a < 0$ with $v-(n+1)a \leq -1$. Rearranging and combining these inequalities, we have

$$-b \leq u \leq -1 \tag{1}$$
$$na \leq v \leq (n+1)a-1 \tag{2}$$

There are b integers u satisfying (1) and there are a integers v satisfying (2). We now have that if c is such that there are exactly n non-negative solutions to $ax+by=c$, then there are integers u and v satisfying (1) and (2) respectively, as well as $au+bv=c$.

Next, we suppose u satisfies (1) and v satisfies (2) and define $c = au+bv$. The inequalities (1) and (2) imply $u < 0$, $u+b \geq 0$, $v-na \geq 0$, and $v-(n+1)a < 0$. Following the reasoning from earlier in this part and in the solution to (c), this means $ax+by=c$ has exactly n non-negative solutions. Moreover, since $n \geq 1$, we have

$$c = au+bv \geq a(-b)+b(na) \geq -ab+ab = 0$$

which says that c is non-negative. [It is worth remarking here that if $n=0$, there are still are exactly ab integers c for which there are n non-negative solutions to $ax+by=c$. However, some of those integers will be negative. With $n \geq 1$, all of the integers c for which there are exactly n non-negative solutions to $ax+by=c$ happen to be non-negative.]

We have that $ax+by=c$ has exactly n non-negative solutions exactly when c takes the form $c = au+bv$ for some integers u satisfying (1) and v satisfying (2). Since there are b choices for an integer u satisfying (1) and a choices for an integer v satisfying (2), there are at most ab values of c that satisfy these conditions. To finish the argument, we must show that we indeed get ab distinct integers when computing $au+bv$ for every possible choice of u satisfying (1) and v satisfying (2).

To do this, we will assume that u_1 and u_2 both satisfy (1), that v_1 and v_2 both satisfy (2), and that $au_1+bv_1 = au_2+bv_2$ and deduce that $u_1 = u_2$ and $v_1 = v_2$. By possibly relabelling, we can assume that $u_1 \geq u_2$. With these assumptions, rearrange $au_1+bv_1 = au_2+bv_2$ to get $a(u_1-u_2) = b(v_2-v_1)$. This means $a(u_1-u_2)$ is a multiple of b. Since $\gcd(a,b)=1$, u_1-u_2 is a multiple of b. However, both u_1 and u_2 are between $-b$ and -1 inclusive, so their difference is smaller than b. We have that $0 \leq u_1-u_2 < b$ is a multiple of b. The only possibility is that $u_1-u_2=0$, or $u_1=u_2$. This means $b(v_1-v_2)=0$ as well, and since $b \neq 0$, $v_1=v_2$.

The question asked for the answer with $n = 2021$, but we have shown that the answer is ab for every integer $n \geq 1$, which includes $n = 2021$.

(e) From the reasoning in (d), we know that the positive integers c with the property that $ax+by=c$ has exactly n non-negative solutions are exactly the integers of the form $au+bv=c$ where $-b \leq u \leq -1$ and $na \leq v \leq (n+1)a-1$.

Observe that there are exactly b possible values of u and a possible values of v, so we need to add ab integers together.

We will do this by examining the u's first, then the v's. Observe that the sum contains exactly a copies of au for every u satisfying $-b \le u \le -1$. Therefore, the "u part" of the sum is

$$\begin{aligned}
&a(-a - 2a - 3a - 4a - \cdots - (b-1)a - ba) \\
&= -a^2(1 + 2 + 3 + \cdots + (b-1) + b) \\
&= -\frac{a^2 b(b+1)}{2}.
\end{aligned}$$

By similar reasoning, the sum contains exactly b copies of the term bv for every v satisfying $na \le v \le (n+1)a - 1 = na + a - 1$. This means the "$v$ part" of the sum is

$$\begin{aligned}
&b\big(bna + b(na+1) + b(na+2) + \cdots + b(na+a-2) + b(na+a-1)\big) \\
&= b^2\big(na + (na+1) + (na+2) + \cdots + (na+a-2) + (na+a-1)\big) \\
&= b^2\big(a(na) + 1 + 2 + 3 + \cdots + (a-2) + (a-1)\big) \\
&= a^2b^2n + b^2(1 + 2 + 3 + \cdots + (a-2) + (a-1)) \\
&= a^2b^2n + \frac{b^2(a-1)a}{2}
\end{aligned}$$

Therefore, the sum we seek is

$$\begin{aligned}
a^2b^2n + \frac{b^2(a-1)a}{2} - \frac{a^2b(b+1)}{2} &= \frac{ab}{2}\left(2abn + b(a-1) - a(b+1)\right) \\
&= \frac{ab}{2}\left(2abn + ab - b - ab - a\right) \\
&= \frac{ab}{2}\left(2abn - a - b\right).
\end{aligned}$$

As promised, we now include proofs of Fact 1 and Fact 2, which were stated between the solutions to parts (b) and (c). The proof of Fact 1 makes use of the fact that if $\gcd(a,b) = 1$, then $ax + by = 1$ always has an integer solution. This is a well known fact from number theory that you may wish to look up.

Fact 1: Suppose a and b are positive integers with $\gcd(a,b) = 1$. For every integer c, the equation $ax + by = c$ has an integer solution.

Proof. There are integers u' and v' such that $au' + bv' = 1$ (see above). Setting $u = cu'$ and $v = cv'$, we have $au + bv = acu' + bcv' = c(au' + bv') = c(1) = c$. $\square$

Fact 2: Suppose a and b are positive integers with $\gcd(a,b) = 1$, that c is an integer, and that $(x,y) = (u,v)$ is an integer solution to $ax + by = c$ (which must exist by Fact 1). For every integer k, the pair $(u + bk, v - ak)$ is a solution to $ax + by = c$. In addition, this gives every integer solution to $ax + by = c$.

Proof. To see that $(u + bk, v - ak)$ is a solution, we can substitute and simplify:

$$\begin{aligned} a(u + bk) + b(v - ak) &= au + abk + bv - abk \\ &= au + bv \\ &= c \end{aligned}$$

since $(x, y) = (u, v)$ is a solution to $ax + by = c$ by assumption.

To see that every solution takes the form $(u + bk, v - ak)$ is slightly trickier and requires use of the fact that $\gcd(a, b) = 1$.

Suppose $(x, y) = (u', v')$ is also a solution to $ax + by = c$. This means $au' + bv' = c$. We also have that $au + bv = c$, so we can subtract to get

$$(au + bv) - (au' + bv') = c - c = 0$$

which can be rearranged and factored to get $a(u' - u) = b(v - v')$.

In the equation above, a, b, $u' - u$, and $v - v'$ are all integers, and so we have that $a(u' - u)$ is a multiple of b. Since $\gcd(a, b) = 1$, $u' - u$ is a multiple of b, which means there is some integer k such that $u' - u = bk$. Substituting this into $a(u' - u) = b(v - v')$ gives $abk = b(v - v')$, and after cancelling b from both sides, we have $v - v' = ak$.

Rearranging $u' - u = bk$ and $v - v' = ak$ to $u' = u + bk$ and $v' = v - ak$ shows that the solution $(x, y) = (u', v')$ takes the form $(u + bk, v - ak)$, as claimed. $\square$

Problem of the Month
Problem 2: November 2021

Problem

A *lattice point* is a point (a, b) in the plane with the property that a and b are both integers. In this problem, we will say that a lattice point $P(a, b)$ is *visible* if $a > 0$, $b > 0$, and the line segment connecting P and the origin does not contain any lattice points other than P and the origin.

(a) How many lattice points $P(a, b)$ with $a \leq 10$ and $b \leq 10$ are visible?

(b) Determine the number of integers b with $b \leq 50$ for which $P(a, b)$ is visible when

 (i) $a = 6$

 (ii) $a = 18$

 (ii) $a = 36$.

(c) Determine how many points $P(a, b)$ with $a \leq 50$ and $b \leq 50$ are visible. There is quite a bit to do by hand, so you may want to use technology to help.

(d) Explain why the following equality is true:

$$\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{5^2}\right)\left(1 - \frac{1}{7^2}\right)\left(1 - \frac{1}{11^2}\right)\cdots \quad = \frac{1}{1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \cdots}$$

The expressions on the left is the infinite product of all expressions of the form $1 - \dfrac{1}{p^2}$ where p is prime.

(e) It is well known that the infinite sum

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \cdots$$

is equal to $\dfrac{\pi^2}{6}$. This fact has many proofs and is originally due to the mathematician Leonhard Euler. You may wish to explore some of these proofs, but the intention in this problem is for you to take the result for granted.

Interestingly, the probability that a randomly chosen point in the first quadrant not on the axes is visible is $\dfrac{6}{\pi^2}$. Explain why this is true.

Note: It is ok to be a bit suspicious of what we mean by "probability" when choosing from an infinite set. Here is a way to think about what is meant in this problem: for a fixed positive integer, n, it is possible to compute the probability that a point $P(a, b)$ with $0 < a \leq n$ and $0 < b \leq n$ chosen randomly is visible. One might call this probability p_n. The question in (e), posed a bit more formally, might be "show that p_n gets very close to $\dfrac{6}{\pi^2}$ as n gets large". If you have seen *limits*, you might want to formalize this further.

Hint

(a) In all parts of this problem, it will be useful to think about how $P(a, b)$ being visible relates to $\gcd(a, b)$.

(b) All three parts have the same answer.

(c) For positive integers u and v, the number of positive multiples of u that are no larger than v is $\left\lfloor \frac{u}{v} \right\rfloor$. You may need to look up the notation $\lfloor x \rfloor$. If you are solving this problem by hand, you might want to first consider how many visible points there are of the form $P(a, b)$ when a is prime.

(d) What is the reciprocal of the sum of the geometric series $1 + \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + \frac{1}{2^8} + \cdots$?

(e) What is the probability that two randomly chosen positive integers are both even? What is the probability that two randomly chosen integers are both multiples of 3?

Problem of the Month

Solution to Problem 2: November 2021

(a) Suppose $P(a,b)$ is visible. Since $a > 0$, we have that $a \neq 0$ and so the line segment connecting P to the origin has equation $y = \frac{b}{a}x$. Now suppose m is a positive common divisor of a and b. Then there are integers a' and b' with $0 < a' \leq a$, $0 < b' \leq b$, $a = a'm$, and $b = b'm$. Then $\frac{b}{a}a' = \frac{b'm}{a'm}a' = b'$, so (a', b') is on the line segment connecting P to the origin. Since P is visible, we cannot have $a' < a$, so $a' = a$ which means $m = 1$. We assumed that m was a positive common divisor of a and b and deduced that $m = 1$. Therefore, if $P(a,b)$ is visible, then $\gcd(a,b) = 1$.

Now suppose $P(a,b)$ is not visible. This means there is some lattice point (a', b') on $y = \frac{b}{a}x$ with $0 < a' < a$. This means $b' = \frac{b}{a}a'$ which can be rearranged to $ab' = a'b$. Since a, b, a', and b' are all integers, we have that the integer $a'b$ is a multiple of the integer a. If $\gcd(a,b) = 1$, then a' is a multiple of a. However, this cannot happen since $a' < a$. Therefore, if $P(a,b)$ is not visible, then $\gcd(a,b) \neq 1$.

We have shown that a point P that is not on the axes is visible exactly when $\gcd(a,b) = 1$. Therefore, counting the visible points $P(a,b)$ with $0 < a \leq 10$ and $0 < b \leq 10$ is the same as counting ordered pairs (a,b) with $0 < a \leq 10$ and $0 < b \leq 10$ such that $\gcd(a,b) = 1$.

The table below has rows indexed by the possible integer values of a from 1 through 10 inclusive and columns indexed by the values of b from 1 through 10 inclusive. The cell in the row corresponding to a and the column corresponding to b contains $\gcd(a,b)$.

	1	2	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1	1	1
2	1	2	1	2	1	2	1	2	1	2
3	1	1	3	1	1	3	1	1	3	1
4	1	2	1	4	1	2	1	4	1	2
5	1	1	1	1	5	1	1	1	1	5
6	1	2	3	2	1	6	1	2	3	2
7	1	1	1	1	1	1	7	1	1	1
8	1	2	1	4	1	2	1	8	1	2
9	1	1	3	1	1	3	1	1	9	1
10	1	2	1	2	5	2	1	2	1	10

By the reasoning above, the number of visible points is equal to the number of 1's in the table above. There are 63 1's in the table, so there are 63 visible points $P(a,b)$ with $0 < a \leq 10$ and $0 < b \leq 10$.

(b) We can answer all three parts of this question at once. Factoring into primes, we have $6 = 2 \times 3$, $18 = 2 \times 3 \times 3$, and $36 = 2 \times 2 \times 3 \times 3$. The only prime numbers that divide 6 are 2 and 3, and the same is true of 18 and 36.

Therefore, for $a = 6$, $a = 18$, and $a = 36$, $\gcd(a, b) = 1$ exactly when 2 is not a divisor of b and 3 is not a divisor of b. This means that the answer to (i), (ii), and (iii) is equal to the number of integers b with $0 < b \leq 50$ that are neither a multiple of 2 nor a multiple of 3.

There are fifty integers b satisfying $0 < b \leq 50$ and exactly half of them are even. Thus, 25 of the values of b are multiples of 2. The largest multiple of 3 that is no larger than 50 is 48. This means the multiples of 3 between 1 and 50 inclusive are 3, 6, 9, and so on up to 3×16. Therefore, 16 of the values of b are multiples of 3.

Each of the totals computed in the previous paragraph, 25 and 16, includes the integers that are multiples of both 2 and 3. An integer is a multiple of both 2 and 3 exactly when it is a multiple of 6. This means the total $25 + 16 = 41$ is equal to the number of values of b that are either a multiple of 2 or a multiple of 3, but it overcounts by the number of multiples of 6 since both 25 and 16 account for the number of multiples of 6.

The largest multiple of 6 that is no larger than 50 is 48, which is equal to 6×8, so there are 8 multiples of 6 that are no larger than 50. Therefore, $25 + 16 - 8 = 33$ integers b with $0 < b \leq 50$ have the property that they are either a multiple of 2, a multiple of 3, or both. We are interested in the number of integers b with $0 < b \leq 50$ that are neither a multiple of 2 nor a multiple of 3, which we can now compute as $50 - 33 = 17$.

For $a = 6$, $a = 18$, and $a = 36$, there are 17 visible points $P(a, b)$ with $0 < b \leq 50$.

(c) In this solution, we will compute the number of visible points with $0 < a \leq 50$ and $0 < b \leq 50$, though some of the calculations will not be shown. With that said, the author promises that the calculation was done entirely by hand, but will not deny that a calculator was used to check them.

By the reasoning from the beginning of the solution to part (a), we wish to count all pairs (a, b) with $\gcd(a, b) = 1$, $0 < a \leq 50$, and $0 < b \leq 50$. We will declare now that a and b are integers satisfying $0 < a \leq 50$ and $0 < b \leq 50$ in this solution to avoid repeating this quantification.

For a fixed a, the number of b with $\gcd(a, b) = 1$ is equal to the number of b such that b has no prime factors in common with a. Therefore, as we saw in part (b), the primes occurring in the prime factorization of a is what matters, not the number of times each prime occurs. To compute this in general, we will compute the number of integers b that *do* have a prime factor in common with a, then subtract the result from 50.

In general, we will need a way to compute the number of multiples of an integer n that are less than or equal to 50. Suppose k is the largest positive integer such that $kn \leq 50$. Then k is the number of multiples of n that are no larger than 50. In other words, what we seek is a general way to compute k from n. To do this, we observe that if k is the largest positive integer such that $kn \leq 50$, then $50 < (k + 1)n$, so we have $kn \leq 50 < (k + 1)n$. Dividing through by n gives $k \leq \dfrac{50}{n} < k + 1$. The quantities k and $k + 1$ are consecutive integers, so we conclude that k is the largest integer that is no larger than $\dfrac{50}{n}$. For example, with $n = 6$, we get that $\dfrac{50}{6} = 8.3333\ldots$, so the largest integer that is no larger than $\dfrac{50}{6}$ is 8. This agrees with what was found in part (b) since there we showed that there are 8 positive multiples of 6 that are no larger than 50.

We now introduce some standard notation. For a real number x, we denote by $\lfloor x \rfloor$ the largest integer that is less than or equal to x. For example, $\lfloor \pi \rfloor = 3$. When x is an integer, $\lfloor x \rfloor = x$. For example, $\lfloor 5 \rfloor = 5$.

We now state a general fact: If u and v are positive integers, then the number of positive multiples of u that are less than or equal to v is $\left\lfloor \frac{v}{u} \right\rfloor$. You might want to think about why this is also true when $v < u$.

We will now continue to address the given question. To start, we will count the number of visible points $P(a, b)$ when a is a prime number.

If a is a prime number, then the number of integers b for which $\gcd(a, b) \neq 1$ is equal to the number of multiples of a between 1 and 50 inclusive. Therefore, if a is prime, then the number of visible points is $50 - \left\lfloor \frac{50}{a} \right\rfloor$.

The prime numbers that are no larger than 50 are

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47$$

For the primes from $a = 29$ to $a = 47$, we have $25 < a < 50$, so $1 = \frac{50}{50} < \frac{50}{a} < \frac{50}{25} = 2$. Thus, for each of the six primes between 29 and 47 inclusive, it must be that $\left\lfloor \frac{50}{a} \right\rfloor = 1$, so there are $50 - 1 = 49$ visible points. This gives a total of $6 \times 49 = 294$ visible points.

For the other nine primes from $a = 2$ through $a = 23$, the table below has the value of a in the left column and the corresponding value of $50 - \left\lfloor \frac{50}{a} \right\rfloor$ in the right column.

a	$50 - \left\lfloor \frac{50}{a} \right\rfloor$
2	25
3	34
5	40
7	43
11	46
13	47
17	48
19	48
23	48

Since $50 - \left\lfloor \frac{50}{a} \right\rfloor$ is the number of visible points when a is prime, we can get the number of visible points with a prime by totaling the values in the right column of the table above and adding this total to 294. This gives $379 + 294 = 673$ visible points $P(a, b)$ when a is prime.

Next, consider the case when $a = 4$. Since $4 = 2^2$, an integer b has $\gcd(4, b) = 1$ exactly when $\gcd(2, b) = 1$. Put differently, b and 4 have a common divisor larger than 1 exactly when b and 2 have a common divisor larger than 1. The same is true of any positive integer power of 2. This means that if a is a power or 2, then there are the same number of visible points $P(a,b)$ as there are visible points $P(2,b)$. Thus, we get 25 visible points for each of $a=4$, $a=8$, $a=16$, and $a=32$. Likewise, the number of visible points when $a=9$ or $a=27$ is the same as when $a=3$, so there are 34 visible points for $a=9$ and $a=27$. By similar reasoning there are 40 visible points when $a=25=5^2$ and 43 visible points when $a=49=7^2$.

