Problem of the Month

Additional Information about Problem 2: November 2023

In this month's problem, you were asked to find four points on various loops in the plane that formed the corners of a square. In all the cases in the problem, such a set of four points exists (luckily for you!). In the beginning of the problem statement, it was mentioned that the problem was inspired by the following more general question: If you draw an arbitrary loop in the plane, is it always possible to find four points on the loop that form the vertices of a square?

This question is sometimes known as the *Square Peg Problem*, and the answer is currently not known. The *Toeplitz Conjecture* is the guess that the answer should be "yes". To the best of our knowledge, the question was first posed in 1911 by Otto Toeplitz in [1] where the conjecture was made.

What is meant by a loop was described informally in the problem statement. More formally, a loop can be described as follows.

Suppose f and g are functions with domain equal to $[0, 1]$ (the set of real numbers x satisfying $0 \leq x \leq 1$). We can define a function γ with domain $[0, 1]$ by $\gamma(x) = (f(x), g(x))$. This means γ is a function that takes real numbers as input but its outputs are points in the plane. The range of γ is called a loop if it satisfies the following conditions:

1. $\gamma(0) = \gamma(1)$ (the loop has to start and end at the same place),
2. If $\gamma(x) = \gamma(y)$ for any $0 < x < 1$, then $x = y$ (the loop doesn't intersect itself, and the loop isn't just a point),
3. The functions $f(x)$ and $g(x)$ are continuous (you can draw the loop without lifting up your pencil).

For example, if $f(x) = \cos(2\pi x)$ and $g(x) = \sin(2\pi x)$, then the loop is exactly the unit circle.

In mathematics, such a loop is called a *Jordan curve*. It is true that every curve you can draw in the informal "don't lift your pencil" manner is a Jordan curve. However, there are Jordan curves that you would not have any hope of drawing. For example, some fractals are Jordan curves. One example is the so-called Koch Snowflake. You might like to to an internet search to see roughly what this curve looks like and to get an idea of why it is impossible to actually draw it.

It has been known since 1944 [2] that every smooth[1] Jordan curve contains four points that are the vertices of a square. Around the late 1970s Herbert Vaughan provided a beautiful topological argument to prove that every Jordan curve contains the vertices of a rectangle. An exposition of this proof can be seen in the video at the following link: https://youtu.be/AmgkSdhK4K8. Tantalizingly, the proof does not give any control over whether or not the rectangle is a square!

Joshua Greene and Andrew Lobb proved in 2021 [3] that every smooth Jordan curve admits four points that are the vertices of a rectangle of any ratio you like! For example, if you want four points that are the vertices of a rectangle with one side 42 times the length of another side, then

[1] "smooth" has a precise meaning, but it essentially means "no corners".

you can find four such points on any smooth Jordan curve! Try to prove this if the curve is a circle or an ellipse.

There has been lots of activity around this problem in recent years. Despite the progress, at the time of writing this the Toeplitz conjecture itself remains intriguingly out of reach!

References

[1] Otto Toeplitz, *Ueber einige Aufgaben der Analysis situs.* Verhandlungen der Schweizerischen Naturforschenden Gesellschaft (1911), no. 4, 197.

[2] L. G. Šnirelman. *On certain geometrical properties of closed curves.* Uspehi Matem. Nauk 10 (1944), pp. 34-44.

[3] Joshua Evan Greene and Andrew Lobb. *The rectangular peg problem.* Ann. of Math. (2) 194.2 (2021), pp. 509-517.