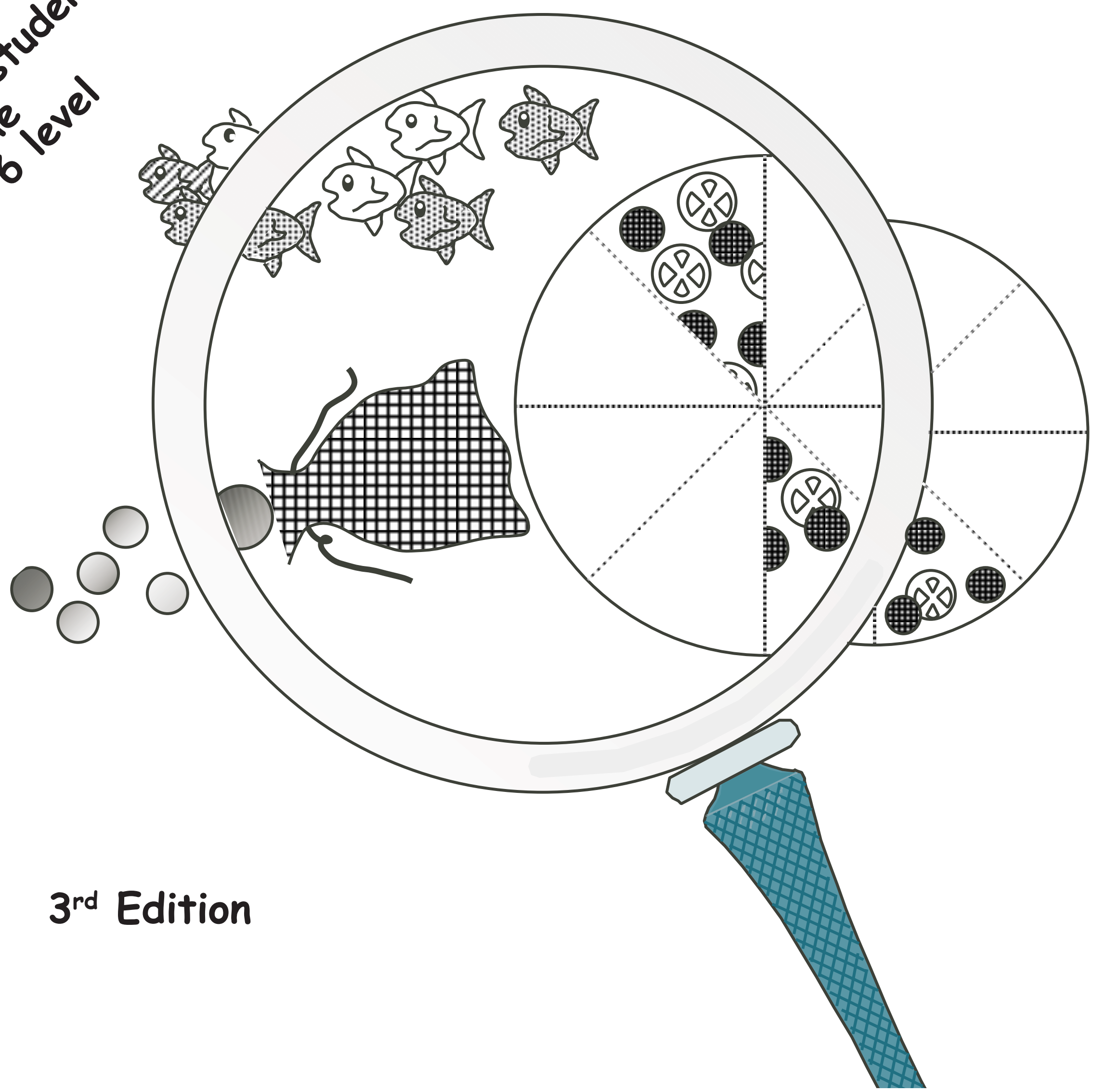
Invitations to Mathematics

Investigations in Probability

"Probability in Action"

Suggested for students at the Grade 6 level



3rd Edition

An activity of
The CENTRE for EDUCATION in MATHEMATICS and COMPUTING
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Preface

The Centre for Education in Mathematics and Computing at the University of Waterloo is dedicated to the development of materials and workshops that promote effective learning and teaching of mathematics. This unit is part of a project designed to assist teachers of Grades 4, 5, and 6 in stimulating interest, competence, and pleasure in mathematics, among their students. While the activities are appropriate for either individual or group work, the latter is a particular focus of this effort. Students will be engaged in collaborative activities which will allow them to construct their own meanings and understanding. This emphasis, plus the extensions and related activities included with individual activities/projects, provide ample scope for all students' interests and ability levels. Related "Family Activities" can be used to involve the students' parents/care givers.

Each unit consists of a sequence of activities intended to occupy about one week of daily classes; however, teachers may choose to take extra time to explore the activities and extensions in more depth. The units have been designed for specific grades, but need not be so restricted. Activities are related to the Ontario Curriculum but are easily adaptable to other locales.

"Investigations in Probability" is comprised of activities which introduce students to basic concepts of probability, techniques used to determine probability, and applications of probability. Everyday encounters with probability in weather forecasting, interpretation of polls, and commercials for various products and lotteries make it imperative that students acquire some basic knowledge of probability if they are to be able to interpret and evaluate such statements, and hence make well-informed decisions.

Acknowledgements

Contributing Teachers

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Mary Thompson (University of Waterloo)

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Table of Contents

Overview

Common Beliefs

The activities in this booklet have been developed within the context of certain values and beliefs about mathematics generally, and about probability specifically. Some of these are described below.

Importance of Probability

Even a cursory glance at newspapers shows the extent to which the language of probability has become important. Individuals need a knowledge of probability to function in our society; consumer reports, cost of living indices, surveys, and samples are a part of everyday life. Nearly all endeavours in the working world require making decisions in uncertain conditions. The goal is to help students develop the critical thinking skills needed to reach sound conclusions based on appropriate data samples.

Instructional Considerations

"Classroom experiences should build on students' natural abilities to solve problems in everyday situations of uncertainty"

NCTM

For example, students learn to play games, and quickly develop a notion of "fairness" which is related to equally likely events. These and other activities develop essential skills for understanding probability — methods of organized counting, comparing results of experiments to theoretical probabilities, using the language of probability correctly — in the context of activities such as dice and spinner games which may be fair or unfair, decoding messages, designing a lottery, and sampling to determine population size.

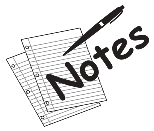
Essential Content

The activities in this unit explore probability on a continuum from zero to one, relate it to experimental frequencies, and examine how it is used in sampling and lotteries. In addition, there are Extensions in Mathematics, Cross-Curricular Activities and Family Activities. These may be used prior to or during the activity as well as following the activity. They are intended to suggest topics for extending the activity, assisting integration with other subjects, and involving the family in the learning process.

During this unit the student will:

- identify probabilities of '0' and '1';
- determine probabilities from simple experiments;
- explore letter frequencies in different languages;
- investigate the nature of lotteries;
- use sampling and marking to determine the size of a population;
- use random numbers to simulate experiments;
- use the language of probability correctly;
- justify opinions with coherent arguments;
- collaborate with other members of a group.

Notes



On the inside of the back cover of this booklet, you will find a chart connecting each activity to Ontario's curriculum expectations.

Overview

Curriculum Expectations

The material in this unit is directly related to Ontario curriculum expectations for Mathematics outlined below. By the end of Grade 6, students will:

- connect the possible events and the probability of a particular event;
- compare experimental results with predicted theoretical results;
- use tree diagrams to record the results of systematic counting;
- show an understanding of probability in making relevant decisions;
- use a knowledge of probability to pose and solve problems.

Assessment

Assessment may be described as the process of gathering evidence about a student's knowledge, skills, and values, and of making inferences based on that evidence for a variety of purposes. These purposes include making instructional decisions, monitoring student progress, evaluating student achievement in terms of defined criteria, and evaluating programs.

To meet these aims, it is necessary to use a variety of assessment techniques in order to:

- assess what students know and how they think and feel about mathematics;
- focus on a broad range of mathematical tasks and taking a holistic view of mathematics;
- assess student performance in a variety of ways, including written and oral, and demonstrations;
- assess the process as well as the product.

Tests are one way of determining what students have learned, but mathematical competence involves such characteristics as communicative ability, problem-solving ability, higher-order thinking ability, creativity, persistence, and curiosity. Because of the nature of the activities it is suggested that a variety of assessment strategies be used. Suggestions include:

- observing students as they work to see if they are applying various concepts; to see if they are working cooperatively; to observe their committment to the tasks;
- assessing the completed project to see if instructions have been followed; to see if concepts have been applied correctly; to see if the language of mathematics has been used correctly;
- assessing the students' descriptions of their completed work to see if mathematical language is used correctly; to see if students understand the concepts used;
- providing opportunities for student self-assessment (Have students write explanations of their understanding, opinion, or feelings about an activity. One technique is to have them write under the headings What I Did, What I Learned, and How I Felt About It. Students could be asked to write a review of one day's activities or of the whole unit's work.);
- selecting an exemplary piece of work to be included in a portfolio for assessment purposes or for sharing with parents.