When $a=1$, $P(a,b)$ is always visible, so there are 50 visible points when $a=1$. We will now recap with a subtotal: if a is prime, a is a power of a prime, or $a=1$, then there are

$$673+4(25)+2(34)+40+43+50=974$$

visible points.

We still need to count the visible points when a takes the values

$$6,10,12,14,15,18,20,21,22,24,26,28,30,33,34,35,36,38,39,40,42,44,45,46,48,50$$

Using the reasoning in part (b), there are exactly 17 visible points $P(a,b)$ if the prime divisors of a are exactly 2 and 3. This accounts for a taking on the values

$$6,12,18,24,36,48$$

so we get $6\times 17=102$ visible points with the property that the prime divisors of a are exactly 2 and 3. We will do a more general computation before adding these to the running total of 974.

Generalizing the ideas to count the number of visible points when the prime divisors of a are exactly 2 and 3, suppose a is an integer with exactly two prime divisors, p and q. The number of integers less than or equal to 50 that are multiples of p is $\left\lfloor \frac{50}{p} \right\rfloor$ and the number of multiples of q is $\left\lfloor \frac{50}{q} \right\rfloor$. Each of these totals counts the common multiples of p and q, but since p and q are distinct primes, their common multiples are exactly the multiples of pq. Thus, the number of integers less than or equal to 50 that are multiples of either p or q is

$$\left\lfloor \frac{50}{p} \right\rfloor+\left\lfloor \frac{50}{q} \right\rfloor-\left\lfloor \frac{50}{pq} \right\rfloor$$

and hence, the number of visible points when the prime divisors of a are exactly p and q is

$$50-\left\lfloor \frac{50}{p} \right\rfloor-\left\lfloor \frac{50}{q} \right\rfloor+\left\lfloor \frac{50}{pq} \right\rfloor.$$

Indeed, with $p=2$ and $q=3$, we get

$$50-\left\lfloor \frac{50}{2} \right\rfloor-\left\lfloor \frac{50}{3} \right\rfloor+\left\lfloor \frac{50}{6} \right\rfloor=50-25-16+8=17.$$

When $p=2$ and $q=5$, there are $50-\left\lfloor \frac{50}{2} \right\rfloor-\left\lfloor \frac{50}{5} \right\rfloor+\left\lfloor \frac{50}{10} \right\rfloor=50-25-10+5=20$ visible points. This is the number of visible points for $a=2\times 5=10$, $a=2^2\times 5=20$, $a=2^3\times 5=40$, and $a=2\times 5^2=50$.

When $p = 2$ and $q = 7$, there are $50 - 25 - 7 + 3 = 21$ visible points, which gives the number of visible points when $a = 14$, $a = 28$, and $a = 42$.

In the table below, the numbers of visible points when a has exactly two prime divisors are summarized. There are four columns in the table. In each row, the cell in the first column contains a prime p, the cell in the second column contains a prime q with $p < q$, the cell in the third column contains the number of visible points for any a whose prime divisors are exactly p and q, and the cell in the fourth column contains a list of the values of a with exactly these two prime divisors. Therefore, to find the number of visible points $P(a, b)$ if a has exactly two prime divisors p and q, locate a in the fourth column and the number of visible points will be the integer in the same row in the third column. Note that pairs (p, q) of primes are accounted for in the table below only if pq has at least one multiple less than or equal to 50.

p	q	$50 - \left\lfloor \frac{50}{p} \right\rfloor - \left\lfloor \frac{50}{q} \right\rfloor + \left\lfloor \frac{50}{pq} \right\rfloor$	a values
2	3	17	$6, 12, 18, 24, 36, 48$
2	5	20	$10, 20, 40, 50$
2	7	21	$14, 28$
2	11	23	$22, 44$
2	13	23	26
2	17	24	34
2	19	24	38
2	23	24	46
3	5	27	$15, 45$
3	7	29	21
3	11	31	33
3	13	32	39
5	7	34	35

The table above contains the number of visible points for every remaining a other than $a = 30$ and $a = 42$. Adding the totals for each of these 24 values of a, we get

$$(6 \times 17) + (4 \times 20) + (2 \times 21) + (3 \times 23) + (3 \times 24) + (2 \times 27) + 29 + 31 + 32 + 34 = 545$$

Adding to our previous total, we get that there are $974 + 545 = 1519$ visible points $P(a, b)$ where a is either 1, is a prime, is a power of a prime, or has exactly two distinct prime divisors. As mentioned above, we now have counted the visible points except for when $a = 30$ and $a = 42$. Notice that these are the only two positive integers less than 50 that have more than two distinct prime divisors.

You may wish to think about a general way to count the number of visible points where a has exactly three distinct prime divisors. It may be useful to read about the *inclusion-exclusion principle*. For this solution, we will just list the values of b that have $\gcd(30, a) = 1$ and $\gcd(42, b) = 1$, respectively. For 30, they are

$$1, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 49$$

and for 42 they are

$$1, 5, 11, 13, 17, 19, 23, 25, 29, 31, 37, 41, 43, 47$$

for totals of 14 visible points for each of $a = 30$ and $a = 42$. Therefore, the total number of visible points $P(a, b)$ with $0 < a \leq 50$ and $0 < b \leq 50$ is $1519 + 14 + 14 = 1547$.

(d) To get an idea of why this is true, we first consider the following even more suspicious looking expression:

$$\left(1 + \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + \cdots\right)\left(1 + \frac{1}{3^2} + \frac{1}{3^4} + \frac{1}{3^6} + \cdots\right)\left(1 + \frac{1}{5^2} + \frac{1}{5^4} + \frac{1}{5^6} + \cdots\right)\cdots$$

This is a product of infinitely many sums. Each sum is an infinite sum of the reciprocals of the *even* powers of a prime.

Although it may require some imagination, consider what would happen if we were to multiply this expression out. Each "term" would be a product of one summand from each parenthetical expression. Suppose, for example, we choose the $\frac{1}{p^2}$ term for each prime p. Then we would get a term

$$\frac{1}{2^2 \times 3^2 \times 5^2 \times 7^2 \times 11^2 \times 13^2 \times 17^2 \times \cdots}$$

and since there are infinitely many primes, the denominator is a product of infinitely many numbers that are greater than 1. This cannot possibly be any finite number, so we can interpret this term as $\frac{1}{\infty}$, which we have little choice but to interpret as being equal to 0.

We run into the same issue any time infinitely many of the "choices" are not equal to 1. Thus, for a term to "contribute" anything to the sum, we can only choose finitely many terms that are different from 1. That is, the expression above is equal to the sum of all terms obtained by choosing a term from each parenthetical expression so that only finitely many of the choices are different from 1. For example,

$$\frac{1}{2^2} \times \frac{1}{7^2} \times \frac{1}{19^6} = \frac{1}{(2 \times 7 \times 19^3)^2}$$

and

$$\frac{1}{7^8} \times \frac{1}{1009^2} = \frac{1}{(7^4 \times 1009)^2}$$

are terms in the sum.

If you think about it every term in the sum will be of the form $\frac{1}{n^2}$. Moreover, given a positive integer n, the prime factorization of n^2 has the form $p_1^{2e_1} p_2^{2e_2} \cdots p_k^{2e_k}$ where $p_1, \ldots, p_k$ are distinct primes and $e_1, \ldots, e_k$ are positive integers. By choosing $\frac{1}{p_i^{2e_i}}$ from the parenthetical expression for the prime p_i and 1 for all others, we get $\frac{1}{n^2}$ as a term in the sum. Since prime factorizations are unique, there is only one way that $\frac{1}{n^2}$ can arise as a term in the sum.

Therefore, it makes some sense that the product of infinite sums above is equal to the sum

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots .$$

Each sum in parentheses above is a geometric series, so we have the following:

$$
\begin{aligned}
&1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\cdots \\
&=\left(1+\frac{1}{2^2}+\frac{1}{2^4}+\frac{1}{2^6}+\cdots\right)\left(1+\frac{1}{3^2}+\frac{1}{3^4}+\frac{1}{3^6}+\cdots\right)\left(1+\frac{1}{5^2}+\frac{1}{5^4}+\frac{1}{5^6}+\cdots\right)\cdots \\
&=\left(\frac{1}{1-\frac{1}{2^2}}\right)\left(\frac{1}{1-\frac{1}{3^2}}\right)\left(\frac{1}{1-\frac{1}{5^2}}\right)\left(\frac{1}{1-\frac{1}{7^2}}\right)\left(\frac{1}{1-\frac{1}{11^2}}\right)\cdots
\end{aligned}
$$

which shows that

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\cdots=\left(\frac{1}{1-\frac{1}{2^2}}\right)\left(\frac{1}{1-\frac{1}{3^2}}\right)\left(\frac{1}{1-\frac{1}{5^2}}\right)\left(\frac{1}{1-\frac{1}{7^2}}\right)\left(\frac{1}{1-\frac{1}{11^2}}\right)\cdots$$

Now take reciprocals of both sides of the equation above to get

$$\frac{1}{1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\cdots}=\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{5^2}\right)\left(1-\frac{1}{7^2}\right)\left(1-\frac{1}{11^2}\right)\cdots$$

(e) It was argued at the beginning of the solution to part (a) that a point $P(a,b)$ is visible exactly when $\gcd(a,b)=1$. Equivalently, $P(a,b)$ is visible exactly when a and b have no prime divisors in common.

We will first discuss the prime $p=2$. Suppose a is chosen randomly in the range $0<a\leq n$ for some integer n. If n is even, then there is exactly a $\frac{1}{2}$ chance that a is a multiple of 2. If n is odd, then there is a $\frac{\frac{n-1}{2}}{n}=\frac{1}{2}-\frac{1}{2n}$ chance that a is a multiple of 2. Notice that in the latter case, the probability is very close to $\frac{1}{2}$ when n is large since the quantity $\frac{1}{2n}$ is close to 0. Thus, if n is large, the probability that a is a multiple of 2 is extremely close to $\frac{1}{2}$, whether n is even or odd.

Now suppose a and b are both between 1 and n inclusive. The probability that a and b are both multiples of 2 is close to $\frac{1}{2}\times\frac{1}{2}$, and this implies that the probability that a and b do *not* have a common divisor of 2 is close to $1-\frac{1}{2^2}$.

We point out that $1-\frac{1}{2^2}$ becomes a better estimate for the probability as n gets larger.

More generally, consider a prime p and some large fixed positive integer, n. Now choose an integer a randomly with $0<a\leq n$. If n happens to be a multiple of p, then the probability that a is a multiple of p is exactly $\frac{1}{p}$. If n is not a multiple of p, then the probability that a is a multiple of p is close to $\frac{1}{p}$ (as with $p=2$, it gets closer as n gets larger). By the same reasoning as with $p=2$, if n is large and $P(a,b)$ is chosen randomly with $0<a\leq n$ and $0<b\leq n$, then the probability that a and b do not have a common divisor of p is close to $1-\frac{1}{p^2}$.

Now consider two different prime numbers p and q. Following reasoning similar to that which is above, if a point $P(a, b)$ is chosen randomly, there is a $\frac{1}{p^2}$ chance that a and b have a common divisor of p, there is a probability of $\frac{1}{q^2}$ that a and b have a common divisor of q, and there is a probability of $\frac{1}{(pq)^2}$ that a and b have a common divisor of both p and q. The latter probability is because a number is a multiple of p and q exactly when it is a multiple of pq.

We can now say that the probability that a and b have either a divisor of p in common and a divisor of q in common should be very close to

$$\frac{1}{p^2} + \frac{1}{q^2} - \frac{1}{(pq)^2}$$

where we subtract $\frac{1}{(pq)^2}$ since it is the probability that an integer is a multiple of both p and q. Therefore, the probability that a and b have neither a divisor of p in common nor a divisor of q in common is

$$1 - \frac{1}{p^2} - \frac{1}{q^2} + \frac{1}{(pq)^2} = \left(1 - \frac{1}{p^2}\right)\left(1 - \frac{1}{q^2}\right).$$

Now let Event 1 be the event that a and b do not have a common divisor of p and Event 2 be the event that a and b do not have a common divisor of q. We have shown that the probability that both Event 1 and Event 2 occur is equal to the product of the probabilities that the events occur individually. In probability theory, we would conclude that Event 1 and Event 2 are independent.

The probability that $\gcd(a, b) = 1$ is equal to the probability that a and b have no prime divisors in common. By the reasoning above, the probability that they have no prime divisors in common is the product of the probabilities for each individual prime. Therefore, the probability is

$$\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{5^2}\right)\left(1 - \frac{1}{7^2}\right)\left(1 - \frac{1}{11^2}\right)\cdots\left(1 - \frac{1}{p^2}\right)\cdots$$

In part (d), we argued that this quantity is the reciprocal of $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$. It was given in part (e) that $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}$. Thus, the probability is $\frac{6}{\pi^2} \approx 0.6079$.

Below is a simple Python script that, for a given positive integer n, returns the proportion of points $P(a, b)$ with $0 < a \leq n$ and $0 < b \leq n$ that are visible.

```
import math
n = int(input())
count = 0
for a in range(1,n+1):
    for b in range(1,n+1):
        if math.gcd(a,b) == 1:
            count += 1
print(float(count)/n**2)
```

Final Remark: You may wonder if a similar analysis can be performed in 3 dimensions. That is, we can consider points in space with coordinates (a, b, c) that are all positive integers and ask whether it is visible. Here, a point being "visible" would again mean there are no points with integer coordinates on the line segment connecting it to the origin.

In fact, a very similar argument can be used to relate the probability that a point (a, b, c) with positive integer coordinates is visible to the quantity

$$1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + \cdots .$$

An interesting fact about the quantity above is that there is no known "closed form" like there is for the sum of the reciprocals of the squares. It is known, however, that it does "equal" something, and that value is about 1.2020569.

You might even stretch your imagination further to wonder about the probability that a point with positive integer coordinates is "visible" in four or more dimensions. It turns out to always be related to the quantity

$$1 + \frac{1}{2^d} + \frac{1}{3^d} + \frac{1}{4^d} + \frac{1}{5^d} + \cdots$$

where d is the dimension. For $d \geq 2$, this quantity always "equals" something, and in fact, it is known that when d is even, it is equal to π^d times a rational number. You may wish to search "Bernoulli numbers" for more information.

These infinite sums are special values of a very famous function known as the "Riemann Zeta Function". This function has been an important object of study in mathematics for well over a century and there are still many unsolved problems involving it.

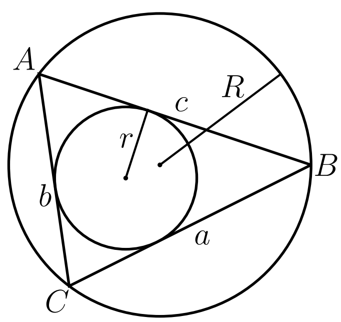
Problem of the Month
Problem 3: December 2021

Problem

Before stating the problem, we will introduce some notation and terminology.

- In $\triangle ABC$, we will denote the length of BC by a, the length of AC by b, and the length of AB by c.
- The *semiperimeter* of $\triangle ABC$ will be denoted by s and is equal to $\dfrac{a+b+c}{2}$.
- The *incircle* of $\triangle ABC$ is the unique circle that is tangent to all three sides of $\triangle ABC$. Its radius is called the *inradius* of $\triangle ABC$ and is denoted by r. An important fact about the incircle is that its centre is at the intersection of the three angle bisectors of the triangle.
- The *circumcircle* of $\triangle ABC$ is the unique circle on which all three of A, B, and C lie. Its radius is called the *circumradius* of $\triangle ABC$ and is denoted by R. An important fact about the circumcircle is that its centre is at the intersection of the perpendicular bisectors of the three sides of the triangle.

The diagram below illustrates some of the information above.



This problem is about right-angled triangles. Most of us are aware of the famous Pythagorean theorem, but there are other interesting properties only satisfied by right-angled triangles.

(a) Suppose $\triangle ABC$ is right-angled at C and that h is the length of the altitude from C to AB. Show that $\dfrac{1}{a^2} + \dfrac{1}{b^2} = \dfrac{1}{h^2}$.

(b) Suppose $\triangle ABC$ is right-angled. Show that $\cos^2\angle A + \cos^2\angle B + \cos^2\angle C = 1$.

(c) Suppose $\triangle ABC$ is right-angled. Show that $s = r + 2R$.

(d) Suppose $\triangle ABC$ satisfies $a^2 + b^2 = c^2$. Prove that $\angle C = 90^\circ$.

(e) Suppose $\triangle ABC$ satisfies $\cos^2\angle A + \cos^2\angle B + \cos^2\angle C = 1$. Prove that $\triangle ABC$ is right-angled.

(f) Suppose $\triangle ABC$ satisfies $s = r + 2R$. Show that $\triangle ABC$ is right-angled. [A solution to this problem will likely require some general identities involving the inradius and circumradius. Some specific useful identities will be given in the hint.]

Hint

There are several ways to approach each of the parts of this problem. The hints below correspond to the solutions that will be provided. You might find solutions that do not use the ideas in these hints.

(a) Apply the Pythagorean theorem to some of the smaller right-angled triangles that appear once the altitude is drawn.

(b) If there is a right angle at A, then what does the equation become?

(c) The important facts in the problem statement about how to find the centres of the incircle and circumcircle may be useful.

(d) Be careful not to confuse the statements "*If* $\angle C = 90°$ *then* $a^2 + b^2 = c^2$" and "*If* $a^2 + b^2 = c^2$ *then* $\angle C = 90°$". The first statement is what is usually considered the Pythagorean theorem. The second statement is its *converse*, and this is the statement this problem asks you to verify. This means that you cannot assume that $\angle C = 90°$ has a right angle; you need to assume $a^2 + b^2 = c^2$ and deduce that $\angle C = 90°$.

(e) Try to use trigonometric identities to prove that $\big(\cos A\big)\big(\cos B\big)\big(\cos(A + B)\big) = 0$.

(f) We found several solutions to this problem and each of them involves significant algebraic manipulation. The simplest solution that we found involved an expression for $8R^2$ in terms of a, b, and c. In our solution, we will use the following two facts that are true of every triangle.

- The quantities rs, $\dfrac{abc}{4R}$, and $\sqrt{s(s-a)(s-b)(s-c)}$ are all equal to the area of $\triangle ABC$.
- The Law of Sines can be extended to the following set of equations:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

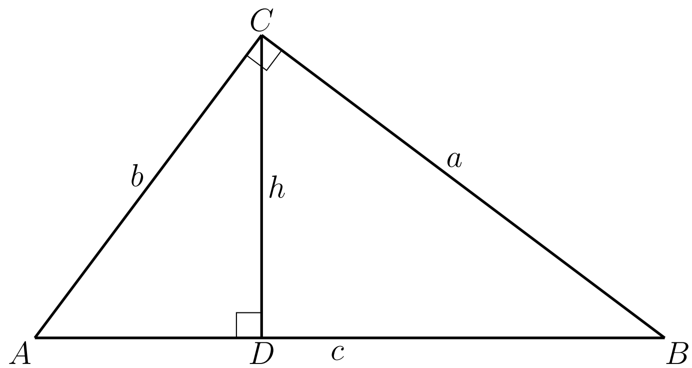
.

