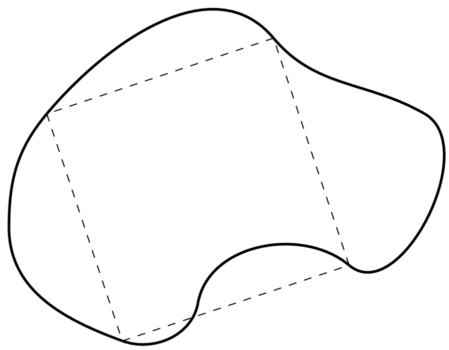
Problem of the Month
Problem 2: November 2023

This month's problem is based on the following general question: If you draw a "loop" in the Cartesian plane, is it always possible to find four points on that loop that are the vertices of a square? For example, the diagram below has a loop (the solid line) and a square drawn (the dashed line) with its four vertices on the loop.



Although it is a bit informal, it should be sufficient to think of a "loop" as a curve that you could draw by starting your pencil somewhere on a page and moving the pencil around the page eventually ending up where it started. Such a loop could be "smooth" (like a circle), "jagged" (like a polygon), or some combination of the two.

(a) In each of parts (i) through (v), find four points on the loop that are the vertices of a square.

- (i) the circle with equation $x^2 + y^2 = 1$
- (ii) the ellipse with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where a and b are fixed positive real numbers
- (iii) the polygon with vertices $(1, 0)$, $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, $(-1, 0)$, $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$, and $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$
- (iv) the boundary of the region enclosed by the parabola with equation $y = -\frac{1}{2}x^2 + \frac{1}{6}x + \frac{16}{9}$ and the line with equation $y = x$
- (v) the boundary of the region enclosed by the parabolas with equations $y = x^2 + \frac{2}{3}x - \frac{4}{3}$ and $y = -x^2 + \frac{2}{3}x + \frac{4}{3}$

(b) Show that for every acute triangle there are exactly three squares whose vertices all lie on the perimeter of the triangle.