Notes

Overview

Prerequisites

It would be helpful, but not essential, if students begin this unit with the following:

- familiarity with simple fractions (e.g., ‘1/4’ means ‘1 out of 4’);
- ability to write equivalent fractions or ratios (e.g., ‘1 out of 4’ is equivalent to ‘2 out of 8’);
- ability to identify prime and composite numbers less than 10;
- some familiarity with simple percent concepts (e.g., 100% = 1; 3 out of 10 = 30%) and the ability to calculate 100% when a lesser percent is known (e.g., If 3 turtles are 10% of the population in a pond, calculate the total number of turtles.)

Logos

The following logos, which are located in the margins, identify segments related to, respectively:

Problem Solving

Communication

Assessment

Overview

Materials

ACTIVITY	MATERIALS
Activity 1 **Exploring Probability**	• Copies of BLMs 1 and 2 for all students • Large chart for data collection from whole class • Coins or two-colour counters
Activity 2 **Probabilities and Codes**	• Copies of BLM 3 • Copies of BLM 4 (optional) • Acetate copies of BLMs 3 and 4 for overhead projector • Markers/playing pieces (chips, buttons, pen caps, ...) • Large chart for data collection from whole class • Dictionaries and Roget's Thesaurus
Activity 3 **Bottle Caps and Lotteries**	• Red, blue, and purple items (12 of each) • Copies of BLM 5 for all students • Copies of BLM 15 (optional)
Activity 4 **Pizzas and Dandelions**	• Copies of BLMs 6, 7, 8 for all students • Bags of marbles, beads, buttons or coloured discs for BLM 6 • Macaroni and food colouring, or other suitable materials for BLM 8 • Copies of BLM 9 and suitable materials (optional)
Activity 5 **Random or Not?**	• Copies of BLMs 10 and 11 for all students • Copies of BLMs 12 and 13 (optional) • Dictionaries and Roget's Thesaurus • Pages from a telephone book • Standard dice or number cubes or spinners • Copies of BLM 15 (optional)

Overview

Letter to Parents

SCHOOL LETTERHEAD

DATE

Dear Parent(s)/Guardian(s):

For the next week or so students in our classroom will be participating in a unit titled “Probability in Action”. The classroom activities will focus on the nature of probability, fair and unfair games, and the use of probability (in the form of letter frequencies) to decode messages, and to sample a population.

You can assist your child in understanding the relevant concepts by working together to look for situations where probability occurs in everyday life.

Various family activities have been planned for use throughout this unit. Helping your child with the completion of these will enhance his/her understanding of the concepts involved.

If you work with probability in your daily work or hobbies, please encourage your child to learn about this so that he/she can describe these activities to his/her classmates. If you would be willing to visit our classroom and share your experience with the class, please contact me.

Sincerely,

Teacher’s Signature

A Note to the Teacher:

If you make use of the suggested Family Activities, it is important to schedule class time for sharing and discussion of results.

Notes

Activity 1: Exploring Probability

Focus of Activity:

- probability on a continuum from zero to one
- equally likely events

What to Assess:

- understanding of the nature of probability
- use of the language of probability
- identification of equally likely events
- calculation of probability as being one or more favourable outcomes out of all possible outcomes
- collaboration with others

Preparation:

- see the table on page 4 for materials
- make copies of BLMs 1 and 2
- prepare a large chart as described below for collecting all the data from BLM 1

Activity:

Before starting the Activity have students collect data on coin (quarter) spinning at home. Before assigning the activity, show how to spin the coin as illustrated on BLM 1. Practice spinning. Discuss reasons why everyone should use the same type of coin and spin it the same way. Distribute copies of BLM 1 and have students spin the coin 20 times and record the number of heads and tails on BLM 1. Instruct students to collect at least four sets of data at home. If necessary, one person could complete more than one chart. Students could complete the first chart on BLM 1 before going home. Ask students to predict the results of all the spins.

Have students report on their experiments. Ask if they were at all surprised by the results. Ask why they think different people had different results.

Because only 20 spins were made by each person, the expected result that half would be heads and half would be tails is unlikely to occur. However, their total results should be closer to the expected distribution. When all the data is collected together from the class, the results should be even closer to the half-and-half distribution.

Collect all data (a composite of all Chart #7s) in a chart such as the one shown:

Family	Number of heads	Number of tails
ANDERSON	55	45
BROWN	33	67
CAMPBELL	50	50

Notes

'BLM' refers to the Black Line Masters, which follow these Activity Notes.

In our discussion of probabilities in these activities, we are assuming perfectly balanced dice, coins, and spinners. In actuality, most dice, coins, and spinners are not perfectly balanced.

Communication

Activity 1: Exploring Probability

Students could use calculators as each student reports on his/her totals, keeping a running total. Alternatively, you might choose to select only 6-10 students from whom to collect data.

Another way to collect the data would be to prepare a blank on chart paper or the blackboard and instruct students to fill in their family data totals as they come in to class.

Compare individual charts, family charts, and class chart. Discuss with students how close each set of results is to a half-and-half distribution.

Introduce the idea of probability, perhaps in the following manner:
“There are two outcomes when I spin a coin. It will land either heads or tails. I can’t choose which way it will fall, and the coin, of course, cannot choose either. Something called ‘luck’ or ‘chance’ or ‘probability’ causes it to fall one way or the other. If I spun the coin 1 million times, it would surprise me to find that it had fallen tails one million times. I would also be surprised if it fell heads one million times. About how many times in one million would you expect the coin to fall heads? to fall tails?”

Assuming the coin is unbiased, i.e. has no preferred way to fall on a single spin, the probability of heads equals the probability of tails. We say these two ‘outcomes’ are ‘equally likely’.

“Because there are two equally likely ways the coin can fall, and 1 of these is a head, we say that the probability of the coin falling heads is ‘1 out of 2’ or ‘$\frac{1}{2}$’. What is the probability of the coin falling tails? of it falling ***either*** heads ***or*** tails?”

Students may have some difficulty answering the last question. However, they should be able to suggest that the coin must land either heads or tails. This can be illustrated by the following, where ‘P(head)’ means ‘the probability of a head’, and ‘P(tail)’ means “the probability of a tail’. *(This notation is a popular, but by no means universal, way to indicate the probability of an outcome.)*

$$\text{P(head)} + \text{P(tail)} = \frac{1}{2} + \frac{1}{2} = 1$$

Any outcome with a probability of ‘1’ is a ‘sure thing’.

Some students may suggest that there is no such thing as a ‘sure’ thing, and that a coin could concievably land on its edge. This could lead to a worthwhile discussion about the nature of probability (which is a mathematical ideal) and whether or not in practice to include rare outcomes (e.g., a coin landing on an

Activity 1: Exploring Probability

edge) in the data. Students should decide on a reasonable way to deal with these issues. They should continue to apply the same conditions throughout this unit.

Another way to view the probability of '1' is to add the probability that a particular outcome occurs and the probability that it does not occur:

P(heads) + P(not-heads) = 1;

P(odd number) + P(not-odd number) = 1;

P(rain today) + P(not-rain today) = 1.

Students should be willing to accept that '1' represents a 'sure thing', or at least an 'almost-sure' thing.

Notes

Either an outcome occurs, or it does not occur.

Present students with some possible experimental outcomes (such as those on BLM 2) and ask if these outcomes are equally likely. You might wish to discuss #1 and #2 with the whole class to be sure that they understand the ideas of 'equally likely', 'more likely' and 'less likely', and can describe outcomes using these descriptions.

Distribute copies of BLM 2 to student pairs/groups, and allow time for them to complete the activity.

Problem Solving

Students should realize that each outcome described in #5 has a probability of 1 or 'almost 1' and is a 'sure thing' (or as close to a sure thing as possible). Have students describe other 'sure things'. For example:

- i) the sun will rise tomorrow;
- ii) I will be older tomorrow than I am now;
- iii) sometime in the next year it will snow somewhere in Canada.

Develop the idea that a probability of '0' or 'almost 0' indicates that an outcome is impossible (or almost impossible), perhaps as follows.

Assessment

Ask students:

"What is the probability of drawing a black ball from the bag in #1?" "What is the probability of <u>not</u> spinning an even number using the spinner in #3c?" "What is the probability of spinning an odd number?"

Since none of these outcomes can occur, we say the probability of each is '0'. Have students describe other outcomes with probabilities of zero. For example:

- i) rolling a '9' using a regular die;
- ii) drawing a red marble from a bag of blue marbles;
- iii) being younger tomorrow than today.

Activity 1: Exploring Probability

Communication

To wrap up the activity, have students write about what they have learned:
"Suppose you are trying to explain today's activities to an absent classmate. What would you say/write?"