Proofs of these facts will not be included in the solution, but they can be easily found online. Better yet, try to prove them for yourself!

Problem of the Month

Solution to Problem 3: December 2021

(a) Suppose the altitude from C intersects side AB at D.



Because $\triangle ADC$ and $\triangle ACB$ are both right-angled and have $\angle A$ in common, we have that $\triangle ADC$ is similar to $\triangle ACB$. This means $\dfrac{DC}{AC} = \dfrac{CB}{AB}$ or $\dfrac{h}{b} = \dfrac{a}{c}$.

Rearranging this equation, we have

$$\frac{1}{h} = \frac{c}{ab}$$

Squaring both sides and using $c^2 = a^2 + b^2$, we have

$$\begin{aligned}\frac{1}{h^2} &= \frac{c^2}{a^2b^2}\\ &= \frac{a^2+b^2}{a^2b^2}\\ &= \frac{a^2}{a^2b^2} + \frac{b^2}{a^2b^2}\\ &= \frac{1}{b^2} + \frac{1}{a^2}\end{aligned}$$

(b) We will assume that $\angle C = 90^\circ$. The argument is similar if $\angle A = 90^\circ$ or $\angle B = 90^\circ$.

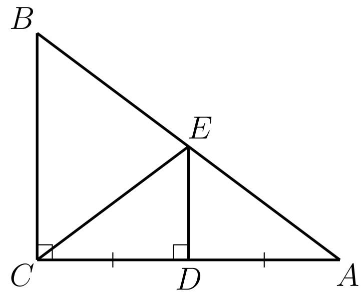
Since $\triangle ABC$ is right-angled at C, we have that $\cos\angle C = \cos 90^\circ = 0$. Therefore,

$$\begin{aligned}\cos^2\angle A + \cos^2\angle B + \cos^2\angle C &= \left(\frac{b}{c}\right)^2 + \left(\frac{a}{c}\right)^2 + 0^2\\ &= \frac{a^2+b^2}{c^2}\\ &= \frac{c^2}{c^2}\\ &= 1\end{aligned}$$

(c) As with part (b), we will assume that the right angle occurs at C since the argument is similar if it is at A or B.

We will first show that the length of the hypotenuse is $2R$ by showing that the centre of the circumcircle is the midpoint of the hypotenuse.

Assume that D is the midpoint of AC. Let E be the point where the perpendicular bisector of AC intersects the hypotenuse of $\triangle ABC$. As well, connect E to C.

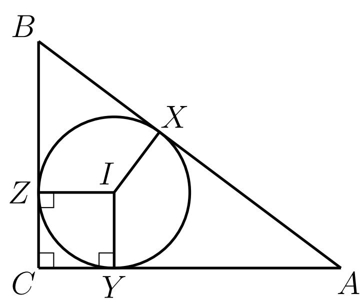


Since they share $\angle A$ and both have a right angle, $\triangle ABC$ and $\triangle AED$ are similar. Since $CD = AD$, we have that $\dfrac{AD}{AC} = \dfrac{1}{2}$, so $\dfrac{AE}{AB} = \dfrac{1}{2}$ because $\triangle ABC$ is similar to $\triangle AED$. Rearranging, we have $AB = 2AE$, which implies that E is the midpoint of AB.

Since $CD = AD$, $\angle CDE = \angle ADE = 90°$, and $\triangle ADE$ and $\triangle CDE$ share side DE, we have that $\triangle ADE$ is congruent to $\triangle CDE$ by side-angle-side congruence. This means $CE = AE$. We now have that E is the midpoint of the hypotenuse of $\triangle ABC$ which means that $AE = BE$. Since $CE = AE$ as well, we have shown that the midpoint of the hypotenuse is equidistant from the three vertices of $\triangle ABC$.

If we draw a circle centred at E with radius equal to AE, it will pass through all three vertices of the triangle. The circumcircle always exists and is the only circle with this property, so this circle is in fact the circumcircle, which means $AE = BE = CE = R$. It follows that the length of the hypotenuse in a right-angled triangle is equal to $2R$.

The next image depicts a right-angled triangle with its incircle, having centre I.



A tangent to a circle is perpendicular to the radius connecting the centre to the point of tangency. Therefore, $\angle IYC = \angle IZC = 90°$. Since $IYCZ$ is a quadrilateral with three right angles, its fourth angle must also be right, so $IYCZ$ is a rectangle. As well, $IZ = IY = r$, which means $IYCZ$ is a square with side length r. Thus, $CZ = CY = r$.

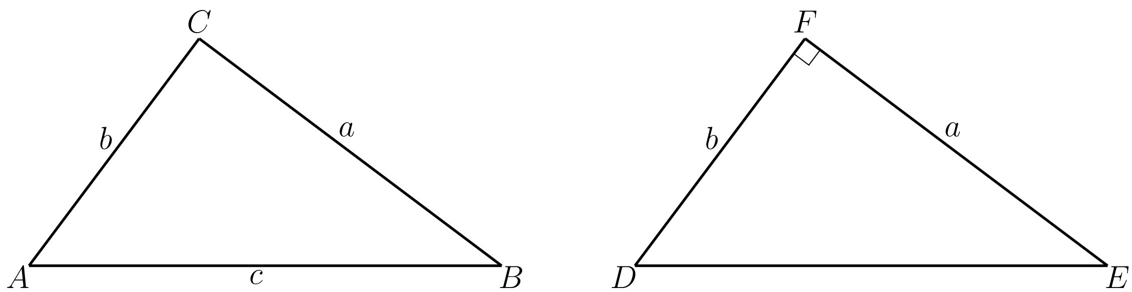
In general, if two tangents to the same circle intersect at some point outside the circle, then the distances from that point to each of the points of tangency are equal. In our case, this implies $BZ = BX$ and $AX = AY$. Using that $BZ = BX$, $AX = AY$, and

$CZ = CY = r$, we get that the perimeter of $\triangle ABC$ is

$$\begin{aligned} AX + AY + BX + BZ + CY + CZ &= AX + AX + BX + BX + r + r \\ &= 2(AX + BX + r) \\ &= 2(AB + r). \end{aligned}$$

Finally, since we also know that $AB = 2R$, we get that the perimeter of $\triangle ABC$ is $2(2R+r)$. Therefore, $s = r + 2R$.

(d) Assume that $a^2 + b^2 = c^2$ and then construct $\triangle DEF$ so that $DF = b$, $EF = a$, and $\angle DFE = 90°$.



By the Pythagorean theorem applied to $\triangle DEF$ (which is right-angled by construction), we have that $a^2 + b^2 = DE^2$. Since $a^2 + b^2 = c^2$, we have $c^2 = DE^2$. Since both c and DE are positive, it follows that $DE = c$. This means $\triangle ABC$ and $\triangle DEF$ are congruent by side-side-side congruence. Therefore, $\angle ACB = \angle DFE = 90°$, so $\triangle ABC$ is right-angled at C.

(e) In this solution, we will use the following identities that hold for all angles θ, x, and y.

$$2\cos^2\theta - 1 = \cos(2\theta) \tag{1}$$
$$\cos(360° - \theta) = \cos\theta \tag{2}$$
$$2\cos(x+y)\cos(x-y) = \cos(2x) + \cos(2y) \tag{3}$$
$$\cos(x+y) + \cos(x-y) = 2\cos x\cos y \tag{4}$$

We will begin with the assumption that $\cos^2\angle A + \cos^2\angle B + \cos^2\angle C = 1$ and deduce several equivalent identities. In an effort to declutter the calculation below, we will drop the "$\angle$" from $\angle A$, $\angle B$, and $\angle C$ and denote them by A, B, and C, respectively. In the calculation that follows, we will use the identities above by referring to their label of (1), (2), (3), or (4). The line labelled by (5) is using the fact that $A + B + C = 180°$.

$$\begin{aligned}
\cos^2 A + \cos^2 B + \cos^2 C &= 1 \\
\frac{\cos 2A + 1}{2} + \frac{\cos 2B + 1}{2} + \frac{\cos 2C + 1}{2} &= 1 \qquad (1) \\
(\cos 2A + 1) + (\cos 2B + 1) + (\cos 2C + 1) &= 2 \\
\cos 2A + \cos 2B + \cos 2C &= -1 \\
\cos 2A + \cos 2B + \cos(2(180° - A - B)) &= -1 \qquad (5) \\
\cos 2A + \cos 2B + \cos(2A + 2B) &= -1 \qquad (2) \\
2\cos(A + B)\cos(A - B) + \cos\big(2(A + B)\big) &= -1 \qquad (3) \\
2\cos(A + B)\cos(A - B) + 2\cos^2(A + B) - 1 &= -1 \qquad (1) \\
2\cos(A + B)\cos(A - B) + 2\cos^2(A + B) &= 0 \\
2\cos(A + B)\big(\cos(A - B) + \cos(A + B)\big) &= 0 \\
4\cos(A + B)\cos A\cos B &= 0 \qquad (4)
\end{aligned}$$

This means that either $\cos \angle A = 0$, $\cos \angle B = 0$, or $\cos(\angle A + \angle B) = 0$. If $\cos \angle A = 0$, then $\angle A = 90°$. If $\cos \angle B = 0$, then $\angle B = 90°$. If $\cos(\angle A + \angle B) = 0$, then $\angle A + \angle B = 90°$, which implies $\angle C = 180° - \angle A - \angle B = 90°$. In all three cases, $\triangle ABC$ has a right angle. Note that $\angle A$, $\angle B$, $\angle C$, and $\angle A + \angle B$ all measure between $0°$ and $180°$, so a cosine of 0 does imply an angle of $90°$.

(f) As mentioned in the hint, the area of $\triangle ABC$ is equal to rs as well as $\frac{abc}{4R}$. Equating these two expressions gives $rs = \frac{abc}{4R}$ which can be rearranged to get

$$4rR = \frac{abc}{s} \qquad (*)$$

By Heron's formula, the area of the triangle is also equal to $\sqrt{s(s-a)(s-b)(s-c)}$. This implies $rs = \sqrt{s(s-a)(s-b)(s-c)}$, so we can square both sides and solve to get

$$r^2 = \frac{(s-a)(s-b)(s-c)}{s} \qquad (**)$$

Starting with the equation, $s = r + 2R$, we can square both sides to get $s^2 = r^2 + 4rR + 4R^2$. We will now solve for $8R^2$ in terms of a, b, and c using this equation, some algebraic manipulation, as well as $(*)$ and $(**)$ above.

$$\begin{aligned}
4R^2 &= s^2 - r^2 - 4rR \\
4R^2 &= s^2 - \frac{(s-a)(s-b)(s-c)}{s} - \frac{abc}{s} && (*), (**) \\
8R^2 &= 2s^2 - \frac{2(s-a)(s-b)(s-c)}{s} - \frac{2abc}{s} \\
&= \frac{1}{s}\big(2s^3 - 2(s-a)(s-b)(s-c) - 2abc\big) \\
&= \frac{1}{s}\big(2s^3 - 2(s^3 - s^2(a+b+c) + s(ab+ac+bc) - abc) - 2abc\big) \\
&= \frac{1}{s}\big(2s^2(a+b+c) - 2s(ab+ac+bc)\big) \\
&= 2s(a+b+c) - 2(ab+ac+bc)
\end{aligned}$$

Now note that $2s = a+b+c$, so in fact

$$\begin{aligned}
8R^2 &= (a+b+c)^2 - 2(ab+ac+bc) \\
&= a^2 + b^2 + c^2 + 2(ab+ac+bc) - 2(ab+ac+bc) \\
&= a^2 + b^2 + c^2.
\end{aligned}$$

In the hint, the *Extended Law of Sines* was given and says that

$$\frac{a}{\sin\angle A} = \frac{b}{\sin\angle B} = \frac{c}{\sin\angle C} = 2R$$

for any triangle. This implies the following three equations

$$\frac{a^2}{4R^2} = \sin^2\angle A \qquad \frac{b^2}{4R^2} = \sin^2\angle B \qquad \frac{c^2}{4R^2} = \cos^2\angle C.$$

Dividing $8R^2 = a^2 + b^2 + c^2$ by $4R^2$, we get

$$\begin{aligned}
2 &= \frac{a^2}{4R^2} + \frac{b^2}{4R^2} + \frac{c^2}{4R^2} \\
&= \sin^2\angle A + \sin^2\angle B + \sin^2\angle C \\
&= (1 - \cos^2\angle A) + (1 - \cos^2\angle B) + (1 - \cos^2\angle C)
\end{aligned}$$

which implies $2 = 3 - (\cos^2\angle A + \cos^2\angle B + \cos^2\angle C)$. This equation can be rearranged to get $\cos^2\angle A + \cos^2\angle B + \cos^2\angle C = 1$.

We have now assumed that $s = r + 2R$ and deduced that $\cos^2\angle A + \cos^2\angle B + \cos^2\angle C = 1$. By part (e), $\triangle ABC$ must be right-angled.

Problem of the Month

Problem 4: January 2022

Problem

The goal of this problem is to work through some techniques that can sometimes help find the roots of polynomials. The statements of some parts of this problem refer to *repeated roots*, which we will now define. Suppose r is a root of the polynomial $p(x)$, that is, $p(r) = 0$. You may already know that if $p(r) = 0$, then $(x - r)$ divides evenly into $p(x)$. We say that r is a repeated root of $p(x)$ if $(x - r)^2$ divides evenly into $p(x)$. For example, 1 is a repeated root of $x^2 - 2x + 1$ because $x^2 - 2x + 1 = (x - 1)^2$, and 2 is a repeated root of $x^4 - 5x^3 + 6x^2 + 4x - 8$ since $x^4 - 5x^3 + 6x^2 + 4x - 8 = (x - 2)^2(x^2 - x - 2)$.

(a) The polynomials $p(x) = 2x^2 - 1275x + 194292$ and $q(x) = x^2 - 635x + 96516$ have a root in common. Determine both roots of both polynomials without using the quadratic formula.

(b) Let $p(x) = x^3 + ax^2 + bx + c$ be a polynomial with a root r. Show that r is a repeated root of $p(x)$ if and only if r is a root of the polynomial $q(x) = 3x^2 + 2ax + b$.

You may recognize $q(x)$ as the *derivative* of $p(x)$. If you are familiar with derivatives, you might want to try to generalize this part.

(c) Suppose $p(x) = x^3 + bx + c$ has roots u, v, and w (which may not all be different). Express the quantity $(u - v)^2(v - w)^2(w - u)^2$ in terms of b and c. This quantity is known as the discriminant of $p(x)$, and this exercise shows that its value can be determined from the coefficients without knowing the roots. Explain how, without knowing any of the roots, it is possible to determine if a cubic of the form $x^3 + bx + c$ has a repeated root.

(d) Consider the polynomial $p(x) = x^3 + ax^2 + bx + c$. Show that the coefficient of x^2 in the polynomial $q(x) = p\left(x - \frac{a}{3}\right)$ is equal to 0. Explain how the roots of $p(x)$ can be found easily if the roots of $q(x)$ are known.

(e) Find all roots of the polynomial $p(x) = x^3 - 135x^2 + 5832x - 81648$.

Hint

In several parts of this problem, it might be useful to review how the roots of a polynomial are related to its coefficients. In particular, how the coefficients arise as combinations of the roots.

(a) Suppose r is a common root of $p(x)$ and $q(x)$. Can you use what you know about $p(r)$ and $2q(r)$ to deduce the value of r without using the quadratic formula?

(b) Assuming there is a repeated root, use the relationships between the roots and the coefficients to compute $q(r)$ where r is the repeated root. For the other direction, you can try something similar or attempt to divide $p(x)$ by $(x-r)^2$, where r is the common root of $p(x)$ and $q(x)$.

(c) Try to show that $(u-v)^2 = -ab - 3uv$, and do not forget what it means for u, v, and w to be roots of $p(x)$!

(d) Think about how the graphs of the polynomials $p(x)$ and $q(x)$ compare to one another.

(e) First use part (d) to find a cubic that has its coefficient of x^2 equal to 0, but whose roots are a translation of those of $p(x)$. Next, use part (c) to show that this new polynomial, and hence, the original one, has a repeated root. Next, use part (b) to find a quadratic that shares that repeated root, and finally use the idea from part (a) to find the repeated root.

Problem of the Month

Solution to Problem 4: January 2022

Before starting the solution, we include a brief discussion on how the roots of polynomials are related to their coefficients.

Consider the quadratic polynomial $ax^2 + bx + c$ where a, b, and c are real numbers with $a \neq 0$. If u and v are the roots of the polynomial, then $u + v = -\frac{b}{a}$ and $uv = \frac{c}{a}$. This is because $ax^2 + bx + c$ has the same roots as $x^2 + \frac{b}{a}x + \frac{c}{a}$, and since u and v are the roots, we must have

$$x^2 + \frac{b}{a}x + \frac{c}{a} = (x-u)(x-v) = x^2 - (u+v)x + uv.$$

By similar reasoning, if $x^3 + ax^2 + bx + c$ has roots u, v, and w, then it must factor as

$$x^3 + ax^2 + bx + c = (x-u)(x-v)(x-w)$$

and expanding, we find that $-a = u + v + w$, $b = uv + vw + wu$, and $c = -uvw$. These are often known as (some of) *Vieta's formulas* and they are very useful when studying the roots of polynomials.

(a) Let r be the common root. Then $2r^2 - 1275r + 194292 = 0$ and $r^2 - 635r + 96516 = 0$. Doubling the second equation gives $2r^2 - 1270r + 193032 = 0$. Subtracting the equation $2r^2 - 1275r + 194292 = 0$ from the equation $2r^2 - 1270r + 193032 = 0$ gives

$$\begin{aligned} 0 &= 0 - 0 \\ &= (2r^2 - 1270r + 193032) - (2r^2 - 1275r + 194292) \\ &= 5r + 193032 - 194292 \\ &= 5r - 1260 \end{aligned}$$

which implies $5r = 1260$. Solving for r gives $r = 252$. From the discussion before the solution, the sum of the roots of $x^2 - 635x + 96516$ is 635, so the other root is $635 - 252 = 383$.

Similarly, the sum of the roots of $2x^2 - 1275x + 194292$ is $\frac{1275}{2}$ and one of the roots is 252, so the other is $\frac{1275}{2} - \frac{504}{2} = \frac{771}{2}$.

(b) Suppose r is a root of $p(x)$.