Extensions in Mathematics:

1. Explore ways that outcomes depend on conditions. For example:
   (a) Stick a bit of clay/plasticine on one side of a coin and spin it several times. Do heads and tails appear to be equally likely?
   (b) Flip a coin rather than spinning it. Whether the coin is showing heads or tails before the flip may have some bearing on the way it falls.

2. Spin a Canadian penny and an American penny. Spin several times. Are both coins equally likely to fall heads and tails? Experiment with coins from other countries.

Cross-curricular Activities:

1. Explore the use of probability in weather forecasts. What is meant by 'a 20% chance of rain'? Where in the world would the probability of rain be close to zero? close to 1?

Family Activities:

1. Use tiddly-winks or bingo chips or two coins. Flip one with the other the way you do for playing Tiddly-Winks. Mark a 10-cm diameter circle on a piece of paper. Draw a starting line 30 cm from the circle. Try to flip a chip/coin into the circle from behind the starting line. Each successful attempt gets a score of one. Allow each person 30 tries and record his/her score.

   Use the scores to answer the following:
   (a) Who, in your family, is most likely to get a high score?
   (b) Who is more likely to get a low score than a high score?
   Give reasons for your answers.

Other Resources

For additional ideas, see annotated "Other Resources" list on page 57, numbered as below.

10. "Dealing with Data and Chance: Addenda Series, Grades 5-8", Judith S. Zawojewski et al.
12. "What Are My Chances?", Creative Publications
21. "Organizing Data and Dealing with Uncertainty", NCTM

Activity 2: Probability and Codes

Focus of Activity:

- exploring letter frequencies

What to Assess:

- accuracy of letter counts
- reasonableness of predictions
- application of letter frequencies to decoding
- collaboration with others

Preparation:

- make copies of BLM 3
- make copies of BLM 4 (optional)
- provide markers for the race game (a marker can be a playing piece from another game, the cap of a pen, a bit of eraser, etc.)
- prepare a chart for all the class data (see below) on blackboard or chart paper; a similar chart for the five most common consonants will also be needed.
- have available dictionaries, books of various sorts, and copies of Roget's Thesaurus

'BLM' refers to the Black Line Masters, which follow these Activity Notes.

Activity:

Every printer and sign maker knows that some letters are used far more frequently in English than others are. The same is true with other languages but different letters are common. If your class includes students with other languages or with other languages in the home, you may wish to incorporate other languages in this Activity.

Letter frequencies can be determined by simple counting (of a wide range of sources) but the use of the race game adds interest for students and ensures careful counting since there will be a feeling of competition.

Distribute copies of BLM 3 to student groups. If groups of 5 are not suitable, fewer students could be in each group and some could choose more than one vowel to track.

You may wish to discuss ways of making the task more manageable. For example, one student could be responsible for 'reading' the vowels in the chosen selection. He/she would simply name each vowel as it appears and the other students would move their markers around the race track.

Once all groups have 'raced' the vowels two or three times, collect all the data in a chart such as the one below. Students could use tally marks to record their own winners as they finish the second or third race, whichever you select. In this partially completed chart, six groups had 'E' coming first and three groups had 'A' coming first.

Activity 2: Probability and Codes

<table>
<tr><th>Vowel / Place</th><th>A</th><th>E</th><th>I</th><th>O</th><th>U</th></tr>
<tr><td>First</td><td>|||</td><td>𝍸 |</td><td></td><td></td><td></td></tr>
<tr><td>Second</td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>Third</td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>Fourth</td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>Fifth</td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>Totals</td><td></td><td></td><td></td><td></td><td></td></tr>
</table>

The letter 'E' is by far the most commonly used letter in English. It will probably win the race the greatest number of times.

The frequencies of vowels will vary depending on the source. The given frequencies are reasonable for the English language.

In order, the next most frequent vowels are A and 0 (tied), I, and U. The observed percentages of occurrence for the five vowels are:

E - 13%,
A - 8%,
O - 8%.
I - 6.5%,
U - 3%.

Because the percentages of A, O, and I are close in value, several student 'races' may have placed these in different orders.

If students are not conversant with percents, relative frequencies can be given as 13 out of 100, 8 out of 100, etc.

Once the class data are collected, have students compare the frequencies shown on the chart with their own group efforts. Reinforce the idea that the more data that is collected, the closer the experimental frequencies will be to the given frequencies above.

Ask students what consonant they think is most commonly used in English. Make a list of the most common consonants according to student belief. You might wish to make a list of all suggestions and then have students vote for the consonants they think are the 'top three'. Then, when these votes have been tabulated, select the five most popular predictions and have the students replace the five vowels with those consonants on the race track. Each group should 'race' these consonants at least twice before recording their data on a large class chart. Ask students:

"What appears to be the most common consonant in English?"
"Is it possible that there might be a more common one?"

Students should be asked to give reasons for their answers.

Activity 2: Probability and Codes

Some students may have suggested ‘S’ as the most common letter, since most dictionaries have more words beginning with ‘S’ than with any other letter. This idea might also give students reasons for labelling such letters as ‘X’ and ‘Z’ as letters that are not common, since there are relatively few words in the dictionary beginning with ‘X’ or ‘Z’.

The‘ top five’ consonants in English are T, N, R, S, and H. Their observed percentages are

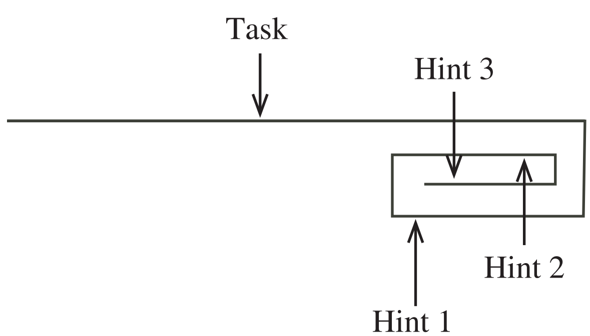
T - 9%
N - 7%
R - 6.5%
S - 6%
H - 5.5%

Activity 2 can be terminated here, or you may wish to continue with the following optional activity.

Optional

Distribute copies of BLM 4: Decoding a Secret Message.

You may wish to fold the hints out of sight before distributing the copies of the task. The page is designed so that you can fold Hint 3 back first, then Hint 2, and then Hint 1. Thus the only part visible to the students is the task itself. (See diagram below showing side view of folded BLM.) Instruct students to try decoding without looking at the hints, but when they get stuck, they should unfold Hint 1 and try the decoding again. This allows students who resist hints because “I want to do it myself” the opportunity to do so, while other students can use the hints as they feel necessary.



Allow time for students to read the instructions. If you have not folded the sheets as suggested above, have the students fold the hints back or cover them with a sheet of paper until they want to use them.

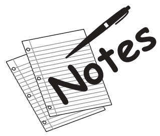
Ask them how knowing letter frequencies might help them in decoding the message.

Students enjoy encoding messages for their classmates to decode. Encourage them to use everyday English and not deliberately to use, say, far more ‘O’s than ‘E’s.

Notes

Problem Solving

Assessment



Activity 2: Probability and Codes

Students could also be asked to write one or two hints to help others decode their messages. Thinking about the hints causes students to think about how the decoding will be done, and may encourage them to write more realistic messages.

Problem Solving

Extensions in Mathematics:

1. Record letter frequencies on graphs. For example,
   (a) graph the number of times each vowel occurs in a given paragraph;
   (b) graph the number of times each vowel 'won' the race, using class results;
   (c) give each vowel a score (e.g., 5 for first, 4 for second, ... 1 for fifth) and graph the total score of each of the vowels.

For different types of graphs, see "Investigations in Data Management: TV - Vice or Virtue?" See order form at the end of this booklet.

2. Explore the frequencies of consonant combinations such as TH, SH, ST, or vowel combinations (diphthongs) such as EA, AI, OU.

3. Students could calculate who has the "most frequent" initials, or name. See "Solutions and Notes" for some data on some famous mathematicians' initials.

Cross-curricular Activities:

1. Ask students for some letter combinations they think are rare, and some that are common. Have them select one of each and write as many words as they can containing that combination of letters. Check with a dictionary.

Communication

2. Try to write an English sentence at least 10 words long
   (i) with no 'E's;
   (ii) with no 'T's;
   (iii) with no 'E's <u>or</u> 'T's.
   Use a dictionary or thesaurus to find words.

3. Students work in pairs or small groups to write a script for a brief dialogue without using particular letters or particular words (e.g., the, these, those, and).

Family Activities:

1. Explore letter frequencies in a different language. For example, is 'E' the most commonly used letter in French? in Greek?

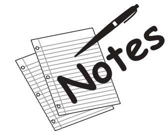
2. Students and their families might like to explore whether or not knowing letter frequencies could help them win on the popular television show "Wheel of Fortune".

Other Resources

For additional ideas, see annotated "Other Resources" list on page 57, numbered as below.