We will first assume r is a repeated root of $p(x)$ and deduce that r is a root of $q(x)$. Since r is a repeated root of $p(x)$, $(x-r)^2$ divides evenly into $p(x)$. This means there must be some other root t such that $(x-r)^2(x-t) = x^3 + ax^2 + bx + c$. From the formulas before the solution, we have that $a = -2r - t$, $b = r^2 + 2rt$, and $c = -r^2t$. Thus

$$\begin{aligned} q(r) &= 3r^2 + 2ar + b \\ &= 3r^2 + 2(-2r - t)r + r^2 + 2rt \\ &= 3r^2 - 4r^2 - 2rt + r^2 + 2rt \\ &= 0 \end{aligned}$$

and so r is a root of $q(x)$. Thus, if r is a repeated root of $p(x)$, then r is a root of $q(x)$.

Now we will assume r is a root of $q(x)$ and deduce that $(x-r)^2$ divides evenly into $p(x)$. Since r is a root of $q(x)$, we have $3r^2+2ar+b=0$, so

$$b = -(3r^2+2ar) \tag{1}$$

As well, we are still assuming that $p(r)=0$, which means $r^3+ar^2+br+c=0$ or $c=-r^3-ar^2-br$. Multiplying $3r^2+2ar+b=0$ through by r and rearranging gives $-br=3r^3+2ar^2$, and substituting this into $c=-r^3-ar^2-br$ gives

$$\begin{aligned} c &= -r^3-ar^2+(3r^3+2ar^2) \\ &= 2r^3+ar^2 \end{aligned} \tag{2}$$

Using these equations, we have

$$\begin{aligned} (x-r)^2(x+a+2r) &= (x^2-2rx+r^2)(x+a+2r) \\ &= x^3+(a+2r-2r)x^2+(r^2-2ar-4r^2)x+(ar^2+2r^3) \\ &= x^3+ax^2-(3r^2+2ar)x+(2r^3+ar^2) \\ &= x^3+ax^2+bx+c && \text{(1) and (2)} \\ &= p(x) \end{aligned}$$

This shows that if r is a root of $p(x)$ and $q(x)$, then $(x-r)^2=x^2-2rx+r^2$ divides evenly into $p(x)$, which means r is a repeated root of $p(x)$.

Here is another solution that uses the following fact: If

$$p(x) = x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1x + a_0$$

is a polynomial of degree n with real coefficients, then there are *complex* numbers $r_1, \ldots, r_n$ such that $p(x)=(x-r_1)(x-r_2)\cdots(x-r_n)$. This is a formulation of the famous *Fundamental Theorem of Algebra*. Proving this theorem is far beyond the scope of this activity, but to readers who recognize $q(x)$ as the *derivative* of $p(x)$ and who know the *product rule*, it offers a more enlightening proof of the fact in this problem.

Suppose $p(x)=(x-r_1)(x-r_2)(x-r_3)$. Observe that $-(r_1+r_2+r_3)=a$ as well as $r_1r_2+r_1r_3+r_2r_3=b$ by Vieta's formulas, which will still hold for complex numbers. Interestingly, even though the roots r_1, r_2, and r_3 may not be real, their sum and the sum of their pairwise products will be real. For reasons that may seem utterly mysterious unless you have seen the product rule, we have the following:

$$\begin{aligned} q(x) &= 3x^2+2ax^2+b \\ &= 3x^2-2(r_1+r_2+r_3)x+r_1r_2+r_1r_3+r_2r_3 \\ &= 3x^2-\big((r_1+r_2)+(r_2+r_3)+(r_3+r_1)\big)x+r_1r_2+r_2r_3+r_3r_1 \\ &= \big(x^2-(r_1+r_2)x+r_1r_2\big)+\big(x^2-(r_2+r_3)x+r_2r_3\big)+\big(x^2-(r_3+r_1)x+r_3r_1\big) \\ &= (x-r_1)(x-r_2)+(x-r_2)(x-r_3)+(x-r_3)(x-r_1) \end{aligned}$$

and from this, the solution falls out almost immediately. The roots of $p(x)$ are r_1, r_2, and r_3. Can you see why this form of $q(x)$ implies that $q(x)$ and $p(x)$ share a root exactly when at least two of r_1, r_2, and r_3 are the same?

(c) Set $D = (u-v)^2(v-w)^2(w-u)^2$. Since u, v, and w are the roots of x^2+bx+c (note that $a=0$), Vieta's formulas imply the equations

$$u+v+w = 0 \tag{1}$$
$$uv+vw+wu = b \tag{2}$$
$$uvw = -c \tag{3}$$

Adding $3uv$ to both sides of (2) and factoring gives $b+3uv = 4uv+w(u+v)$. From (1), we also have that $w=-u-v$. Substituting this into $b+3uv = 4uv+w(u+v)$ gives

$$\begin{aligned} b+3uv &= 4uv+w(u+v) \\ &= 4uv-(u+v)(u+v) \\ &= 4uv-u^2-2uv-v^2 \\ &= -(u^2-2uv+v^2) \\ &= -(u-v)^2 \end{aligned}$$

from which it follows that $-b-3uv = (u-v)^2$. A very similar calculation shows that $-b-3vw = (v-w)^2$ and $-b-3wu = (w-u)^2$. Using these three equations, we have

$$\begin{aligned} D &= (u-v)^2(v-w)^2(w-u)^2 \\ &= (-3uv-b)(-3vw-b)(-3wu-b) \\ &= -(b+3uv)(b+3vw)(b+3wu) \\ &= -\left(b^3+3b^2(uv+vw+wu)+9b(u^2vw+uv^2w+uvw^2)+27u^2v^2w^2\right) \\ &= -\left(b^3+3b^2(b)+9buvw(u+v+w)+27(uvw)^2\right) && (2) \\ &= -\left(4b^3+9buvw(0)+27(-c)^2\right) && (1) \text{ and } (3) \\ &= -4b^3-27c^2 \end{aligned}$$

(d) If we were to expand

$$p\left(x-\frac{a}{3}\right) = \left(x-\frac{a}{3}\right)^3 + a\left(x-\frac{a}{3}\right)^2 + b\left(x-\frac{a}{3}\right) + c,$$

the x^2 term must come from $\left(x-\frac{a}{3}\right)^3$ and $a\left(x-\frac{a}{3}\right)^2$. The x^2 term coming from $\left(x-\frac{a}{3}\right)^3$ is $-3\left(\frac{a}{3}x^2\right) = -ax^2$ and the x^2 term coming from $a\left(x-\frac{a}{3}\right)^2$ is ax^2. Their sum is 0, so the coefficient of x^2 in the polynomial $p\left(x-\frac{a}{3}\right)$ must be 0.

Suppose r is a root of $q(x)$. Then $q(r)=0$, which means $p\left(r-\frac{a}{3}\right)=0$. This means $r-\frac{a}{3}$ is a root of $p(x)$. As well, if t is a root of $p(x)$, then $t+\frac{a}{3}$ is a root of $q(x)$ since

$$q\left(t+\frac{a}{3}\right) = p\left(t+\frac{a}{3}-\frac{a}{3}\right) = p(t) = 0$$

Thus, the roots of $p(x)$ are exactly the roots of $q(x)$ with $\frac{a}{3}$ subtracted from them. If you have studied horizontal translations of functions, you may notice that $q(x)$ is a horizontal translation of $p(x)$ to the right by $\frac{a}{3}$, which gives a geometric explanation of why the roots of one polynomial are just the roots of the other after a horizontal shift.

This may not seem like an important observation, but it is of both algebraic and historical significance. In a theoretical sense, it tells us that if we can understand the roots of cubics without a quadratic term, then we can understand the roots of *every* cubic. Just like there is a "quadratic formula" that will produce the exact roots of any quadratic polynomial in terms of its coefficients, there is a "cubic formula". The general cubic formula is quite complicated, but its specialized version in the case where the quadratic term is missing is much simpler.

In fact, this specialized formula was discovered many years before the general formula was, which may seem surprising since, as observed in this question, the only barrier between the two appears to be a simple translation. Indeed, the observation in this problem is the one found by Cardano that finally generalized the specialized cubic formula to handle more general cubics. By today's standards, the observation is as simple as it seems. However, one must keep in mind that the techniques of modern algebra were not available in the 16^{th} century. What seems today like a quick algebraic substitution and manipulation was a geometric observation that, by today's standards, would seem contrived and unnecessary.

Cardano's formula will find the roots of any cubic in terms of its coefficients. However, because of the need to extract roots of negative and sometimes complex numbers, mathematicians of the day, in a sense, did not know how to use the formula to its full potential. You might wish to do some research on the history of the cubic formula.

(e) We will consider the polynomial

$$q(x) = p\left(x - \frac{a}{3}\right) = p\left(x - \frac{-135}{3}\right) = p(x+45)$$

We will not show the calculations, but it it can be checked that

$$(x+45)^3 - 135(x+45)^2 + 5832(x+45) - 81648 = x^3 - 243x - 1458$$

and so we will find the roots of $x^3 - 243x - 1458$ and then subtract -45 from them to get the roots of the given polynomial.

To find the roots of this simpler cubic, we first calculate its discriminant. Here, $b = -243$ and $c = -1458$. So

$$-4b^3 - 27c^2 = -4(-243)^3 - 27(-1458)^2 = 57395628 - 57395628 = 0$$

and by part (c), the polynomial $x^3 - 243x - 1458$ must have a repeated root.

By part (b), that repeated root, r, must also be a root of the quadratic $3x^2 - 243$. If $3r^2 - 243 = 0$, then $r^2 - 81 = 0$, which means $r = \pm 9$. [At this point, we could test both possibilities to see which is the root of the cubic of interest, but we will go through the calculation to demonstrate how this sort of technique works more generally.] Since $r^2 - 81 = 0$, $r^3 - 81r = 0$. We also have that $r^3 - 243r - 1458 = 0$. Subtracting this equation from $r^3 - 81r = 0$ gives $162r + 1458 = 0$, so $r = -\frac{1458}{162} = -9$.

We now have that -9 is a root of $x^3 - 243x - 1458$, and in fact, it must be a repeated root. This means we can factor $(x+9)^2$ out of the polynomial. Indeed, a bit of polynomial division reveals that

$$x^3 - 243x - 1458 = (x^2 + 18x + 81)(x - 18)$$

This means the final root of $x^3 - 243x - 1458$ is 18, so the roots of the original polynomial are $-9 + 45 = 36$, which is a repeated root, and $18 + 45 = 63$. You can check that the polynomial in the problem factors as

$$(x - 36)(x - 36)(x - 63) = x^3 - 135x^2 + 5832x - 81648$$

While it may have been just as well to use some other technique like the rational roots theorem, the point being made here is that there are potentially a few simplifying tricks when it comes to finding roots of polynomials. The first is the substitution at the beginning to change our focus to a polynomial with one of its coefficients equal to 0 (can you see how this is done for polynomials of higher degree?). The next is to check the discriminant to detect repeated roots. If there is a repeated root, then, in principle, it is easier to find than a non-repeated root because of the trick involving the *derivative* (this also generalizes to higher degree polynomials). There are many algorithms known for factoring polynomials, especially those with integer coefficients. They are surprisingly efficient because of tricks like these.

Problem of the Month

Problem 5: February 2022

Problem

In each part of this problem, there is a hallway containing of K doors numbered consecutively from 1 to K that are all initially closed. To *toggle* a door means to open it if it is closed and to close it if it is open. We will also use the notation that for a positive integer n, $\tau(n)$ is equal to the number of positive integer factors of n. For example, $\tau(1) = 1$ since 1 has exactly one positive factor and for a prime number p, we always have $\tau(p) = 2$ since prime numbers have exactly two positive factors. For another example $\tau(10) = 4$ since it has four positive integer factors, 1, 2, 5, and 10.

(a) In this part, $K = 100$. 100 "steps" are performed as follows:

- In step 1, every door that is numbered with a multiple of 1 is toggled.
- In step 2, every door that is numbered with a multiple of 2 is toggled.
- In step 3, every door that is numbered with a multiple of 3 is toggled.

In step n, every door that is numbered with a multiple of n is toggled. After all 100 steps are performed, which doors are open?

(b) In this part, $K = 100$. As with part (a), 100 steps are performed with one step for each integer n from 1 through K. This time, in step n, each door that is numbered with a multiple of n is toggled n times. For example, in step 5, each door that is numbered with a multiple of 5 is to be toggled 5 times. After all 100 steps are performed, which doors are open?

(c) In this part, $K = 2^9 \times 3^4 \times 5^{13} \times 7^{12}$. As with parts (a) and (b), a step is performed for each positive integer n from 1 through K. In step n, every door that is numbered by a multiple of n is toggled $\tau(n)$ times. For example, in step 5, every door that is numbered by a multiple of 5 is toggled $\tau(5) = 2$ times.

After all K steps are performed, is the door numbered with K open or closed?

Hint

(a) Solving this problem will come down to counting how many positive factors each door number has. It might be useful to determine the number of factors of the first few positive integers and see if you notice a pattern. Try determining the number of factors of the integers from 1 through 20.

(b) In this part, whether a door is open or closed only depends on how many odd positive factors it has.

(c) The following general fact may be useful in this or the other parts: Every integer n can be expressed in the form $n = p_1^{e_1}p_2^{e_2}\cdots p_k^{e_k}$ where the p_i are distinct prime numbers and the e_i are positive integers. An integer d is a positive factor of n if and only if it can be expressed in the form $d = p_1^{f_1}p_2^{f_2}\cdots p_k^{f_k}$ where $0 \leq f_i \leq e_i$ for each i. This means that $\tau(n) = (e_1+1)(e_2+1)\cdots(e_k+1)$. Can you see why?

Problem of the Month

Solution to Problem 5: February 2022

(a) After the first 100 steps are performed, a door will be open if it was toggled an odd number of times and closed if it was toggled an even number of times. Therefore, to determine whether a door is open or closed after all 100 steps, we need to determine how many times it has been toggled.

To gain intuition, we will look at a few particular doors. Door 4 is toggled in Step 1, Step 2, Step 4, and no other steps. This is because 1, 2, and 4 are the only positive factors of 4. This means Door 4 will be open after all 100 steps.

Door 10 is toggled in Step 1, Step 2, Step 5, and Step 10. No other positive integers are factors of 10, so there are no other steps in which Door 10 is toggled. Therefore, Door 10 is toggled four times, so it is closed after all 100 steps.

Recall from the problem statement that $\tau(n)$ is equal to the number of positive factors of n. Door n is toggled in Step k if and only if n is a multiple of k. Put differently Door n is toggled in Step k if and only if k is a factor of n. Thus, Door n is toggled exactly $\tau(n)$ times. Note that since $n \leq 100$ and 100 steps are performed, Step k will occur for every factor k of n.

Combining this observation with the earlier discussion, Door n will be open after all 100 steps if and only if $\tau(n)$ is odd. We will now argue that $\tau(n)$ is odd if and only if n is a perfect square.

Fix a positive integer n and suppose m is a factor of n with $1 \leq m < \sqrt{n}$. Then $\dfrac{n}{m}$ is a factor of n with the property that $\sqrt{n} < \dfrac{n}{m} \leq n$. Similarly, if m is a factor of n with the property that $\sqrt{n} < m \leq n$, then $\dfrac{n}{m}$ is a factor of n with the property that $1 \leq \dfrac{n}{m} < \sqrt{n}$. Notice that $m \times \dfrac{n}{m} = n$, so either way, if m is a positive factor of n that is not equal to $\sqrt{n}$, then $\left(m, \dfrac{n}{m}\right)$ is a factor pair for n with one factor less than $\sqrt{n}$ and the other greater than $\sqrt{n}$.

By the previous paragraph, for any positive integer n, there are an even number of positive factors of n that are different from $\sqrt{n}$. This means that the integer n has an odd number of positive factors if and only if $\sqrt{n}$ is a factor of n. If n is a perfect square, then $\sqrt{n}$ is an integer and $\sqrt{n}\sqrt{n} = n$, so $\sqrt{n}$ is a factor of n. If n is not a perfect square, then $\sqrt{n}$ is not an integer, so it cannot be a factor of n. Therefore, the integer n has an odd number of positive factors if and only if it is a perfect square. This means that Door n will be open after all 100 steps if and only if n is a perfect square.

(b) Similar to part (a), Door n gets toggled at Step d for every factor d of n. However, this time it gets toggled d times for each factor d of n. Therefore, the total number of times that a door gets toggled is equal to the sum of its positive factors.

Since a door is open after all 100 steps if and only if it has been toggled an odd number of times, we need to determine for which positive integers n the sum of the factors of n is odd.

We will consider two cases.

Case 1: Suppose n is odd. In this situation, every positive factor of n is odd. The sum of odd integers is odd if there is an odd number of them being added together and even otherwise. Thus, the positive factors of an odd integer have an odd sum if and only if there is an odd number of them. In other words, if n is odd, then Door n is open after all 100 steps if and only if $\tau(n)$ is odd. By part (a), Door n is open if and only if n is a perfect square.

Case 2: Suppose n is even. This means there is a positive integer k and an odd positive integer m such that $n = 2^k m$. The number of even factors will not affect whether the sum of the factors is odd or even since the sum of any number of even factors is always even. This means to determine if the sum of the positive factors is even or odd, we need only consider the odd factors. If there is an odd number of odd factors, then the sum of the positive factors will be odd. Otherwise, it will be even.

If d is an odd factor of $n = 2^k m$, then d must be a factor of m. Conversely, if d is a factor of m, then d is an odd factor of n. Thus, the number of odd factors of $n = 2^k m$ is equal to the number of odd factors of m. We are assuming that m is odd, so it has an odd number of positive factors if and only if it is a perfect square (by the previous case). Therefore, n has an odd number of odd factors if and only if m is a perfect square.

Putting the cases together, we have that every positive integer n can be written in the form $2^k m$ where k is a *non-negative* integer and m is an odd positive integer. Door n will be open after all 100 steps if and only if m is a perfect square.

Thus, Door n will be open after all 100 steps if and only if n is the product of a power of 2 and an odd perfect square. This description can be simplified even more. Suppose $n = 2^k m$ where k is non-negative and m is an odd perfect square. This means $m = r^2$ for some r. If k is even, then $n = 2^k m = 2^k r^2 = \left(2^{\frac{k}{2}} r\right)^2$, so n itself is a perfect square. If k is odd, then $k - 1$ is even, so $n = 2^k m = 2\left(2^{k-1} r^2\right) = 2\left(2^{\frac{k-1}{2}} r\right)^2$, so n is two times a perfect square.

We can now say that Door n will be open after all 100 steps if and only if n is a perfect square or n is two times a perfect square.

(c) For this part, we still need to identify whether Door n, after n steps, has been toggled an even or an odd number of times. It will be open if and only if it has been toggled an odd number of times.

In the first n steps, Door n gets toggled $\tau(d)$ times for every factor d of n. From part (a), we know that $\tau(d)$ is even unless d is a perfect square. By reasoning we have used earlier, this means that Door n will be open after n steps if and only if an odd number of perfect squares divide n.