19. "Mode Code" David B. Spangler

Activity 3: Bottle Caps and Lotteries



Focus of Activity:

- the use of tree diagrams to list possible outcomes

What to Assess:

- use of tree diagrams to determine all possible outcomes
- logical statements in solving the problems
- collaboration with others

Preparation:

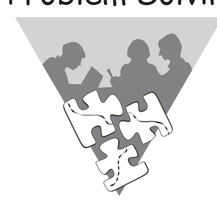
- provide items in red, blue, and purple to represent the bolts in the problem
- make copies of BLM 5
- make copies of BLM 15 if you wish the students to have notes on tree diagrams

'BLM' refers to the Black Line Masters, which follow these Activity Notes.

Activity:

Refer to contests such as those involving letters or pictures in the lids of soft drink bottles or printed on box tops. Ask students if they have ever run across such a contest or if they have tried collecting all the necessary elements. Tell them that this Activity will explore the probabilities of winning such events.

Problem Solving



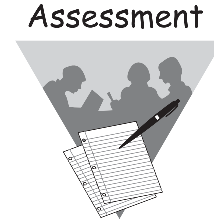
Present the following scenario:

> "Suppose a robot, RTD3, needs a pair of bolts every day to replace worn pieces. His manager likes to use 2 bolts of the same colour each day. All the bolts, red and blue, are in a box on a high shelf. The manager reaches in each day, hoping to get two matching bolts. What is the least number of bolts he must pull out in order to be sure of having 2 of the same colour? Could he do this by taking out 2? Why or why not?"

You may wish to model this for students using blocks or scraps of paper of different colours in an envelope. Have a students draw a 'bolt' without looking into the envelope. Have another student draw a second 'bolt'. If the colours do not match, have a third student draw a 'bolt'.

Repeat several times and record the number of draws needed. Students may at first be surprised to find that they never need a fourth draw.

Assessment

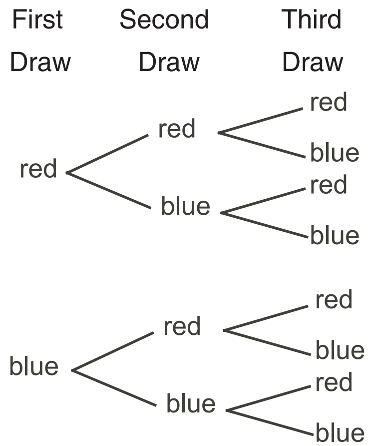


Eventually, they should be able to reason as follows:

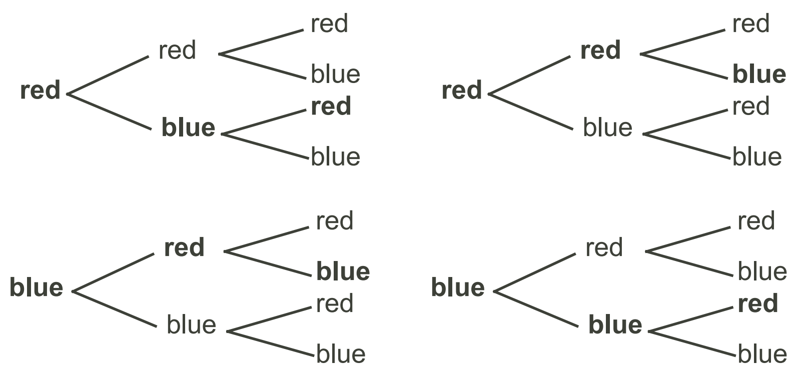
> "Suppose the first bolt is red.
> Suppose the second bolt is blue.
> Since the third bolt has to be either red or blue it will always match one of the first two and only three draws are needed."

Activity 3: Bottle Caps and Lotteries

A tree diagram is another way to illustrate all possible outcomes (draws of three bolts):



It can then be seen that every possible combination of 3 bolts has at least two of one colour. For example, follow the branches in bold type.

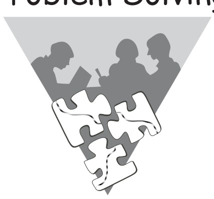


Ask students:

> "How many of the eight possible combinations of three bolts have 2 red bolts? How many have 2 blue bolts? Did you expect this? Why?"

For more on tree diagrams, see BLM 15.

Problem Solving



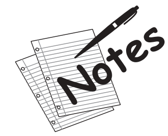
After discussing this scenario, present a variation on the problem:

> "Suppose there were 3 colours of bolts in the box (red, green, and purple). What is the minimum number of bolts the manager would need to draw out in order to be sure of getting a matching pair?"

Allow students a few minutes for thinking or talking this out with group members before describing possible solutions to the whole class.

If a third colour is added, the manager needs to draw only 4 to be sure of having a matched pair. Students could reason thus:

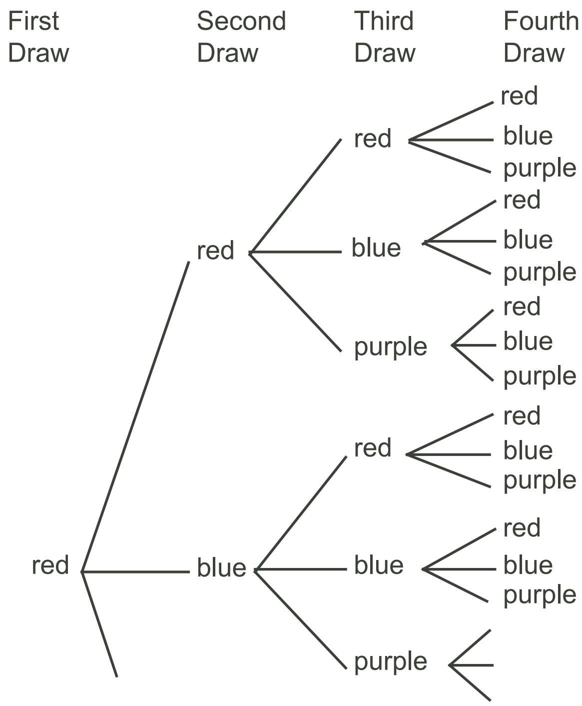
> "Even if the first three bolts are all different colours, the 4th one must match one of the first three."



Activity 3: Bottle Caps and Lotteries

A tree diagram can be used to indicate all possibilities. With three choices for each draw, there are far more possibilities than with two colours. A partial tree is shown below.

If you choose to use tree diagrams in this activity, students should be made aware that sometimes the tree diagram can become unwieldly.



Present a third variation on the problem.

"One day the manager decides to use two bolts of different colours.
(a) What is the least number he must draw from the box of red and blue bolts to be sure this happens?
(b) What is the least number he must draw from the box of red, blue, and purple bolts to be sure this happens?
Do you have all the information you need to solve this problem? Why or why not?"

Problem Solving

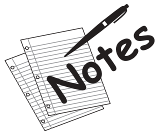
Allow students time to discuss this in pairs/groups. If some students need help getting started, give some direction with questions such as:

"Suppose the first, second, and third bolts are blue. Could the fourth bolt be blue? Could the fifth?"

Communication

Students should eventually realize that they need to know how many of each colour are in the box. Once they come to that conclusion, give them the needed information:

The first box holds 12 of each colour (red and blue).
The second box holds 9 each of red, blue, and purple.



Activity 3: Bottle Caps and Lotteries

Although it is unlikely to happen,the manager could pull out all 12 blue bolts in a row before getting a red one. Thus, the minimum number of bolts he needs to draw from the first box is 13.

For the second box, he needs to draw 10 to be sure of getting two different colours. Ask what minimum number must be drawn if the manager wants three bolts of different colours.The answer is 19 bolts. That is, he could draw all the red (9 bolts) and all the blue (9 bolts more) before drawing a purple to give three different colours. Thus 19 is the minimum number that must be drawn to be certain that he will have one of each colour.)

Problem Solving

Note that solutions given above are for the first draw only. For subsequent draws, one must take into account that some bolts have been removed from the box. For the first problem, the manager is using one of each colour, so there will be 11 of each colour left for the next draw, if the manager returns unused bolts to the box after obtaining one of each colour. Thus, the minimum number the manager needs to draw on the second day to be certain of getting at least one of each colour is 12. Students could follow this problem to its logical end when there will be only 1 red and 1 blue bolt left in the box.

Communication

The bolt problems may not be realistic situations, but they build a foundation for exploring other problems such as those on BLM 5 and in Extension 1. Students should work in pairs or small groups for these.

These three problems have been carefully structured so that solving each helps to solve the next, starting with number 1 on BLM 5.

If students do not finish during the class period, this would be a good opportunity to involve other family members. Students could ask their families for their suggestions and report back to their groups the next day.

When students finish, have them report to the class. Each group should be prepared to justify their solutions. See “Solutions and Notes”, for possible responses.

Students could be asked to write their solutions, along with their reasons. Such a solution would be a good item for a portfolio.

Since Extension 1 is fairly lengthy, you may wish to make a copy for each student or group.