Every positive integer n can be written uniquely in the form $p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$ where the p_i are distinct prime numbers and the e_i are positive integers. A positive integer d is a factor of n if and only if $d = p_1^{f_1} p_2^{f_2} \cdots p_k^{f_k}$ for some integers f_i with $0 \leq f_i \leq e_i$ for each i. For d to be a perfect square, each of the f_i must be even. To count the number of factors of

n that are perfect squares, we can count how many even integers there are from 0 to e_i inclusive for each i, then take the product of these values. This is because we obtain a perfect square factor for every choice of even integers f_i, and the choices are independent.

The product of positive integers is odd if and only if all of the integers being multiplied together are odd. Thus, for $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$ to have an odd number of perfect-square factors, it must be true that for each i there is an odd number of even integers between 0 and e_i inclusive. For a fixed i, we consider four cases. In each case, we will determine how many even positive integers there are between 0 to e_i inclusive.

Case 1: $e_i = 4r$ for some integer r. In this case, $0, 2, 4, \ldots, 4r-2, 4r$ are the even integers from 0 to e_i inclusive. There are $\frac{4r}{2} + 1 = 2r + 1$ of them, so in this case, there is an odd number of choices for f_i.

Case 2: $e_i = 4r + 1$ for some integer r. In this case, $0, 2, 4, \ldots, 4r - 2, 4r$ are the even integers from 0 to e_i inclusive. By the same calculation as Case 1 above, there is an odd number of choices for f_i.

Case 3: $e_i = 4r+2$ for some integer r. In this case, the even integers from 0 to e_i inclusive are $0, 2, 4, \ldots, 4r, 4r + 2$. There are $2r + 2$ integers in this list, so there is an even number of choices for f_i.

Case 4: $e_i = 4r+3$ for some integer r. In this case, the even integers from 0 to e_i inclusive are the same as those in Case 3, so there is again an even number of choices for f_i.

By the reasoning given before we considered the cases, we get that Door n will be open after n steps if and only if each e_i is either a multiple of 4 or one more than a multiple of 4.

For the given integer $K = 2^9 3^4 5^{13} 7^{12}$, each exponent has this property, so Door K will be open after K steps.

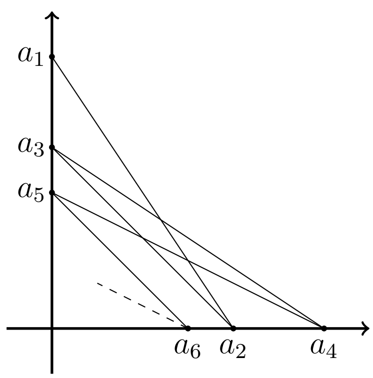
Problem of the Month
Problem 6: March 2022

Problem

In this problem, we will explore the following construction: Start with the positive real number $a_1 = 1$ and an infinite sequence $m_1, m_2, m_3, \ldots$ of negative slopes that are all distinct. For $n \geq 1$, we define a_{n+1} from a_n as follows.

- For odd n, a_{n+1} is the x-intercept of the line with slope m_n through $(0, a_n)$.
- For even n, a_{n+1} is the y-intercept of the line with slope m_n through $(a_n, 0)$.

The diagram below illustrates this. The line through $(0, a_1)$ and $(a_2, 0)$ has slope m_1, the line through $(a_2, 0)$ and $(0, a_3)$ has slope m_2, and so on.



(a) Suppose that $m_n = -\dfrac{1}{2^n}$ for all $n \geq 1$.

(i) Compute a_2, a_3, a_4, and a_5.

(ii) Find a general formula for a_n. You will likely need a separate formula for even n and odd n. Describe what happens to a_n as n gets large.

(b) Suppose that $m_n = -\dfrac{1}{2^{\frac{1}{2^n}+1}}$ for all n. [The exponent in the denominator is $\frac{1}{2^n} + 1$]

(i) Find a general formula for a_n.

(ii) Describe what happens to a_n as n gets large.

(c) Let u and v be arbitrary positive real numbers with $u \neq 1$. Give a sequence of slopes so that the sequence $a_1, a_3, a_5, a_7, \ldots$ approaches u and the sequence $a_2, a_4, a_6, a_8, \ldots$ approaches v. Remember that the sequence of slopes should not contain any repetitions.

(d) Suppose $m_n = -\dfrac{1}{n}$ for all $n \geq 1$.

(i) Find an integer n so that $a_n < \dfrac{1}{100}$.

(ii) Find an integer n so that $a_n > 100$.

Hint

Before attempting any of the problems, it might useful to show that a_{n+1} can be expressed in terms of a_n and m_n.

(b) In parts (i) and (ii), try computing the first few a_n and looking for a pattern. Do you notice a familiar type of series forming in the exponents?

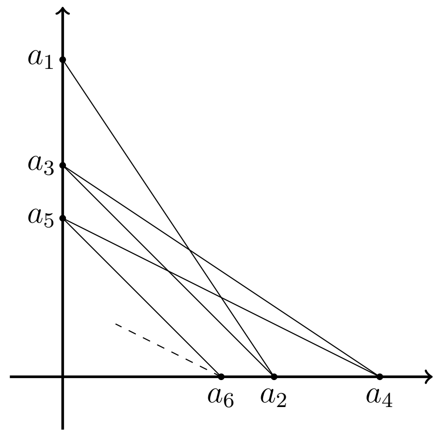
(b) If you are comfortable with logarithms, you might find that it simplifies some calculations to define $A_n = \log_2(a_n)$ and work with the A_n instead. If you can find a general formula for A_n, then you can find a general formula for a_n by using that $a_n = 2^{A_n}$.

(c) Use the idea from part (b) to construct the sequence of slopes. What happens when you change the $\frac{1}{2^n} + 1$ in the exponent to $\frac{1}{2^n} + c$ for some $c \neq 1$?

(d) To start, find a general formula for a_n. A separate formula for even n and odd n will probably be useful. For odd n, try to show that $(a_n)^2$ is less than $\frac{1}{n-1}$. Can you do something similar for even n?

Problem of the Month
Solution to Problem 6: March 2022

As suggested in the hint, we will first show how a_{n+1} can be obtained directly from a_n and m_n. We will consider the even and odd cases separately. Consider the diagram from the problem statement:



When n is odd, we have by definition that the slope of the line through $(0, a_n)$ and $(a_{n+1}, 0)$ is m_n. This means $m_n = \dfrac{a_n - 0}{0 - a_{n+1}} = \dfrac{a_n}{-a_{n+1}}$. Solving for a_{n+1} gives $a_{n+1} = -\dfrac{a_n}{m_n}$.

When n is even, the slope of the line through $(0, a_{n+1})$ and $(a_n, 0)$ has slope m_n. This means $m_n = \dfrac{a_{n+1} - 0}{0 - a_n} = \dfrac{a_{n+1}}{-a_n}$, which implies $a_{n+1} = -a_n m_n$.

Putting these cases together, we get that

$$a_{n+1} = \begin{cases} -\dfrac{a_n}{m_n} & \text{if } n \text{ is odd} \\ -a_n m_n & \text{if } n \text{ is even} \end{cases} \tag{1}$$

Note that since the m_n are negative, they are non-zero. As well, $a_1 = 1$ is nonzero, so every a_n is nonzero since each is obtained from the previous by either multiplying or dividing by a non-zero slope.

(a) (i) Using Equation (1) and that $a_1 = 1$, we get

$$a_2 = -\frac{a_1}{m_1} = -\frac{1}{-\frac{1}{2}} = 2 \qquad a_3 = -a_2 m_2 = -2\left(-\frac{1}{2^2}\right) = \frac{1}{2}$$

$$a_4 = -\frac{a_3}{m_3} = -\frac{\frac{1}{2}}{-\frac{1}{2^3}} = 4 \qquad a_5 = -a_4 m_4 = -4\left(-\frac{1}{2^4}\right) = \frac{1}{4}$$

(ii) Based on the calculations in (i), we might guess that the sequence $a_1, a_2, a_3, a_4, \ldots$ is

$$1, 2, \frac{1}{2}, 2^2, \frac{1}{2^2}, 2^3, \frac{1}{2^3}, 2^4, \frac{1}{2^4}, \ldots$$

and after a bit of thought, this can be expressed more precisely as

$$a_n = \begin{cases} \dfrac{1}{2^{\frac{n-1}{2}}} & \text{if } n \text{ is odd} \\ \\ 2^{\frac{n}{2}} & \text{if } n \text{ is even} \end{cases} \tag{2}$$

We will verify that Equation (2) holds using *mathematical induction*. Using the calculations in part (i), it can be checked that Equation (2) holds when $n = 1$, $n = 2$, $n = 3$, $n = 4$, and $n = 5$. We will now show that if Equation (2) is true for some positive integer k, then it is true for $k+1$. By the principle of mathematical induction, this will imply that it is true for all positive integers.

Suppose k is a positive integer and that Equation (2) is true for $n = k$. We will consider two cases:

Case 1: k is odd.
Since Equation (2) holds for $n = k$, we have that $a_k = \dfrac{1}{2^{\frac{k-1}{2}}}$. By Equation (1), we also have that $a_{k+1} = -\dfrac{a_k}{m_k}$. Using these equations and that $m_k = -\dfrac{1}{2^k}$, we can calculate

$$a_{k+1} = -\frac{a_k}{m_k} = -\frac{\frac{1}{2^{\frac{k-1}{2}}}}{-\frac{1}{2^k}} = \frac{2^k}{2^{\frac{k-1}{2}}} = 2^{k-\frac{k-1}{2}} = 2^{\frac{k+1}{2}}$$

and since k is odd, $k+1$ is even, so the above calculation shows that Equation (2) holds when $n = k+1$.

Case 2: k is even.
Since Equation (2) holds for $n = k$, we have that $a_k = 2^{\frac{k}{2}}$. By Equation (1), we also have that $a_{k+1} = -a_k m_k$. Similar to Case 1, we get

$$a_{k+1} = -a_k m_k = -2^{\frac{k}{2}}\left(-\frac{1}{2^k}\right) = 2^{\frac{k}{2}-k} = \frac{1}{2^{\frac{k}{2}}} = \frac{1}{2^{\frac{(k+1)-1}{2}}}$$

and since $k+1$ is odd, this shows that Equation (2) holds for $k+1$.

As mentioned above, this establishes that Equation (2) holds for all positive integers n.

As n gets large, the terms a_n are getting larger and larger without bound for even n and are approaching 0 for odd n.

(b) (i) In this part, we will make use of logarithms to simplify some of the calculations. In fact, you might want to go back and try part (a) again using logarithms since they are useful there too!

Specifically, we will define $A_n = \log_2(a_n)$ for each $n \geq 1$. You may wish to spend some time convincing yourself that a_n is always positive which implies that there is no issue with taking its logarithm.

With A_n defined, we will apply Equation (1) with logarithm rules to get related equations for A_n. When n is odd, we get

$$\begin{aligned}
A_{n+1} = \log_2(a_{n+1}) &= \log_2\left(-\frac{a_n}{m_n}\right) \\
&= \log_2(a_n) - \log_2(-m_n) \\
&= A_n - \log_2\left(\frac{1}{2^{\frac{1}{2^n}+1}}\right) \\
&= A_n + \log_2\left(2^{\frac{1}{2^n}+1}\right) \\
&= A_n + \frac{1}{2^n} + 1
\end{aligned}$$

When n is even, we get

$$\begin{aligned}
A_{n+1} = \log_2(a_{n+1}) &= \log_2\left(a_n(-m_n)\right) \\
&= \log_2(a_n) + \log_2\left(\frac{1}{2^{\frac{1}{2^n}+1}}\right) \\
&= A_n - \frac{1}{2^n} - 1
\end{aligned}$$

Keep in mind that m_n is negative, so $-m_n$ is positive. Putting these equations together, we get

$$A_{n+1} = \begin{cases} A_n + \frac{1}{2^n} + 1 & \text{if } n \text{ is odd} \\ \\ A_n - \frac{1}{2^n} - 1 & \text{if } n \text{ is even} \end{cases} \tag{3}$$

Now observe that $A_1 = \log_2(a_1) = \log_2(1) = 0$. From this observation and Equation (3) we get

$$\begin{aligned}
A_2 &= A_1 + \frac{1}{2} + 1 \\
&= \frac{1}{2} + 1
\end{aligned}
\qquad
\begin{aligned}
A_3 &= A_2 - \frac{1}{2^2} - 1 \\
&= \frac{1}{2} - \frac{1}{2^2}
\end{aligned}$$

$$\begin{aligned}
A_4 &= A_3 + \frac{1}{2^3} + 1 \\
&= \frac{1}{2} - \frac{1}{2^2} + \frac{1}{2^3} + 1
\end{aligned}
\qquad
\begin{aligned}
A_5 &= A_4 - \frac{1}{2^4} - 1 \\
&= \frac{1}{2} - \frac{1}{2^2} + \frac{1}{2^3} - \frac{1}{2^4}
\end{aligned}$$

Remark: At this point, especially if you are uncomfortable with logarithms, you may want to verify directly that $a_2 = 2^{\frac{1}{2}+1}$, $a_3 = 2^{\frac{1}{2}-\frac{1}{2^2}}$, $a_4 = 2^{\frac{1}{2}-\frac{1}{2^2}+\frac{1}{2^3}+1}$, and $a_5 = 2^{\frac{1}{2}-\frac{1}{2^2}+\frac{1}{2^3}-\frac{1}{2^4}}$.

From the emerging pattern, we guess that

$$A_{2r+1} = \frac{1}{2} - \frac{1}{2^2} + \frac{1}{2^3} - + \cdots + \frac{1}{2^{2r-1}} - \frac{1}{2^{2r}}$$

for all $r \geq 0$ and that

$$A_{2r} = 1 + \frac{1}{2} - \frac{1}{2^2} + \frac{1}{2^3} - + \cdots - \frac{1}{2^{2r-2}} + \frac{1}{2^{2r-1}}$$

for all $r \geq 1$.

Using a trick for finding the sum of a *geometric series*, if we fix $r \geq 1$ and set

$$X = \frac{1}{2} - \frac{1}{2^2} + \frac{1}{2^3} - + \cdots - \frac{1}{2^{2r-2}} + \frac{1}{2^{2r-1}}$$

then we have

$$2X = 1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + - \cdots - \frac{1}{2^{2r-3}} + \frac{1}{2^{2r-2}}.$$

When the two equations above are added, most of the terms cancel and we are left with $3X = 1 + \frac{1}{2^{2r-1}}$ or $X = \frac{1}{3}\left(1 + \frac{1}{2^{2r-1}}\right)$.

We now use this formula to refine our guesses to

$$\begin{aligned} A_{2r+1} &= X - \frac{1}{2^{2r}} \\ &= \frac{1}{3}\left(1 + \frac{1}{2^{2r-1}}\right) - \frac{1}{2^{2r}} \\ &= \frac{1}{3}\left(1 + \frac{2}{2^{2r}} - \frac{3}{2^{2r}}\right) \\ &= \frac{1}{3}\left(1 - \frac{1}{2^{2r}}\right) \end{aligned}$$

for $r \geq 0$, and similarly

$$\begin{aligned} A_{2r} &= 1 + X \\ &= 1 + \frac{1}{3}\left(1 + \frac{1}{2^{2r-1}}\right) \\ &= \frac{1}{3}\left(4 + \frac{1}{2^{2r-1}}\right) \end{aligned}$$

and these two guesses can be combined to get

$$A_n = \begin{cases} \frac{1}{3}\left(1 - \frac{1}{2^{n-1}}\right) & \text{if } n \text{ is odd} \\ \\ \frac{1}{3}\left(4 + \frac{1}{2^{n-1}}\right) & \text{if } n \text{ is even} \end{cases} \tag{4}$$

It is easily checked that Equation (4) holds for $n = 1$, $n = 2$, $n = 3$, and $n = 4$. As in part (a), we will use induction to prove that Equation (4) holds for all positive integers n.

Assume that Equation (4) holds for some positive integer k. If k is odd, Equation (4) means that $A_k = \frac{1}{3}\left(1 - \frac{1}{2^{k-1}}\right)$. Using Equation (3), we get

$$\begin{aligned} A_{k+1} &= A_k + \frac{1}{2^k} + 1 \\ &= \frac{1}{3}\left(1 - \frac{1}{2^{k-1}}\right) + \frac{1}{2^k} + 1 \\ &= \frac{1}{3}\left(1 - \frac{1}{2^{k-1}} + \frac{3}{2^k} + 3\right) \\ &= \frac{1}{3}\left(4 - \frac{2}{2^k} + \frac{3}{2^k}\right) \\ &= \frac{1}{3}\left(4 + \frac{1}{2^k}\right) \end{aligned}$$

which confirms that Equation (4) holds for the even integer $k+1$. If k is even, a similar calculation shows that Equation (4) holds for $k+1$. By mathematical induction, Equation (4) holds for all positive integers n.

Since $A_n = \log_2(a_n)$, we have $2^{A_n} = a_n$. Therefore,

$$a_n = \begin{cases} 2^{\frac{1}{3}\left(1 - \frac{1}{2^{n-1}}\right)} & \text{if } n \text{ is odd} \\ 2^{\frac{1}{3}\left(4 + \frac{1}{2^{n-1}}\right)} & \text{if } n \text{ is even} \end{cases} \tag{5}$$

(ii) For large values of n, the quantity $\dfrac{1}{2^{n-1}}$ gets very close to 0. Thus, for large odd values of n, a_n gets very close to $2^{\frac{1}{3}} = \sqrt[3]{2}$. For large even values of n, a_n gets very close to $2^{\frac{4}{3}} = 2\sqrt[3]{2}$.

Notice that $a_1, a_3, a_5, \ldots$ is a sequence of y-intercepts. As observed above, this sequence approaches the quantity $\sqrt[3]{2}$. Similarly, the sequence $a_2, a_4, a_6, \ldots$ is a sequence of x-intercepts and it approaches $2\sqrt[3]{2}$. The line through the points $(0, \sqrt[3]{2})$ and $(2\sqrt[3]{2}, 0)$ has slope $-\frac{1}{2}$. For large n, $\dfrac{1}{2^n}$ is close to 0, so m_n is close to $-\frac{1}{2^1} = -\frac{1}{2}$, which is exactly the slope computed in the previous sentence. You might want to think about why these values are the same.

(c) We will generalize the idea from part (b) above. For now, fix a positive number $b \neq 1$ and a real number c and set $m_n = -\dfrac{1}{b^{\frac{1}{2^n}+c}}$. By defining $A_n = \log_b(a_n)$, an almost identical calculation to the one in part (b) shows that

$$a_n = \begin{cases} b^{\frac{1}{3}\left(1 - \frac{1}{2^{n-1}}\right)} & \text{if } n \text{ is odd} \\ b^{\frac{1}{3}\left(3c + 1 + \frac{1}{2^{n-1}}\right)} & \text{if } n \text{ is even} \end{cases} \tag{6}$$

for every positive integer n.