Extensions in Mathematics:

1. The Super Saver Store had a contest. Each customer was allowed to draw three letters from a large drum containing the letters of the store’s name: S,U,P,E,R,A,V,T, and O. If the customer drew three ‘S’s (the store’s initials), that customer would get a 50% discount on any purchases just made.

Activity 3: Bottle Caps and Lotteries

(a) The drum will hold 900 letters. If you were asked by the owner to design a contest that would be neither very easy nor very difficult to win, how would you do this?

(b) If the contest were to be very easy to win, how would you do this?

(c) How could you run an honest contest but still make it very difficult to win?

(d) Would your answers to (a), (b), and (c) change if the letters were returned to the drum after every draw? Explain.

This is not a trivial problem and students will need some careful reasoning. Since "easy to win" and "hard to win" are value judgements, expect student answers to vary.

Family Activities:

1. Examine the probabilities of winning a local lottery. Determine how many prize-winning possibilities there are.

2. Investigate a variety of lotteries. Why are some very expensive (i.e., $100) and some inexpensive (i.e., $2)?

Other Resources

For additional ideas, see annotated "Other Resources" list on page 57, numbered as below.

10. "Dealing with Data and Chance: Addenda Series, Grades 5-8", Judith S. Zawojewski et al.
13. "Truth or Coincidence?", Daniel J. Brohier
20. "Cat and Mouse", David Lannen
22. "Calendar Mathematics", Lorna J. Morrow

Notes

Activity 4: Pizzas and Dandelions

Focus of Activity:

- sampling to determine/predict characteristics of a population

What to Assess:

- accurate record keeping
- logical reasons given for predictions

Preparation:

'BLM' refers to the Black Line Masters, which follow these Activity Notes.

- make copies of BLMs 6, 7 and a copy of BLM 6 for use with the overhead
- make copies of BLM 8
- make copies of BLM 9 (optional)
- prepare bags of marbles/beads/coloured disks for BLM 6. Each bag should contain the same collection of items. Solutions are given for bags containing 5 red, 3 green, and 1 blue item.
- prepare materials for use with BLM 8. For example, a container of pieces of macaroni with a few pieces coloured with food colouring; several white bread ties and a few coloured ones; several buttons with holes, and a few with shanks; several plain animal crackers and a few marked with food colouring.
- prepare materials for use with BLM 9 (optional)(See Extensions.)

Activity:

The general rubric in "Suggested Assessment Strategies" (following Solutions and Notes) has been adapted for this problem, as a sample to assist you in assessing your students.

Communication

Tell students that each group is going to be given a bag of marbles and, without looking into the bag, will be expected to predict the contents of the bag.

Student groups will be asked to share results with one other group. This will work well if the class is divided into an even number of groups.

Show BLM 6 on the overhead, directing students to the 9 statements below the chart. Explain that they are to take 3 marbles at a time from the bag, record the colours, and tell if each of statements A to I is true for that sample.

Display three marbles, 2 red and 1 blue. These colours are already recorded as RRB in the square labelled "Sample (i)" on the chart. Ask if statement A is true (no; it is false because the marbles in the sample are not all the same). Repeat for the other statements, recording T or F under "Sample (i)". Students should give reasons why a statement is true or false for a particular sample. For the sample suggested above (R R B), statements C, D, E, and I are true. The others are false.

Some of statements A to I are open to interpretation. For example, statement C could be taken to mean "There is exactly 1 red marble" or "There is at least one red marble". Similarly, for statements E, H, and I. This is worth discussing if students raise the question. The class should then decide on the interpretation that will be used for this activity. The second interpretation is the one used for given solutions.

Activity 4: Pizzas and Dandelions

You may wish to draw a second sample and label each statement as true or false (record as "Sample (ii)"), if students are having difficulty determining the truth or falsity of the individual statements. Once they are comfortable with the process, distribute copies of BLM 6 and tell students to complete six trials. For each trial, they should draw 3 marbles without looking, record the colours of this draw as one of the trials, mark each statement as true or false for that sample, and then return the marbles to the bag.

Once students have completed the 6 trials, ask them to use their recorded results to predict (in their groups) the makeup and number of the marbles in the bag. Some students may need some help with this. Direct them to pieces of data they should be using by asking such questions as:

"How many colours do you think there are? Why?"
"Are there more of one colour than of another? What makes you think so?"

Problem Solving

Students should be able to make such statements as:
"We think there are three colours because we drew three different colours in all our samples."
"We drew only red and green so we think there are only two colours in the bag."
"We think there are three colours, but we never had more than 1 blue, so we think there is only 1 blue in the bag."

Tell the students that there are 9 marbles in the bag, and ask if they want to change their predictions.

Communication

Once the students have written their predictions and reasons, have the groups pair up. Each pair of groups should exchange their predictions and reasons, then consider all their data together and adjust their predictions if necessary.

Groups should present their conclusions to the class. Using all the data, the class should make final predictions about the contents of the bag, and then open the bags to check. If predictions are not accurate, students should suggest why.

This type of sampling is similar to the type used by scientists who want to determine the makeup of a population. Give students examples, such as:

"A biologist might wish to know how many red squirrels and how many grey squirrels there are in a forest"
"A person planning to open a toy store might want to know how many 5-6 year olds, how many 7-10 year olds, and how many 11-14 year olds there are in the neighborhood near his planned store."

Ask students to suggest other examples.

A simpler type of sampling is used to determine the size of a population. For example, a biologist might want to know how many trout there are in a lake; a statistician might want to know how many adults live in a particular city; an insurance agent might want to know how many people over 60 live in her sales area.

Notes

Activity 4: Pizzas and Dandelions

BLM 7 presents some simple sampling problems. Distribute BLM 7 to pairs/groups and allow them time to think about and discuss the problem. If they seem to be having difficulty, you may want to complete #1 a) together. Students should see that they should

(i) count the number of pieces in each ‘sample’, and
(ii) calculate the average.

Problem Solving

Since there are 8 slices of pizza, the number of pieces on the whole pizza should be about eight times as much as the average number of pieces on one slice. In #1 a), the average is about 5 pieces per slice. Students should realize that this assumes the pepperoni slices to be evenly distributed.

Note that in #1 c), it is possible that both pizzas in #1 a) and b) had the same number of pepperoni slices since

(i) we do not know if the pepperoni slices were evenly distributed, and
(ii) pizza makers try to put the same number of slices on each pizza.

Allow students time to calculate/estimate the total number of dandelions in #2 a) and b).

Ask them to suggest other situations in which this type of sampling might be used.

If students suggest such things as estimating the number of bears in a forest, point out that situations like these involve things that move (e.g., bears, unlike pieces of pepperoni and dandelions, do not tend to stay in one spot) and that the sampling technique needs to be different.

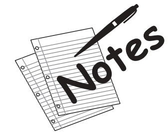
BLM 8 shows a different way of sampling that is often used in studying wildlife, if you want to determine the size of a population. Draw a group, count the number of items in the group, tag them in some way, and replace them. This tagging is necessary because the animals being studied don’t stay put the way pepperoni slices do. The next step is to wait for your tagged animals to mix in with the non-tagged animals. Then draw a sample which we hope will have both tagged and untagged animals. The ratio of tagged to untagged animals in the sample is assumed to be close to the ratio of tagged to untagged animals in the whole population. Since we know the total number of tagged animals, we can estimate the total number of animals.

Ideally, these ratios should be the same:

$$\frac{\textit{tagged in sample}}{\textit{sample size}} = \frac{\textit{total tagged}}{\textit{total population}}$$

For example, suppose we tag 10 animals and release them. After a suitable wait for them to mix with the whole population, we take a sample of 10, and discover that 2 of these 10 are tagged. This suggests that 2 out of 10 animals in the total population are tagged. Since we initially tagged a total of 10 animals, we conclude that $\frac{2}{10}$ or $\frac{1}{5}$ or 20% of the population is 10 animals. Therefore we estimate that the whole population contains about 50 animals, as follows:

Activity 4: Pizzas and Dandelions



20% of the population is 10 animals, so
100% is $5 \times 20\%$ or 5×10 or 50 animals

Distribute BLM 8 and read through it with students. Work through #1-5 as an example, using one of the containers of 'wolves'. Then have pairs or groups of students complete the activity. Once students have made a final estimate (#8) they should count the total number of items in the container to see how close their 'estimated' total was.

Ask why biologists might want to take more than one sample, as the students did. Elicit the idea that more samples should give estimates closer to the actual number.

Ask students to describe other situations in which this type of sampling would be useful.

Cross-Curricular Activities:

1. A different type of sampling is used when trying to determine the makeup of a population, for example, to determine the number of red, grey, and black squirrels in a certain area.

   Distribute copies of BLM 9 and have students work in pairs/groups to complete the Activity.