As n goes to infinity, $\dfrac{1}{2^{n-1}}$ goes to 0, so the sequence $a_1, a_3, a_5, a_7, \ldots$ approaches $b^{\frac{1}{3}} = \sqrt[3]{b}$ and the sequence $a_2, a_4, a_6, a_8, \ldots$ approaches $b^{c+\frac{1}{3}}$. Thus, to answer this question, we can solve the system of equations

$$u = \sqrt[3]{b}$$
$$v = b^{c+\frac{1}{3}}$$

The first equation implies $b = u^3$. Notice that since $u \neq 1$, we indeed have $b \neq 1$, so the construction above will work. Substituting $b = u^3$ into the second equation gives $v = u^{3c+1}$. Taking $\log_u$ of both sides (which can be done since u and v are both positive) and solving gives $c = \frac{1}{3}\left(\log_u(v) - 1\right)$. Thus, we get the desired result by taking

$$m_n = -\frac{1}{b^{\frac{1}{2^n}+c}}$$

with $b = u^3$ and $c = \frac{1}{3}\left(\log_u(v) - 1\right)$.

(d) Using Equation (1), we can compute the following first few values of a_n.

$$\begin{aligned}
a_1 &= 1 & a_2 &= 1 & a_3 &= \frac{1}{2} \\
a_4 &= \frac{1 \times 3}{2} & a_5 &= \frac{1 \times 3}{2 \times 4} & a_6 &= \frac{1 \times 3 \times 5}{2 \times 4} \\
a_7 &= \frac{1 \times 3 \times 5}{2 \times 4 \times 6} & a_8 &= \frac{1 \times 3 \times 5 \times 7}{2 \times 4 \times 6} & a_9 &= \frac{1 \times 3 \times 5 \times 7}{2 \times 4 \times 6 \times 8}
\end{aligned}$$

(i) Consider the following:

$$\begin{aligned}
(a_5)^2 &= \frac{1 \times 3 \times 1 \times 3}{2 \times 4 \times 2 \times 4} \\
&= \frac{1 \times 3}{2^2} \times \frac{3}{4^2} \\
&= \frac{3}{4} \times \frac{3}{4^2} \\
&< \frac{3}{4^2}
\end{aligned}$$

We can do similar calculations with a_7 and a_9 to get

$$\begin{aligned}
(a_7)^2 &= \frac{1 \times 3 \times 5 \times 1 \times 3 \times 5}{2^2 \times 4^2 \times 6^2} \\
&= \frac{1 \times 3}{2^2} \times \frac{3 \times 5}{4^2} \times \frac{5}{6^2} \\
&= \frac{3}{4} \times \frac{15}{16} \times \frac{5}{6^2} \\
&< \frac{5}{6^2}
\end{aligned}
\qquad
\begin{aligned}
(a_9)^2 &= \frac{1 \times 3}{2^2} \times \frac{3 \times 5}{4^2} \times \frac{5 \times 7}{6^2} \times \frac{7}{8^2} \\
&= \frac{3}{4} \times \frac{15}{16} \times \frac{35}{36} \times \frac{7}{8^2} \\
&< \frac{7}{8^2}
\end{aligned}$$

Which shows that for $k = 2$, $k = 3$, and $k = 4$, we have

$$(a_{2k+1})^2 < \frac{2k-1}{(2k)^2} \tag{7}$$

By two applications of Equation (1), we get that $a_{2k+3} = a_{2k+1} \times \frac{2k+1}{2k+2}$ for every k. If we assume that Inequality (7) holds for some positive integer $k \geq 2$, we have

$$\begin{aligned}
(a_{2k+3})^2 &= (a_{2k+1})^2 \frac{(2k+1)^2}{(2k+2)^2} \\
&< \frac{2k-1}{(2k)^2} \times \frac{(2k+1)^2}{(2k+2)^2} && \text{(Inequality (7))} \\
&= \frac{(2k-1)(2k+1)}{(2k)^2} \times \frac{2k+1}{(2k+2)^2} \\
&= \frac{(2k)^2-1}{(2k)^2} \times \frac{2(k+1)-1}{(2(k+1))^2} \\
&< \frac{2(k+1)-1}{(2(k+1))^2}
\end{aligned}$$

which says that Inequality (7) holds for $k+1$. Since the odd integers are exactly those of the form $2k+1$ for some integer k, we have shown that $(a_n)^2 < \frac{n-2}{(n-1)^2}$ when $n \geq 3$ is odd. Since $n-2 < n-1$, we can simplify further to get that get that $(a_n)^2 < \frac{n-1}{(n-1)^2} = \frac{1}{n-1}$. It follows that $a_n < \frac{1}{\sqrt{n-1}}$ when $n \geq 3$ is odd. Setting $n = 10001$, we get that

$$a_{10001} < \frac{1}{\sqrt{10001-1}} = \frac{1}{\sqrt{10000}} = \frac{1}{100}$$

Note that $n = 10001$ is not the smallest n for which $a_n < \frac{1}{100}$. You may want to write a computer program to find the very first n for which $a_n < \frac{1}{100}$.

The inequality $a_n < \frac{1}{\sqrt{n-1}}$ for odd n shows us that the sequence $a_1, a_3, a_5, a_7, \ldots$ is approaching 0. This is because the quantity $\sqrt{n-1}$ goes to infinity as n goes to infinity, so its reciprocal goes to 0. The term a_n is between 0 and something that is getting closer and closer to 0, so it must also go to 0.

(ii) In much the same way as part (i), it can be shown that for even n, we have

$$\begin{aligned}
(a_n)^2 &= \frac{3^2}{2 \times 4} \times \frac{5^2}{4 \times 6} \times \frac{7^2}{6 \times 8} \times \cdots \times \frac{(n-3)^2}{(n-4) \times (n-2)} \times \frac{(n-1)^2}{2 \times (n-2)} \\
&> \frac{(n-1)^2}{2 \times (n-2)} \\
&> \frac{(n-2)^2}{2(n-2)}
\end{aligned}$$

which implies $a_n > \frac{\sqrt{n-2}}{\sqrt{2}}$ when n is even. If we take $n = 20002$, we get

$$a_{20002} > \frac{\sqrt{20002-2}}{\sqrt{2}} = \sqrt{10000} = 100$$

again, $n = 20002$ is not the smallest n for which $a_n > 100$.

Problem of the Month

Problem 7: April 2022

Problem

In this problem, a 3-*factorization* of a positive integer n is a triple (a, b, c) of positive integers such that $abc = n$. Two 3-factorizations will be regarded as the same if one of them can be obtained by reordering the integers in the other. For example, $(1, 2, 3)$ and $(2, 3, 1)$ are the same 3-factorization of 6. The *sum* of the 3-factorization (a, b, c) is $a + b + c$.

(a) Suppose n has two different 3-factorizations (a, b, c) and (d, e, f) with the same sum. Prove that at most one of these 3-factorizations contains the integer 1.

(b) Suppose n has two different 3-factorizations with the same sum. Prove that n has at least four prime factors. [We allow for repetition here. For instance, $24 = 2 \times 2 \times 2 \times 3$ has four prime factors, even though it only has two *distinct* prime factors.]

(c) Find the smallest integer n that has two different 3-factorizations with the same sum.

(d) Find an infinite family $n_1, n_2, n_3, \ldots$ of positive integers satisfying

- For each i, n_i has two different 3-factorizations with the same sum.
- For each i and j, $\gcd(n_i, n_j) = 1$.

(e) Challenge: Can you find an integer with three different 3-factorizations having the same sum? Can you find infinitely many such integers? Some direction on this will be given in the hint. You may wish to try to write a computer program to get started.

Hint

(a) Assuming that $a+b+c=d+e+f$, $abc=def$, and $a=d=1$, try to prove that either $b=e$ and $c=f$ or $b=f$ and $c=e$.

(b) Show that n cannot have two different 3-factorizations with the same sum if

(i) n is prime.

(ii) n is the product of two prime numbers.

(iii) n is the product of 3 prime numbers.

Part (a) can be used to eliminate much of the case work.

(c) The answer is less than 50. Part (b) can be used to narrow the search considerably.

(d) Use the prime factorization of the integer found in part (c) as a hint to find more integers.

(e) The smallest integer that has three different 3-factorizations with the same sum is 1200. The relevant 3-factorizations are $(4, 15, 20)$, $(5, 10, 24)$, and $(6, 8, 25)$. The next few positive integers that have three different 3-factorizations with the same sum are 1386, 1680, 1872, 2880, 2970, 3024, 3264, 3360, 3600, 3960, and 4320. These were found using a computer search, and the same computer search suggests that such integers are not especially rare (despite seeming a bit tricky to find by hand). Perhaps trying to find a pattern in the list above will lead to an infinite family or to some other interesting observations!

Problem of the Month

Solution to Problem 7: April 2022

(a) Assume that $abc = def$ and that $a + b + c = d + e + f$. To prove the claimed fact, we will further assume that $a = d = 1$ and deduce that either $b = e$ and $c = f$ or $b = f$ and $c = e$. This will prove that the 3-factorizations (a, b, c) and (d, e, f) are the same. Keep in mind that the integers in a 3-factorizations are positive.

With $a = d = 1$, the assumed equations become $bc = ef$ and $b + c = e + f$. Squaring both sides of the second equation leads to $b^2 + 2bc + c^2 = e^2 + 2ef + f^2$. Since $bc = ef$, $4bc = 4ef$, and if we subtract this from $b^2 + 2bc + c^2 = e^2 + 2ef + f^2$ we get $b^2 - 2bc + c^2 = e^2 - 2ef + f^2$ which factors as $(b - c)^2 = (e - f)^2$. Taking square roots, this means $|b - c| = |e - f|$, so either $b - c = e - f$ or $b - c = f - e$.

Suppose $b - c = e - f$. Adding $b + c = e + f$ gives $2b = 2e$ or $b = e$. Similarly, if $b - c = f - e$, then $2b = 2f$ or $b = f$. Thus, either $b = e$ or $b = f$. If $b = e$, then $b + c = e + f$ implies $c = f$. If $b = f$, then $b + c = e + f$ implies $c = e$. Therefore, either $b = e$ and $c = f$ or $b = f$ and $c = e$.

We have shown that if two 3-factorizations of an integer each contain a 1 and have the same sum, then they must be the same 3-factorization. Therefore, it is impossible for two different 3-factorizations of an integer to have the same sum and both contain 1.

(b) Suppose n is either prime or is the product of 2 prime numbers. Then any 3-factorization of n must contain a 1, so part (a) implies that it is impossible for two different 3-factorizations of n to have the same sum.

We now suppose $n = pqr$ where p, q, and r are prime, some or all of which may be equal. The only 3-factorization of n that does not include a 1 is (p, q, r), so if there are two 3-factorizations of n with the same sum, then part (a) implies that one of these 3-factorizations is (p, q, r).

The other 3-factorizations of n are $(1, 1, pqr)$, $(1, p, qr)$, $(1, q, pr)$, and $(1, r, pq)$. We will show that none of the equations

$$\begin{aligned}
p + q + r &= 1 + 1 + pqr \\
p + q + r &= 1 + p + qr \\
p + q + r &= 1 + q + pr \\
p + q + r &= 1 + r + pq
\end{aligned}$$

can be satisfied when p, q, and r are prime.

First, suppose $p + q + r = 1 + 1 + pqr$ for some prime numbers p, q, and r. By possibly relabelling, we can assume that $r \geq q$ and $r \geq p$. Since p and q are both prime, $p \geq 2$ and $q \geq 2$, so $pq \geq 4$, which means $pqr \geq 4r$. Therefore, $1 + 1 + 4r \leq 1 + 1 + pqr$, but we are assuming that $1 + 1 + pqr = p + q + r$, so we have $1 + 1 + 4r \leq p + q + r$. Since $r \geq q$ and $r \geq p$, this means $2 + 4r \leq 3r$, but this is impossible since r is a positive integer. Therefore, it is impossible that $p + q + r = 1 + 1 + pqr$ when p, q, and r are prime.

Now assume that $p + q + r = 1 + p + qr$ for some integers p, q, and r. This simplifies to $qr - q - r + 1 = 0$ or $(q-1)(r-1) = 0$, which means that either $q = 1$ or $r = 1$. Since 1 is not a prime number, the integers p, q, and r cannot all be prime, and so the equation cannot hold if p, q, and r are all prime.

By symmetry, $p + q + r = 1 + q + pr$ and $p + q + r = 1 + r + pq$ also cannot be satisfied when p, q, and r are all prime.

We have now shown that if n is prime, the product of two prime numbers, or the product of three prime numbers, then it cannot have two different 3-factorizations with the same sum. Therefore, if n has two different 3-factorizations with the same sum, then it must have at least four prime factors.

(c) By part (b), we can restrict our search to positive integers that are the product of at least four prime numbers. It can be checked that the first five positive integers with this property are 16, 24, 32, 36, and 40. In fact, 36 is the integer we seek, but we will go through the possibilities above in order to verify that 36 is indeed the smallest.

The 3-factorizations of 16 are $(1,1,16)$, $(1,2,8)$, $(1,4,4)$, and $(2,2,4)$. Their sums are 18, 11, 9, and 8, no two of which are the same, so 16 does not have two different 3-factorizations with the same sum.

The 3-factorizations of 24 are $(1,1,24)$, $(1,2,12)$, $(1,3,8)$, $(1,4,6)$, $(2,2,6)$, and $(2,3,4)$. Their sums are 26, 15, 12, 11, 10, and 9, no two of which are the same, so 24 does not have two different 3-factorizations with the same sum.

The 3-factorizations of 32 are $(1,1,32)$, $(1,2,16)$, $(1,4,8)$, $(2,2,8)$, and $(2,4,4)$. Their sums are 34, 19, 13, 12, and 10, no two of which are the same, so 32 does not have two different 3-factorizations with the same sum.

Among the 3-factorizations of 36 are $(1,6,6)$ and $(2,2,9)$, both of which have a sum of 13. They are different 3-factorizations, and so we have shown that 36 is the smallest positive integer that has two different 3-factorizations with the same sum.

(d) By part (b), each of the n_i needs to be the product of at least four prime numbers. The factorization of 36 is $2^2 3^2$, so we will try to generalize this.

We consider an integer of the form x^2y^2 for some integers x and y, both larger than 1. We will not assume that x and y are prime. Among the 3-factorizations of x^2y^2 are (x, x, y^2) and $(1, xy, xy)$. If their sums are equal, then $2x + y^2 = 1 + 2xy$, which can be rearranged to get $y^2 - 2xy + 2x - 1 = 0$ and then factored as $(y-1)(y-2x+1) = 0$. From here, we can see that the equation will be satisfied if $y = 2x - 1$. Thus, if $y = 2x - 1$, then (x, x, y^2) and $(1, xy, xy)$ have the same sum. As well, as long as we assume that $x > 1$, then $y > 1$ as well, and these two 3-factorizations are guaranteed to be different since one of them contains 1 and the other does not.

Thus, for each integer $x > 1$, the integer $x^2(2x-1)^2$ has two distinct 3-factorizations with the same sum. The table below summarizes the first few examples.

x	$x^2(2x-1)^2$	$(x, x, (2x-1)^2)$	$(1, x(2x-1), x(2x-1))$
2	36	$(2, 2, 9)$	$(1, 6, 6)$
3	225	$(3, 3, 25)$	$(1, 15, 15)$
4	784	$(4, 4, 49)$	$(1, 28, 28)$
5	2025	$(5, 5, 81)$	$(1, 45, 45)$

However, we need to do something more to get the infinite family we seek since, for example, 2025 is a multiple of 225, so $\gcd(225, 2025) = 225 \neq 1$.

To finish the problem, we will use the following fact: For any positive integer n, the integer $M = (n+1)^2(2n+1)^2$ satisfies $\gcd(n, M) = 1$.

To see this, suppose $k \geq 1$ is a factor of both n and $(n+1)^2(2n+1)^2$. This means there are integers a and b with $n = ka$ and $(n+1)^2(2n+1)^2 = 4n^4 + 12n^3 + 13n^2 + 6n + 1 = kb$. Thus,

$$\begin{aligned} 1 &= kb - (4n^4 + 12n^3 + 13n^2 + 6n) \\ &= kb - n(4n^3 + 12n^2 + 13n + 6) \\ &= kb - ka(4n^3 + 12n^2 + 13n + 6) \\ &= k[b - a(4n^3 + 12n^2 + 13n + 6)] \end{aligned}$$

and this shows that 1 is a multiple of k because $b - a(4n^3 + 12n^2 + 13n + 6)$ is an integer. Therefore, the only possible value of k is 1, so $\gcd(n, (n+1)^2(2n+1)^2) = 1$.

This fact implies that if $k > 1$ is any divisor of n, then $\gcd(k, (n+1)^2(2n+1)^2) = 1$ since, if k and $(n+1)^2(2n+1)^2$ have a divisor greater than 1 in common, then so do n and $(n+1)^2(2n+1)^2$. We will use this to construct the infinite family of integers we seek.

We start with $n_1 = 2^2(2(2)-1)^2 = 36$, which has two different 3-factorizations with the same sum. Next, set $n_2 = (n_1+1)^2(2(n_1+1)-1)^2$, which has two distinct 3-factorizations since it is of the form $x^2(2x-1)^2$ with $x = n_1 + 1$. Expanding the second parenthetical expression, we have $n_2 = (n_1+1)^2(2n_1+1)^2$, and from the fact above, $\gcd(n_1, n_2) = 1$.

Next, set $n_3 = (n_1n_2+1)^2(2(n_1n_2+1)-1)^2 = (n_1n_2+1)^2(2n_1n_2+1)^2$. By construction, n_3 has two different 3-factorizations with the same sum. Moreover, $\gcd(n_1n_2, n_3) = 1$, so $\gcd(n_1, n_3) = \gcd(n_2, n_3) = 1$.

Continuing in this way, for each $k \geq 2$ we can define n_{k+1} from $n_1, \ldots, n_k$ by setting

$$n_{k+1} = (n_1n_2\cdots n_k + 1)^2(2n_1n_2\cdots n_k + 1)^2$$

By construction, n_{k+1} will have two different 3-factorizations with the same sum. As well, $\gcd(n_1n_2\cdots n_k, n_{k+1}) = 1$, so $\gcd(n_i, n_{k+1}) = 1$ for all $i \leq k$.

(e) We will outline two different ways to build infinite families of positive integers that have three different 3-factorizations with the same sum.

The first approach is to take any known positive integer that has three different 3-factorizations with the same sum and multiply this integer each of the perfect cubes. For example, as given in the hint, the integer 1200 has 3-factorizations $(4, 15, 20)$, $(5, 10, 24)$, and $(6, 8, 25)$, each with a sum of 39. For every positive integer n, the positive integer $1200n^3$ has 3-factorizations $(4n, 15n, 20n)$, $(5n, 10n, 24n)$, and $(6n, 8n, 25n)$. These three factorizations are all different and have the sum $39n$. Each positive integer gives a different value of $1200n^3$, so this indeed gives an infinite family of positive integers, each of which has three different 3-factorizations with the same sum.