2. This activity uses an inflatable or light-weight globe. If you do not have one, sketch the continents and other major land masses on a beach ball.
   Students toss the ball to each other. Whenever someone catches the ball he/she looks to see if his/her right thumb is on land or water.
   Record the number of times a thumb is found on land and the number of times the thumb is on water.
   Ask students what conclusions they can draw from this. (e.g., "There is more water than land on earth." "The probability of landing on water is greater than the probability of landing on land.")
   Students could investigate the percent of land and water on earth or the percent of land and water for individual continents or countries using encyclopedias or geography books.
   Students could try to find photos of earth taken from space by astronauts or satellites. Then they could compare their findings with the percentage of "thumb-landings" on land versus water in the activity.

Communication

Other Resources:

For additional ideas, see annotated "Other Resources" list on page 57, numbered as below.

12. "What Are My Chances?" Creative Publications
18. "Data and Chance", NCTM

Activity 5: Random or Not?

'BLM' refers to the Black Line Masters, which follow these Activity Notes.

Focus of Activity:
- random numbers and the meaning of randomness

What to Assess:
- accuracy of data collection
- understanding of the concept of randomness
- collaboration with others

Preparation:
- make copies of BLMs 10 and 11.
- make copies of BLMs 12 and 13 (optional)
- have available dictionaries, and copies of Roget's Thesaurus
- provide pages from a telephone book
- provide a standard die for each student pair/group. If dice are not available, provide copies of BLM 14 and have students construct spinners.
- make copies of BLM 16 if you wish students to have notes on random numbers

Activity:

Tell students that previous activities have involved selecting items "at random". Ask what they think is the meaning of the word "random". Have students investigate the word in dictionaries or a thesaurus. (See BLM 16) Have students describe events that might be called 'random' if the dictionary meaning is used.

Mathematicians use the word in a precise way. Sometimes a set of random numbers is used to select items to be sure that they are not chosen in any deliberate way. In a set of random numbers every number stands an equal chance of occurring and the occurrence of any number is not dependent on the occurrence of any other number.

For example, in a list of 200 random one-digit numbers, we would expect to find approximately twenty of each digit from 0 to 9, and these would be scattered throughout the list of 200 digits with no apparent pattern.

In order to explore the idea of randomness further, distribute BLM 10 to students and instruct them to complete the first set of numbers by writing "0" or "1" in each box, starting at the top left and working across the rows one at a time. They should try to write the numbers at random (i.e. without any pattern). You may wish to give a time limit so that students do not have time to count the digits as they are writing or to plan their placement.

Activity 5: Random or Not?

When students have completed the task have them count the number of each digit (#2 on BLM 10). If the list is random, ‘0’ should occur approximately as often as ‘1’. Have students write replies to #3 on the back of BLM 10.

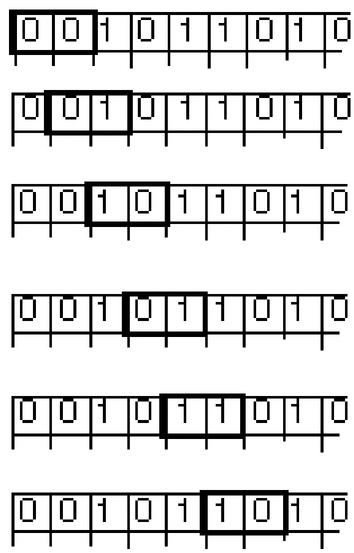
Ask students how many of them had approximately the same number of zeros and ones. If the digits were equally distributed there would be 24 of each. In practice, a 23 to 25 split or 22 to 26 or even 21 to 27 should be considered as “approximately the same number of each”.

Communication

You may wish to collect data from all pairs/groups on the chalkboard, and compare the total number of zeros to the total number of ones. If one of these appears more often than the other, ask students why this might have happened. What were they thinking as they wrote the digits? Was something making them think of one number more often than the other?

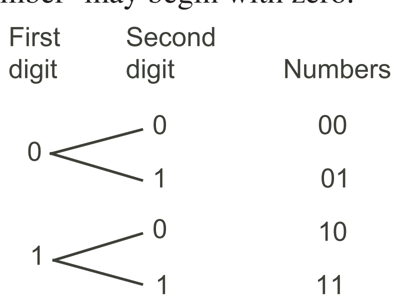
Counting the numbers of zeros and ones is one test of randomness, i.e. testing for an equal distribution of items (the zero and one).

A second test involves counting the two-digit numbers in the list. This helps to verify that there is not a repeating pattern. The method of grouping the terms is shown on the right. Notice that each individual digit is used as the first digit of a two-digit number, so that the two-digit numbers overlap. The last digit in a row can combine with the first digit of the next row.



Ask students how many two-digit numbers there will be in the chart. They should see that every digit in the chart, except for the last one, is the first digit of a two-digit number. There should, therefore, be 47 two-digit numbers, or 48 if the last digit in the chart is paired with the first digit in the chart to give the 48th two-digit number.

Ask students how many <u>different</u> two-digit numbers there are. You might wish to suggest the use of a tree diagram to count the possibilities. Notice that, for this activity, a ‘number’ may begin with zero.

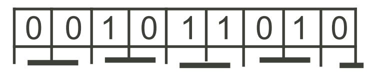


Notes

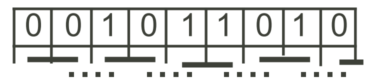
Activity 5: Random or Not?

Have students count the number of times each two-digit number occurrs in their own charts. This may be easier to do if students work in pairs and select one of their charts to test. Keep the other chart to test for three-digit numbers (see below).

One way to keep track of the numbers is to underline consecutive two-digit numbers in one colour and count them.



Then underline the 'overlapping' two-digit numbers in another colour and count them.

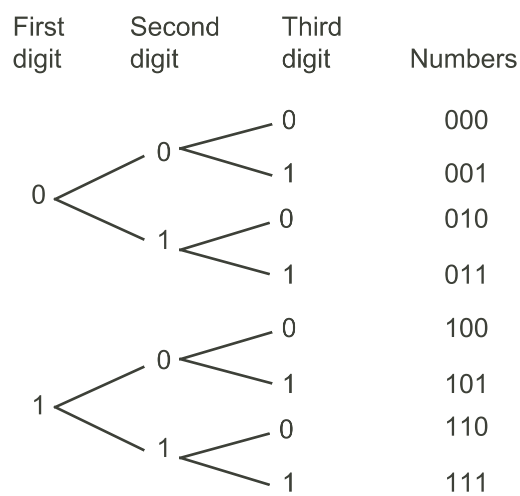


If the sequence is random, then we would expect each of the four two-digit numbers to occur about $\frac{1}{4}$ of the time. That is, the probability of '00' is the same as the probability of '01' or of '10' or of '11'. Or, symbolically:

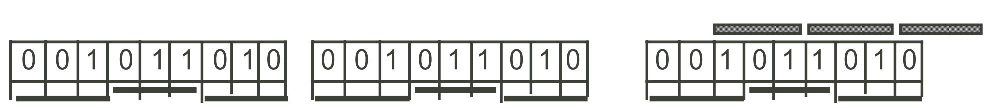
$$P(00) = P(01) = P(10) = P(11).$$

Compare the results across the class. A few students may be close to random but most probably will not be. Although '11' and '00' are as likely as '01' and '10', many people, in trying to write random numbers, try not to have two of the same digit together very often because they somehow feel that "this is not random".

Next, have the students count all the three-digit numbers in their charts. As with the two-digit numbers, they should first identify all possible different three-digit numbers.



Colour can be used once again to identify the three-digit numbers for easy reading. Notice that the third colour is placed above the numbers for clarity.



Students will probably find that this test shows that their numbers are not random. The numbers '000' and '111' will probably be much rarer than any of the other three-digit numbers.

Activity 5: Random or Not?

Distribute copies of BLM 11. Students should work in pairs or groups to complete it. BLM 11 illustrates two possible methods of generating random numbers. Students should test both techniques by counting the single digits and the two-digit numbers in the charts. The die should generate numbers that prove to be random. Many telephone pages will, as well, but some may show a bias.

Optional

Random numbers can be used to simulate experiments. See Cross-curricular Activities.

Extensions in Mathematics:

1. Students should construct a chart similar to the one on BLM 11 in which to record the rolls of the die. Since they will be recording six different digits, rather than just 0 and 1, the chart should have space for more than 48 digits. Alternatively, each group can record 48 digits and then two groups or the whole class can combine their results before testing for randomness.

2. Generate a set of random numbers using a programmable calculator or a computer. See "Other Resources" (#16) for an activity using Logo software.

3. Describe how you could use a coin to generate a set of random numbers.

Cross-curricular Activities:

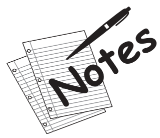
1. A simulation exercise using random number tables is outlined on BLM 12. This simulation involves analyzing a lottery based on #1, BLM 5, from Activity 3. An extensive random number table is given on BLM 13.

2. A politician wants to poll his constituents. How could he/she use a random number table to assist in selecting the sample to whom the questionnaire will be sent? Hint: The politician will have a list of names and addresses of people in his/her riding.