In the second approach, for each positive integer n we define $p(n)$ to be the integer $p(n) = n(n+2)(n+4)(n+5)(n+6)(n+7)$. The integer $p(n)$ has 3-factorizations

$$\begin{aligned}
\big(n(n+7),(n+2)(n+4),(n+5)(n+6)\big) &= (n^2+7n, n^2+6n+8, n^2+11n+30)\\
\big(n(n+6),(n+2)(n+5),(n+4)(n+7)\big) &= (n^2+6n, n^2+7n+10, n^2+11n+28)\\
\big(n(n+5),(n+2)(n+7),(n+4)(n+6)\big) &= (n^2+5n, n^2+9n+14, n^2+10n+24)
\end{aligned}$$

and observe that the sums of these 3-factorizations, respectively, are

$$\begin{aligned}
(n^2+7n)+(n^2+6n+8)+(n^2+11n+30) &= 3n^2+24n+38\\
(n^2+6n)+(n^2+7n+10)+(n^2+11n+28) &= 3n^2+24n+38\\
(n^2+5n)+(n^2+9n+14)+(n^2+10n+24) &= 3n^2+24n+38
\end{aligned}$$

which are all the same. For example, when $n=1$, we get $p(1) = 1\times3\times5\times6\times7\times8 = 5040$ and the 3-factorizations are $(8,15,42)$, $(7,18,40)$, and $(6,24,35)$.

The calculations above show that, for each n, $p(n)$ has three 3-factorizations of n with the same sum. However, there is a possibility that these 3-factorizations are not different. In fact, this problem will never occur, and as long as $n \neq 8$, not only will the given 3-factorizations of $p(n)$ be different, the nine integers occurring in them will all be different.

Consider the nine integers in the 3-factorizations. Each is a quadratic in n with a leading coefficient of 1 and none of the quadratics are the same. If an integer n has the property that two of the nine integers are the same, then we have $n^2+an+b = n^2+cn+d$ for some a, b, c, and d. The n^2 cancels, so in fact we have a linear equation in n. Thus, for each pair of the nine integers, there is *at most* one integer n for which those two integers are equal. For example, $n^2+7n = n^2+7n+10$ implies $0=10$, so there are no integers n that will make $n^2+7n = n^2+7n+10$ equal to each-other. As another example, if $n^2+6n+8 = n^2+10n+24$, then $-16 = 4n$, which implies $n=-4$, which is not a positive integer. In fact, of all of the 36 possible such equations, $n^2+7n = n^2+6n+8$ is the only one with a positive integer solution, which is $n=8$. Therefore, for every positive integer n, the given construction gives an integer that has three different 3-factorizations with the same sum.

For a hint as to how this construction was discovered, the key was to find a set of 5 positive integers with the following property: there are three different ways to choose four of the integers and break those four into two pairs so that the sum of the products of those pairs is the same. For example, the list 2, 4, 5, 6, 7 has this property because $2\times4+5\times6$, $2\times5+4\times7$, and $2\times7+4\times6$ are all equal to 38. Can you find another such set of five integers?

One final thought on this construction. For every integer n, either n is even or $n+5$ is even, which implies that $p(n)$ is even for all integer n. This means that $\gcd(p(n),p(m)) \geq 2$ for all positive integers m and n, so the infinite family generated in this way will not satisfy the condition in part (d). At the time of writing, we still have not found an infinite list $n_1, n_2, n_3, \ldots$ of positive integers satisfying $\gcd(n_i, n_j) = 1$ for all $i \neq j$, each of which has three different 3-factorizations with the same sum.

Problem of the Month
Problem 8: May 2022

Problem

In each part of this problem, a unit cube is positioned with its centre at the origin and is rotated about the x-axis so that it sweeps out a new "solid of revolution". To visualize this solid, you might imagine a cube being rotated very quickly about a fixed axis to produce an illusion of the solid. [This is the same phenomenon as when a rotating propellor or fan blade looks like a disk.] For example, if the cube is originally positioned so that the x-axis passes through the centres of two opposite faces, then the solid of revolution is a cylinder.

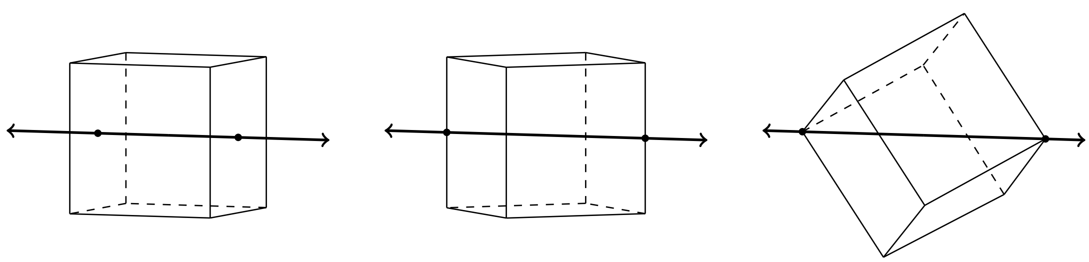
The solid of revolution depends on the original position of the cube. In each part, information is given to describe the original position of the cube and the goal is to describe the region in the (x, y)-plane intersected by the solid of revolution.

(a) The cube is positioned so that the x-axis passes through the centres of two opposite faces. As mentioned in the preamble, the solid is a cylinder.

(b) The cube is positioned so that the x-axis passes through the midpoints of two opposite edges of the cube (that is, two edges that are parallel and are not edges of the same face).

(c) The cube is positioned so that the x-axis passes through two opposite vertices of the cube (that is, two vertices that are not on a common face).

Below, from left to right, are diagrams of the original position of the cube for parts (a), (b), and (c), respectively. In order to avoid clutter in the diagrams, only the x-axis is included.



Notes:

- In the solutions, regions in the (x, y)-plane will have descriptions of the form "the region between $x = a$ and $x = b$ above the graph of $y = f(x)$ and below the graph of $y = g(x)$. You may have some other way of describing the regions.
- Solids of revolution are studied in calculus. If you already know some calculus and would like an added challenge, you might like to try to compute the area of the regions you find, or even the volumes of the solids of revolution.

Hint

(a) The cross sections of the cube in this part are all unit squares. What length in the square is equal to the diameter of the base of the cylinder swept out by the rotating the cube?

(b) In this part, the cross sections are always rectangles, but their dimensions vary depending on where the cross section is taken. Here is a link to a GeoGebra applet to help visualize the rotating cube and the cross sections. The "Rotate" option will cause the cube to rotate around the axis. The "Show Cross Section" option will show the cross section when the cube is sliced by a plane perpendicular to the axis. The plane can be moved to see different cross sections. The "Show Trace" option will show the solid traced out by the cube as it rotates.

(c) In this part, the cross sections are either triangles or hexagons, depending on how close the cross section is taken to the vertices that are fixed on the axis. The cube has 120° rotational symmetry, which means that if it is rotated by 120°, it occupies exactly the same space that it occupied before it was rotated. The cross sections must also have 120° rotational symmetry. It may be useful to think about what sorts of triangles and hexagons can have such symmetry. This is a link to another GeoGebra applet that works in essentially the same way as the one for part (b).

Problem of the Month

Solution to Problem 8: May 2022

In each part, the solid will be a figure that has rotational symmetry about the x-axis. For each x-value, this means if we slice the solid by a plane perpendicular to the x-axis at that x-value, the cross section of the solid will be a circle with its centre on the x-axis. Thus, to describe the solid of revolution in each part, we need to determine, for each x-value, the radius of this cross sectional circle. To do this, we will examine the corresponding cross sections of the cube. The radius of the circular cross section of the solid of revolution will be the distance from the x-axis to the point in the cross section of the cube that is farthest from the x-axis. The GeoGebra applets provided in the hint may be useful for visualizing these cross sections.

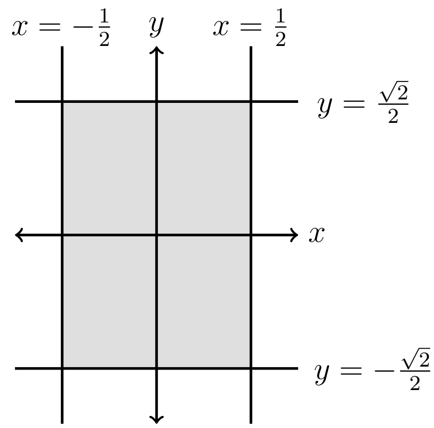
The approach in each part will be as follows:

- Determine the range of x-values occupied by the cube, which will be called I.
- For each $a \in I$, describe the cross section of the cube when it is sliced by the plane with equation $x = a$. [That is, the plane perpendicular to the x-axis that intersects the x-axis at $x = a$.]
- Let f be the function with domain I so that for each $a \in I$, $f(a)$ is the largest possible distance to the x-axis from a point in the cross section of the cube at $x = a$.
- The solid of revolution is that which has circular cross sections with radius $f(a)$ at each $a \in I$.
- The region in the xy-plane is the set of points with $x \in I$ that are above the graph of $y = -f(x)$ and below the graph of $y = f(x)$.

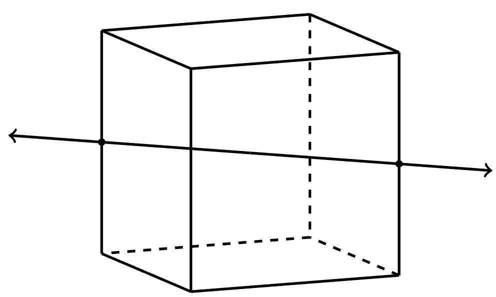
(a) Since the cube is centred at the origin and its sides have length 1, the cube is initially positioned along the interval $I = \left[-\frac{1}{2}, \frac{1}{2}\right]$. Because the x-axis is perpendicular to two faces of the cube, when we slice the cube by any plane that is perpendicular to the x-axis, the cross section is a unit square with its centre on the x-axis.

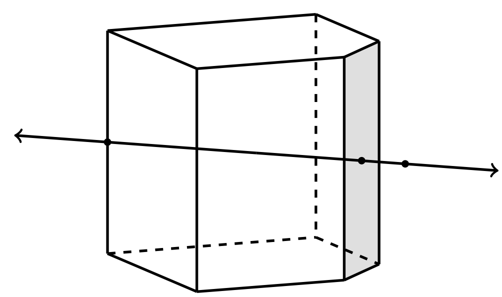
In any square, the points that are farthest from the centre are the four vertices. The distance from the centre to a vertex is half the length of the diagonal of the square. By the Pythagorean theorem, the length of the diagonal of a unit square is $\sqrt{1^2 + 1^2} = \sqrt{2}$, so the distance from the centre of the square to a vertex is $\frac{\sqrt{2}}{2}$.

Thus, for any $a \in \left[-\frac{1}{2}, \frac{1}{2}\right]$, we have that $f(a) = \frac{\sqrt{2}}{2}$. The solid of revolution is the cylinder that is parallel to the x-axis that has radius $\frac{\sqrt{2}}{2}$ and height 1. The cylinder intersects the xy-plane in a rectangle, and that rectangle is the set of points that are bound by the horizontal lines with equations $y = \frac{\sqrt{2}}{2}$ and $y = -\frac{\sqrt{2}}{2}$ on the interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$. This region is pictured below.

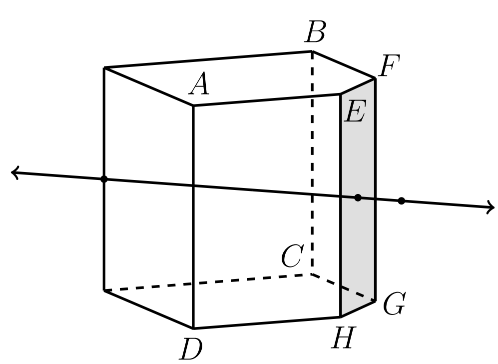


(b) The distance between the midpoints of two opposite edges of a cube is the same as the length of the diagonal of any face. By the computation in part (a), this length is $\sqrt{2}$. Since the cube is centred at the origin, the interval I in this part is $\left[-\dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2}\right]$. In the diagram below, the cube is seen in its original position. On the left, the entire cube is pictured. On the right, a piece has been removed to show a generic cross section, which is shaded.





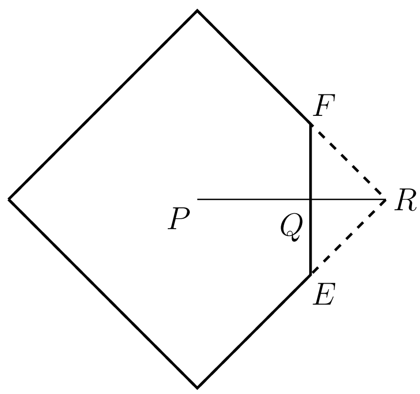
The shaded cross section appears to be a rectangle. To explain why it is indeed a rectangle, we first note that the plane with equation $x = a$ with $a > 0$ intersects four faces of the cube, and each of these intersections gives a line segment. This means that the cross section is a quadrilateral. To see that this quadrilateral is indeed a rectangle, we will label some points on the surface of the cube. The vertices of the cube that are on the plane with equation $x = 0$ will be labelled A, B, C, and D with A at the "top front", B at the "top back", C at the "bottom back", and D at the "bottom front". As well, the points where the plane with equation $x = a$ intersect the edges of the cube will be labelled E, F, G, and H in such a way that segments AE, BF, CG, and DH all lie on edges of the cube.



The plane through A, B, C, and D is perpendicular to the x-axis because of the way the cube is positioned. Therefore, the plane through A, B, C, and D is parallel to the plane with equation $x = a$. Hence, AD, BC, FG, and EH are all parallel. By similar reasoning, EF is parallel to HG, so $EFGH$ is a parallelogram. As well, AD is perpendicular to the top face of the cube, which means EH is perpendicular to the top face of the cube. This means that EH is perpendicular to any line through E and another point in the top face. Hence, EH is perpendicular to EF. A parallelogram with a right angle must be a rectangle, which shows that $EFGH$ is a rectangle. Also note that $AEHD$ is a rectangle for similar reasoning.

By symmetry, the centre of the rectangular cross section is on the x-axis. Thus, the circular cross section at $x = a$ of the solid of revolution has a radius equal to the distance from the centre of the rectangular cross section of the cube to any of its four vertices. This radius is half the length of the diagonal of the rectangular cross section, which can be found using the Pythagorean theorem once we know the side lengths, so it remains to determine the dimensions of the cross section at $x = a$, which we expect to depend on the value of a. By symmetry, it is enough to consider $a > 0$.

As noted above, $AEHD$ is a rectangle, so $EH = AD = 1$, which is independent of the value of a. The length of EF does depend on the value of a. Below is a diagram of the top face of the cube. The centre of the top face has been labelled by P, the corner of the top face that was removed in the previous diagram is labelled by R, and the point where the line segment PR intersects EF is labelled by Q.



The length of PR is equal to $\dfrac{\sqrt{2}}{2} = \dfrac{1}{\sqrt{2}}$ because it is half the length of the diagonal of a unit square. As well, PQ has length a by assumption, and EF is perpendicular to PQ because PQ is parallel to the x-axis and EF is perpendicular to the x-axis. Therefore, QR is an altitude of $\triangle FQR$. The line connecting the centre of a square to one of its vertices must be an angle bisector, which means that $\angle PRF = \angle PRE = 45°$. Since $\angle FQR = \angle EQR = 90°$, we also have $\angle QFR = \angle QER = 45°$, which means that $\triangle FQR$ and $\triangle EQR$ are both isosceles. Therefore, $QE = QR = QF$, but $QR = \dfrac{1}{\sqrt{2}} - a$, so

$$EF = QE + QF = 2QR = 2\left(\frac{1}{\sqrt{2}} - a\right) = \sqrt{2} - 2a$$

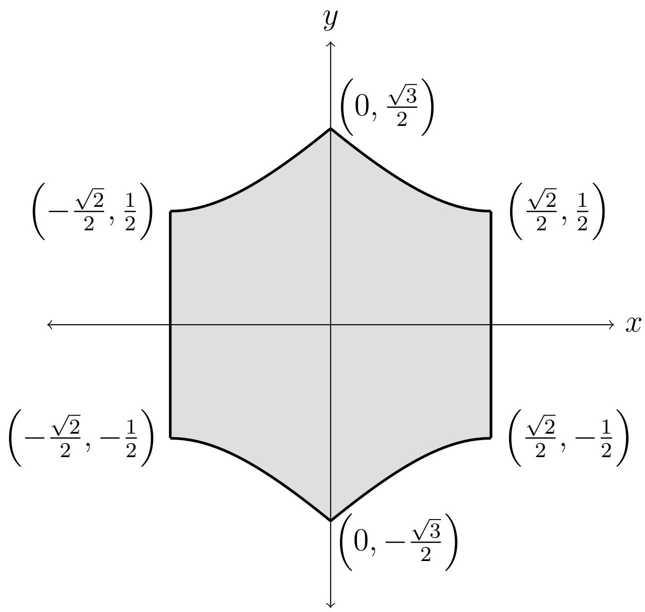
Therefore, the diagonal length of the rectangular cross section at $x = a$ is

$$\sqrt{EH^2 + EF^2} = \sqrt{1^2 + (\sqrt{2} - 2a)^2} = \sqrt{3 - 4\sqrt{2}a + 4a^2}$$

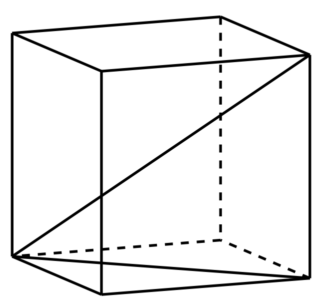
For $x \geq 0$, we have that $f(x) = \dfrac{1}{2}\sqrt{3 - 4\sqrt{2}x + 4x^2}$. By symmetry, the function f on the interval should be an *even* function on I, which means $f(x) = f(-x)$. This means that we can replace x by $-x$ to determine $f(x)$ when $x < 0$. After doing this, we find that $f(x)$ is defined *piecewise* by

$$f(x) = \begin{cases} \dfrac{1}{2}\sqrt{3 + 4\sqrt{2}x + 4x^2} & \text{if } -\dfrac{\sqrt{2}}{2} \leq x < 0 \\ \dfrac{1}{2}\sqrt{3 - 4\sqrt{2}x + 4x^2} & \text{if } 0 \leq x \leq \dfrac{\sqrt{2}}{2} \end{cases}$$

Below is a diagram of the region above the graph of $y = -f(x)$ and below the graph of $y = f(x)$ on the interval $\left[-\dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2}\right]$.



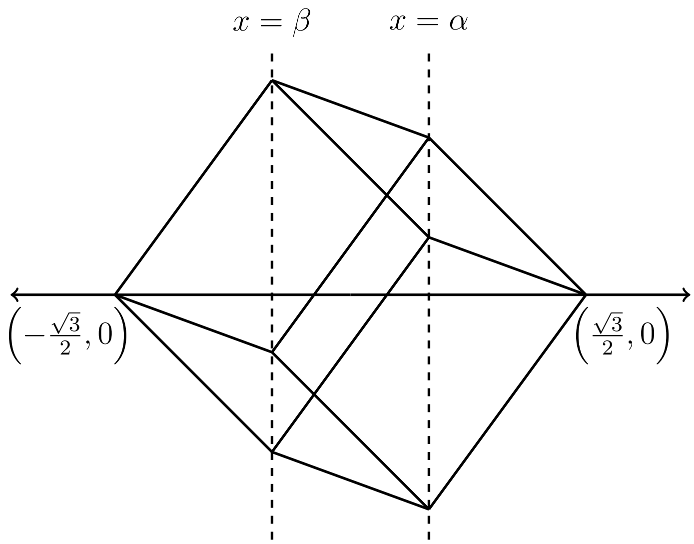
(c) In this part, we observe that the distance between two opposite vertices of a cube is the length of the hypotenuse of a right-angled triangle with one leg equal to an edge of the cube and one leg equal to the diagonal of a face of the cube. This is pictured below.



The length of the diagonal of a unit square is $\sqrt{2}$, so the distance between two opposite vertices of the cube is $\sqrt{(\sqrt{2})^2 + 1^2} = \sqrt{3}$. Thus, in this part, the interval is $I = \left[-\dfrac{\sqrt{3}}{2}, \dfrac{\sqrt{3}}{2}\right]$.

In this part, the cross sections come in two "types". If x is close enough to 0, then the cross section is a hexagon. Otherwise, the cross section is an equilateral triangle.

Consider the vertices of the cube that are to the right of the origin. By rotational symmetry, if the cube is rotated $120°$ around the x-axis, these vertices (other than those on the x-axis) will take each other's positions. Therefore, they must all have the same x-coordinate, so there is some $\alpha > 0$ such that the plane with equation $x = \alpha$ passes through all three of these vertices. Similarly, there is $\beta < 0$ so that the plane with equation $x = \beta$ passes through all three of the vertices of the cube that are not on the x-axis and have a negative x-coordinate. The diagram below is of a cube positioned with two opposite vertices on the x-axis but viewed at an angle perpendicular to the x-axis. This gives some indication of the different cross sections and where they change type. The dashed vertical lines are meant to represent the planes with equations $x = \alpha$ and $x = \beta$.



Suppose $\alpha < a < \dfrac{\sqrt{3}}{2}$. The plane with equation $x = a$ intersects three faces of the cube, so the cross section of the cube at $x = a$ is a triangle. The rotational symmetry of the cube implies that this triangle has $120°$ rotational symmetry about the x-axis, and such a triangle must be equilateral since it must have three equal angles. Similarly, if $-\dfrac{\sqrt{3}}{2} < a < \beta$, then the cross section at $x = a$ is also an equilateral triangle.

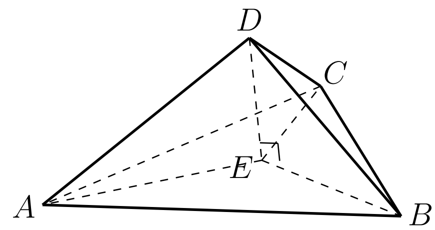
For $\beta < a < \alpha$, the plane with equation $x = a$ intersects all 6 faces of the cube. The plane intersects each face in a line segment, so the cross section must be a hexagon since each of these line segments will be a side of the cross section. There is no reason to expect it to be a regular hexagon, but it will have $120°$ rotational symmetry, which will be used later.

We will delay computing the values of α and β, though we will observe that, by symmetry, $\alpha = -\beta$, and it suffices to analyze the cross sections at $x = a$ for $a > 0$.

To analyze the triangular cross sections, we will use the following fact.

Fact 1: Suppose tetrahedron $ABCD$ has equilateral base $\triangle ABC$ and its other three faces satisfy $\angle ADB = \angle BDC = \angle CDA = 90°$ and $AD = BD = CD$. If E is the point in $\triangle ABC$ so that DE is the altitude of the tetrahedron from D, then $DE = \dfrac{AD}{\sqrt{3}}$ and $AE = \sqrt{2}DE$.

Proof. By symmetry, $AE = BE = CE$.



Since $\triangle ADB$ is isosceles and right-angled with hypotenuse AB, we get that $AB = \sqrt{2}AD$. We also have that $\triangle AEB$, $\triangle BEC$, and $\triangle CEA$ are all congruent by side-side-side congruence. As well, $\angle AEB + \angle BEC + \angle CEA = 360°$, so since they are equal by congruence, they are all equal to $120°$. Because $AE = BE$, it follows that $\triangle AEB$ is isosceles and that $\angle ABE = \dfrac{180° - 120°}{2} = 30°$.

Using the Sine law, we have $\dfrac{AE}{\sin 30°} = \dfrac{AB}{\sin 120°}$, from which it follows that

$$AE = \frac{AB \sin 30°}{\sin 120°} = \frac{AB}{\sqrt{3}} = \frac{\sqrt{2}AD}{\sqrt{3}}$$

We can now use the Pythagorean theorem on $\triangle ADE$ to get that

$$DE = \sqrt{AD^2 - AE^2} = \sqrt{AD^2 - \left(\frac{\sqrt{2}}{\sqrt{3}}AD\right)^2} = AD\sqrt{1 - \frac{2}{3}} = \frac{AD}{\sqrt{3}}$$

which is one of the claims in the fact. The other now follows by rearranging the equation above to get $AD = \sqrt{3}DE$ then substituting into $AE = \dfrac{\sqrt{2}AD}{\sqrt{3}}$ to get

$$AE = \frac{\sqrt{2}AD}{\sqrt{3}} = \frac{\sqrt{2}(\sqrt{3}DE)}{\sqrt{3}} = \sqrt{2}DE$$

$\square$

We can now compute the value of α as well as $f(a)$ for each a with $\alpha < a < \dfrac{\sqrt{3}}{2}$.

When we take the cross section at $x = a$ with $\alpha \leq a < \dfrac{\sqrt{3}}{2}$, we have already argued that the cross section is an equilateral triangle. Taking such a cross section "removes" a tetrahedral corner of the cube with this cross section as its base. By rotational symmetry and the fact that the faces of a cube are squares, the other three faces of this "removed" tetrahedron are isosceles right-angled triangles. Thus, the tetrahedron satisfies the conditions of Fact 1. Moreover, points E and D are on the x-axis and $f(a)$ is equal to the length of AE.

When $a = \alpha$, AD is an edge of the cube, so $AD = 1$ which gives $DE = \dfrac{AD}{\sqrt{3}} = \dfrac{1}{\sqrt{3}}$ by Fact 1. As well, $\alpha = \dfrac{\sqrt{3}}{2} - DE$, which means

$$\alpha = \frac{\sqrt{3}}{2} - DE = \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{3}} = \frac{1}{2\sqrt{3}}$$

For any a with $\frac{1}{2\sqrt{3}} \le a < \frac{\sqrt{3}}{2}$, the tetrahedron has $DE = \frac{\sqrt{3}}{2} - a$, and since $AE = \sqrt{2}DE$, we get

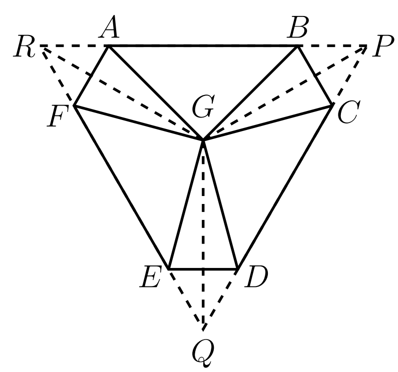
$$f(a) = AE = \sqrt{2}DE = \sqrt{2}\left(\frac{\sqrt{3}}{2} - a\right) = \frac{\sqrt{3}}{\sqrt{2}} - \sqrt{2}a$$

While we have not yet determined how to compute $f(x)$ for all $x \in I$, we do now have for $x \in \left[\frac{1}{2\sqrt{3}}, \frac{\sqrt{3}}{2}\right)$ that $f(x) = \frac{\sqrt{3}}{\sqrt{2}} - \sqrt{2}x$. Notice that at $x = \frac{\sqrt{3}}{2}$, $f(x) = 0$, which makes sense. You may want to think about this.

Next we will examine the hexagonal cross sections for $0 \le a < \frac{1}{2\sqrt{3}}$. We will use the following fact.

Fact 2: Suppose that $ABCDEF$ is a hexagon that has opposite sides parallel (that is, AB and DE are parallel, BC and EF are parallel, and CD and FA are parallel) and has a point G in its interior so that the hexagon has $120°$ rotational symmetry about G. Then G is equidistant from all six vertices of the hexagon.

Proof. Below is a diagram of such a hexagon with AB and DC extended to meet at P, CD and FE extended to meet at Q, and EF and BA extended to meet at R. Point G is also connected to each vertex of the hexagon as well as to P, Q, and R.



The fact that the hexagon has $120°$ rotational symmetry means that it also has $240°$ rotational symmetry. This means that it has $120°$ rotational symmetry both clockwise and counterclockwise. A clockwise rotation will send A to the position of C, B to D, C to E, D to F, E to A, and F to B. Since the rotation is around G, this implies that $GA = GC = GE$ and $GB = GD = GF$, as well as $AB = CD = EF$ and $BC = DE = FA$. Finally, by the way P, Q, and R are defined, the rotational symmetry also implies that $\triangle PQR$ has $120°$ rotational symmetry about G. From an earlier argument, this implies $\triangle PQR$ is equilateral and that G is equidistant from P, Q, and R.

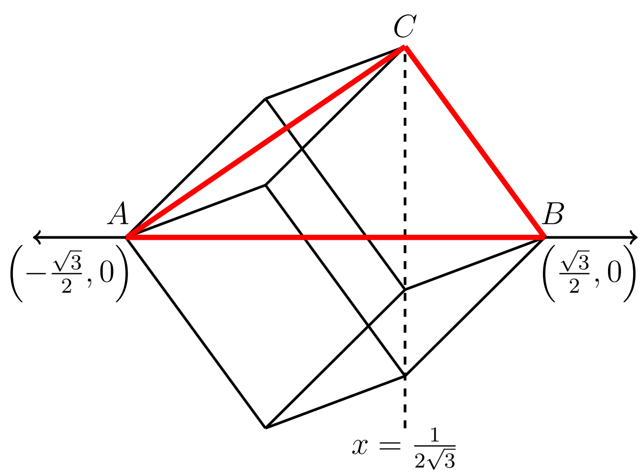
By properties of parallel lines, we get that $\angle RAF = \angle EDQ = \angle BPC$, but from the previous paragraph we have $\angle BPC = \angle ARF = 60°$, so $\triangle RAF$ has two angles equal to $60°$. Therefore, it is equilateral.

Since $GP = GR = GQ$ and $PQ = QR = RP$, it must be that $\triangle GPQ$, $\triangle GQR$, and $\triangle GRP$ are all congruent by side-side-side congruence. It follows that $\angle GRQ = \angle GRP$, and since their sum is $60°$, they are both equal to $30°$. Let H be the point of intersection of GR and AF. We know that $\angle RAF = 60°$, and so it follows that $\angle RHA = 90°$. As well, $\triangle FRH$ and $\triangle ARH$ are congruent by side-angle-side congruence, so $FH = AH$. Since $\angle RHA = 90°$, so do each of $\angle FHG$ and $\angle AHG$, so we now conclude that $\triangle FHG$ and $\triangle AHG$ are congruent by side-angle-side congruence. This means $GA = GF$, and since $GA = GC = GE$ and $GB = GD = GF$, it follows that G is equidistant from all six vertices of the hexagon. $\square$

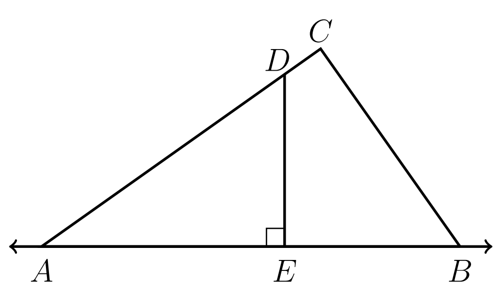
The cube has $120°$ rotational symmetry about the x-axis, and so if we take any a with $0 \leq a < \dfrac{1}{2\sqrt{3}}$, the hexagonal cross section must also have $120°$ rotational symmetry about the point where the plane with equation $x = a$ intersects the x-axis. As well, opposite faces of the cube are parallel, so opposite sides of the hexagonal cross section must also be parallel. By Fact 2, the six vertices of the cross section are equidistant from the x-axis.

This means that $f(a)$ is the distance from the x-axis to any of the six points where the plane with equation $x = a$ intersects an edge of the cube.

Consider the diagram of the cube below. The vertices of the cube that are on the x-axis are labelled by A and B, one of the vertices of the cube with x-coordinate equal to $\dfrac{1}{2\sqrt{3}}$ is labelled by C, and $\triangle ABC$ is in bold red.



We know that $AB = \sqrt{3}$, $AC = \sqrt{2}$ and $BC = 1$. Suppose $0 \leq a < \dfrac{1}{2\sqrt{3}}$ and consider the cross section at $x = a$. Let E be the point on the x-axis at $x = a$, which implies that E is on AB. As well, let D be the point at which the plane with equation $x = a$ intersects AC. The plane is perpendicular to the x-axis, which means that $\angle AED = 90°$.



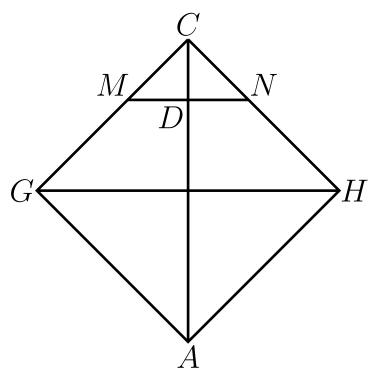
The length of AE is $\dfrac{\sqrt{3}}{2}+a$, and $\triangle AED$ is similar to $\triangle ACB$ since they share an angle at A and $\angle AED = \angle ACB = 90^\circ$. Therefore, $\dfrac{AD}{AE} = \dfrac{AB}{AC} = \dfrac{\sqrt{3}}{\sqrt{2}}$ which gives

$$AD = \frac{\sqrt{3}}{\sqrt{2}}\left(\frac{\sqrt{3}}{2}+a\right) = \frac{3}{2\sqrt{2}} + \frac{\sqrt{3}a}{\sqrt{2}}$$

As well, $\dfrac{ED}{AE} = \dfrac{CB}{AC} = \dfrac{1}{\sqrt{2}}$ and so

$$ED = \frac{1}{\sqrt{2}}\left(\frac{\sqrt{3}}{2}+a\right) = \frac{\sqrt{3}}{2\sqrt{2}} + \frac{a}{\sqrt{2}}$$

Next, let G and H be the other two vertices of the cube that are on the same face as A and C and let M and N be the points where the plane with equation $x = a$ intersects GC and HC, respectively.



The plane with equation $x = -\dfrac{1}{2\sqrt{3}}$ contains both G and H and is parallel to the plane with equation $x = a$. Since segments GH and MN are themselves in the same plane, they must be parallel. It follows that $\triangle CMN$ is an isosceles right-angled triangle. By an argument used in part (b), it follows that $MD = CD = ND$. We have that $AC = \sqrt{2}$ and $AD = \dfrac{3}{2\sqrt{2}} + \dfrac{\sqrt{3}a}{\sqrt{2}}$, which means

$$MD = ND = CD = AC - AD = \sqrt{2} - \left(\frac{3}{2\sqrt{2}} + \frac{\sqrt{3}a}{\sqrt{2}}\right) = \frac{1}{2\sqrt{2}} - \frac{\sqrt{3}a}{\sqrt{2}}$$

Now consider $\triangle EMN$ with D on MN. By the fact from earlier about hexagons, we already know that $EM = EN$. We have just shown that $MD = ND$. Since they also share side ED, we get that $\triangle EDN$ and $\triangle EDM$ are congruent by side-side-side congruence. Thus, $\angle EDM = \angle EDN$ and their sum is 180°, so $\triangle EDN$ is right-angled at D. Therefore, the length of EN, which is $f(a)$, can be computed using the Pythagorean theorem.

$$\begin{aligned}
f(a) = EN &= \sqrt{ED^2 + ND^2} \\
&= \sqrt{\left(\frac{\sqrt{3}}{2\sqrt{2}} + \frac{a}{\sqrt{2}}\right)^2 + \left(\frac{1}{2\sqrt{2}} - \frac{\sqrt{3}a}{\sqrt{2}}\right)^2} \\
&= \sqrt{\frac{3}{8} + \frac{\sqrt{3}a}{2} + \frac{a^2}{2} + \frac{1}{8} - \frac{\sqrt{3}a}{2} + \frac{3a^2}{2}} \\
&= \sqrt{\frac{1}{2} + 2a^2}
\end{aligned}$$

We can now define $f(x)$ on $\left[0, \frac{\sqrt{3}}{2}\right]$ as a piecewise function:

$$f(x) = \begin{cases} \sqrt{\frac{1}{2} + 2x^2} & \text{if } 0 \le x < \frac{1}{2\sqrt{3}} \\ \frac{\sqrt{3}}{\sqrt{2}} - \sqrt{2}x & \text{if } \frac{1}{2\sqrt{3}} \le x \le \frac{\sqrt{3}}{2} \end{cases}$$

To extend f to all of I, we observe that, like in part (b), $f(x) = f(-x)$. Thus, we can define $f(x)$ on $\left[-\frac{\sqrt{3}}{2}, 0\right]$ by substituting $x = -x$. Note that since $(-x)^2 = x^2$, the function definition is the same for $-\frac{1}{2\sqrt{3}} < x \le 0$ as it is for $0 \le x < \frac{1}{2\sqrt{3}}$. Thus, we get that

$$f(x) = \begin{cases} \frac{\sqrt{3}}{\sqrt{2}} + \sqrt{2}x & \text{if } -\frac{\sqrt{3}}{2} \le x \le -\frac{1}{2\sqrt{3}} \\ \sqrt{\frac{1}{2} + 2x^2} & \text{if } -\frac{1}{2\sqrt{3}} < x < \frac{1}{2\sqrt{3}} \\ \frac{\sqrt{3}}{\sqrt{2}} - \sqrt{2}x & \text{if } \frac{1}{2\sqrt{3}} \le x \le \frac{\sqrt{3}}{2} \end{cases}$$

Below is a diagram of the region above the graph of $y = -f(x)$ and below that of $y = f(x)$.