3. Explain how a random number table could be used for each of the following:
   (a) choosing numbers for a lottery ticket;
   (b) selecting people on one street to receive free samples of a new product;
   (c) choosing students in your school to answer a questionnaure about home work.

Family Activities:

1. Students could have family members try to write random zeroes and ones as on BLM 10. They could have a "contest" to see which family member came closest to writing random numbers.



Activity 5: Random or Not?

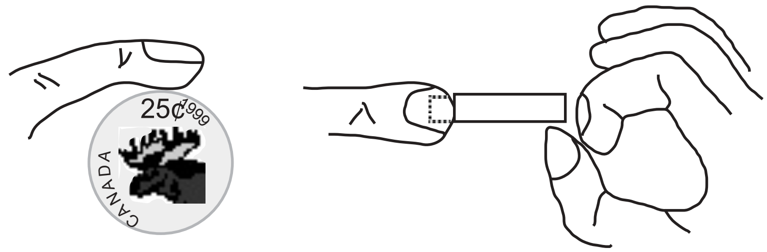
Other Resources:

For additional ideas, see annotated “Other Resources” list on page 57, numbered as below.

8. “Making Sense of Data: Addenda Series, Grades K-6”, Mary Lindquist et al.
14. “Exploring Random Numbers”, William R. Speer
16. “Looking at Random Events with Logo Software”, Thor Charischak and Robert Berkman

BLM 1: Quarters, Heads, and Tails

For this experiment you will need to spin a quarter as shown in the diagrams, and observe whether it lands with 'heads' or 'tails' facing up.



Hold the coin upright with one index finger, and flick it with the other index finger.

Spin the coin 20 times and record, in Chart #1, the number of heads (H) and tails (T) that occur. Ask other members of your family to spin the coin 20 times each. Record the heads and tails in Charts 2 to 6. Be sure to collect at least 4 sets of data

Chart #1:

	Tally	Number
H		
T		

Chart #2:

	Tally	Number
H		
T		

Chart #3:

	Tally	Number
H		
T		

Chart #4:

	Tally	Number
H		
T		

Chart #5:

	Tally	Number
H		
T		

Chart #6:

	Tally	Number
H		
T		

Write the totals from Charts 1 to 6 in the chart below, and find their sums.

Chart #7:

from Chart	Number of heads	Number of tails
#1		
#2		
#3		
#4		
#5		
#6		
Totals		

BLM 2: Equally Likely Events and Sure Things

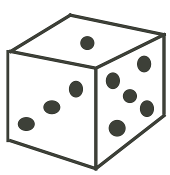
1. Imagine you are drawing marbles one at a time from a bag containing 1 red, 1 blue, 2 yellow, and 3 green marbles. Each time, you draw the marble without looking into the bag, and you return the marble after recording its colour.

   What is the probability of each of the following outcomes?
   (a) drawing a red marble
   (b) drawing a blue marble
   (c) drawing a green marble
   (d) drawing a yellow marble
   (e) drawing a red <u>or</u> a blue marble

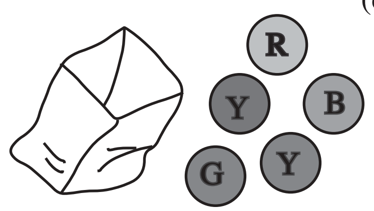
2. Which of the outcomes in #1 are equally likely?

3. Each illustration below suggests an experiment.
   (i) Describe each experiment.
   (ii) Give two equally likely outcomes for each experiment.

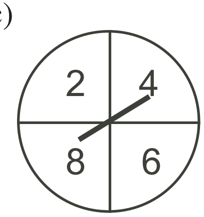
(a)



(b)



(c)



(d)

4. For each experiment in #3 give two outcomes that are not equally likely. For example, using the die in (a), rolling an even number is more likely than rolling a 4. This can be written as P(even number) $\neq$ P(4).

5. Give the probability of each of the following outcomes:
   (a) drawing a red, blue, yellow, or green marble from the bag in 3(b)
   (b) spinning an even number using the spinner in 3(c).
   (c) rolling an even number <u>or</u> an odd number using a regular die as in 3(a).

BLM 3: Bet on Your Favourite Vowel!

This is an activity for 5 players. If you do not have 5 in your group, then at least one of you will have to 'race' with two of the vowels.

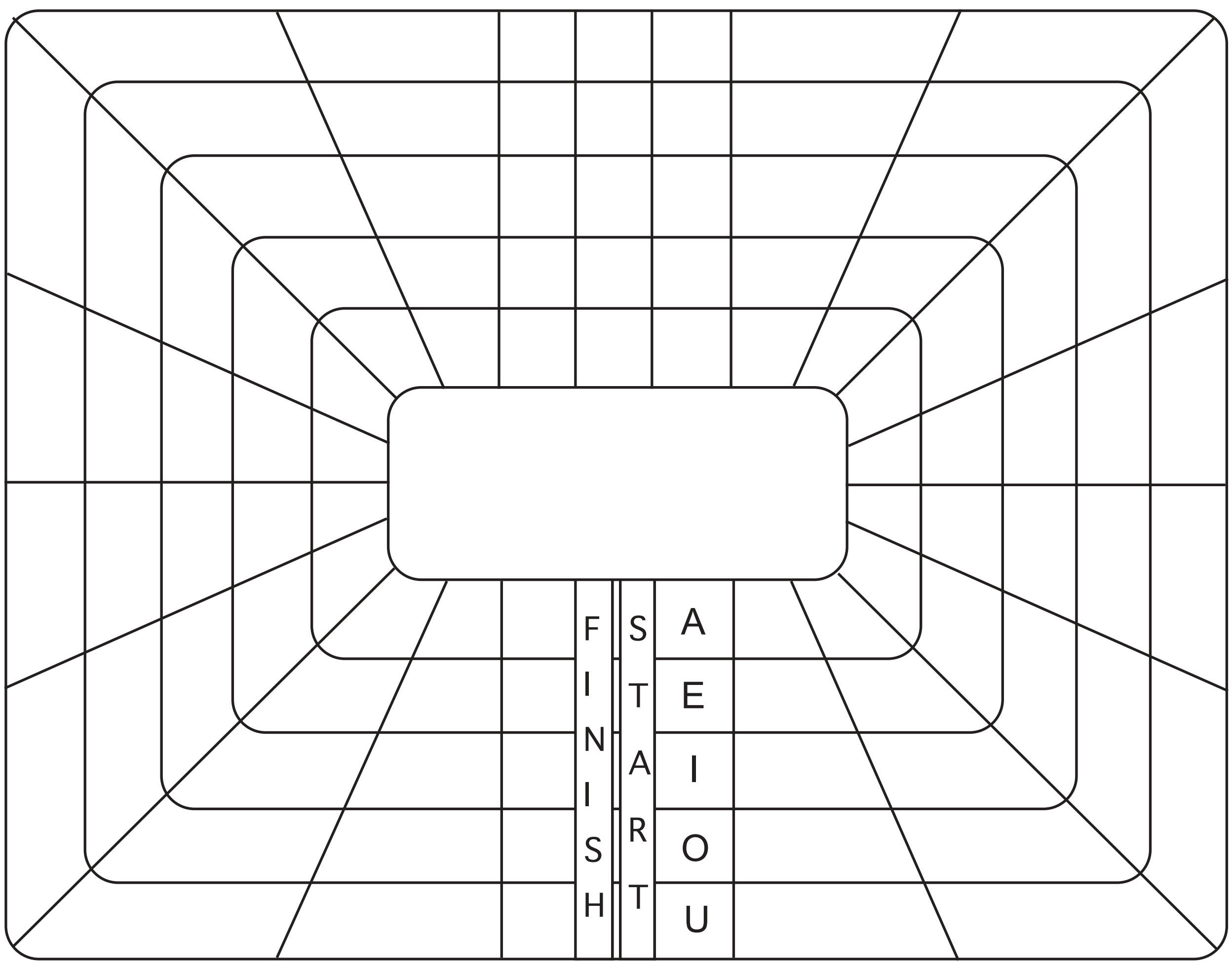
1. Each of you should pick one of the vowels and place a marker on that vowel at the START of the race.

2. Select a page from any book — fiction, history, drama, science, or any other.

3. Count the vowels letter by letter. Each time your vowel occurs, move your marker one space along the race track. Play until at least 3 vowels have finished the race.

Before beginning, predict which vowel you think will win. Why? What vowel will be second? Why do you think so?

4. Which vowel did win? Which one came second? third? If you were to play till all the vowels reached the FINISH line, which vowel do you think would be last? Why?



5. Try the race again using a different book. Did you get the same results? Did other groups in your class get the same results?

BLM 4: Decoding a Secret Message

Code-breakers (cryptologists) use information such as letter frequencies to break simple substitution codes. In the sentence below, each letter stands for another. For example, the letter **D** will stand for the same letter every time it occurs. Use what you know about letter frequencies to decode the message.

DMRA EXCEFX GC RCQ JRCL QBMQ QBX NQOGA CS

DMQBXDMQTUN UMR BXFE QBXD WXMG CW LWTQX

M UCGX.

There are three hints below this dotted line. Do not look at the hints until you need them. Then look at one hint at a time, and again try decoding the message before looking at the next hint.

- -

Hint 1: A one-letter word in English is either **I** or **A**. (Can you think of any other one-letter words?) In this message the one-letter word is **A.** This means that the letter **M** can be changed to the letter **A** wherever it occurs.

- -

Hint 2: QBX stands for the most common three-letter English word. What do you think it is? It is the word “the”. Thus, **Q** stands for **T**, **B** stands for **H**, and **X** stands for **E**. What word might **QBMQ** stand for?or **QBXD?**

- -

Hint 3: DMQBXDMQTUN stands for the name of a subject you study in school. You are using that subject to decode this message.

BLM 5: Lotteries

1. Suppose you have a bag of marbles in different colours, and you are asked "What is the least number of marbles you must draw to be certain you have at least one of each colour?"

   (a) What information about the marbles do you need to answer this question?

   (b) If there are four colours, and six of each colour, what is the least number of marbles you must draw to be certain you have at least one of each colour?

   (c) If the bag contains 10 black marbles, 8 red marbles, 5 green marbles, and 2 orange marbles, what is the least number of marbles you must draw to be certain you have at least one of each colour?

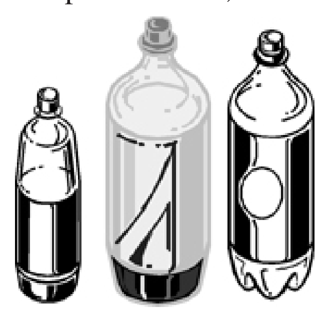
   (d) Toby said the answer to (c) was 16 marbles, because you might draw the 2 orange, the 5 green, and the 8 red before drawing any black marbles. Do you agree with Toby's answer? Why or why not?

2. The Frothy Cola Company decided to run a contest using letters under the bottle caps of their new Licorice flavoured cola. They put the letters F, R, O, T, H, and Y under the caps, one letter per cap. Any customer who found all six letters won 6 cases of Frothy.

   The president of the company, Mr. Froth, wanted some idea of the probable number of winners, so he asked his advertising manager, Ms. Cola, for an estimate.

   She told him the number of bottles of Frothy Licorice Cola a person would have to buy to be absolutely sure of winning.

   Mr. Froth was surprised because he thought the number would be '6' — one with the letter 'F', one with the letter 'R' and so on. But Ms. Cola knew something about frequencies and probabilities, so she told him that
   12 million bottles had a F under the lid;
   10 million bottles had an R under the lid;
   5 million bottles had an O under the lid;
   2 million bottles had a T under the lid;
   1 million bottles had an H under the lid;
   but only 100 bottles had a Y under the lid.



   (a) If all the bottles were purchased during the contest, how many winners could there be? Would there necessarily be that many winners? Why or why not?

   (b) What number had Ms. Cola given Mr. Frothy? That is, how many bottles would you have to buy to be certain of winning?

BLM 6: Do You Have All Your Marbles?

Below the chart below are several statements about a bag of marbles.
1. Draw a sample of three marbles, and record the colours .

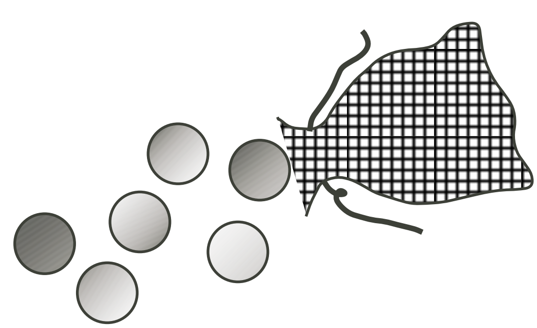
2. Decide if statement A is true or false for the sample drawn, and write 'T' or 'F' in the chart.

3. Continue for the other statements.

4. Draw four more samples and complete the chart for each.

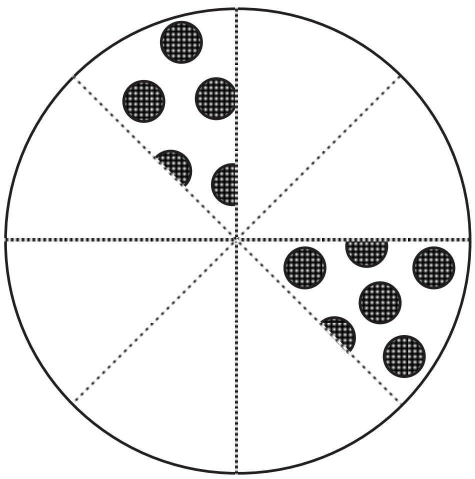
Colours ——>	Sample (i) RRB	Sample (ii)	Trial #1	Trial #2	Trial #3	Trial #4	Trial #5
Statement A							
Statement B							
Statement C							
Statement D							
Statement E							
Statement F							
Statement G							
Statement H							
Statement I							

Statement A: All marbles are the same colour.
Statement B: All marbles are red.
Statement C: There is 1 red marble.
Statement D: The marbles are not all the same.
Statement E: There are 2 red marbles.
Statement F: There is 1 red marble and 2 marbles that are not red.
Statement G: None of the marbles is blue.
Statement H: There is 1 green marble.
Statement I: There is 1 blue marble.

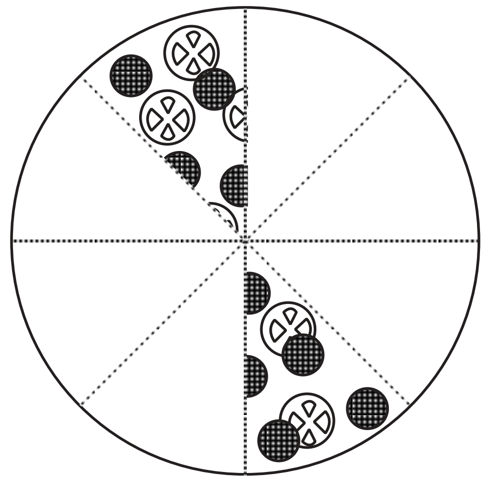


BLM 7: Counting Pepperoni

1. (a) Two pieces of pizza are shown below. Each circle on the pizza represents a slice of pepperoni. Estimate the number of slices of pepperoni that were on the whole pizza.



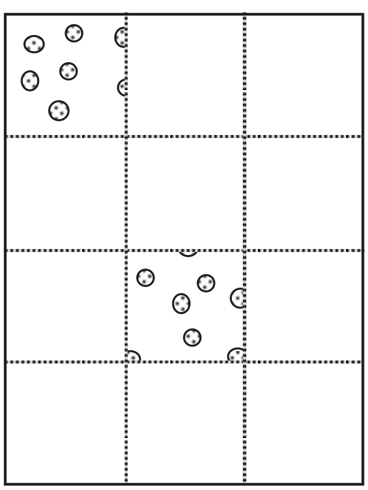
(b) Two pieces of pizza are shown below. Each shaded circle represents a slice of pepperoni, and each of the other circles represents a slice of tomato. Estimate the number of slices of pepperoni and of tomato that were on the whole pizza.



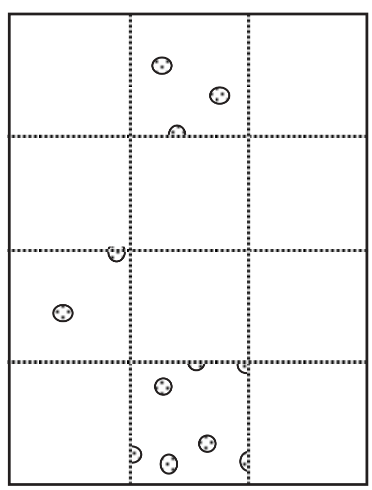
(c) Is it possible that both pizzas had the same number of pepperoni slices? Explain.

2. The diagrams below show dandelions growing in different sections of a lawn. Approximately how many dandelions are growing in each whole lawn?

(a)



(b)



(c) Is it possible that both lawns have the same number of dandelions? Explain.

(d) How few dandelions could each lawn have? Explain why you think this.

(e) What additional information do you need in order to be sure that your answers to (a) and (b) are accurate?

BLM 8: How Many Wolves?

You will need counters of some kind. For example, two types of macaroni, coloured and uncoloured. You will need 50 - 100 uncoloured and 10 coloured. You will also need a container from which to (draw) ‘capture’ groups of 10 pieces of macaroni.

For this Activity, the macaroni pieces represent wolves. You are a wildlife biologist and you want to know how many wolves are in the forest (the container).

1. Reach into the container without looking, and ‘capture’ 10 ‘wolves’.
2. Replace each uncoloured piece with a coloured piece. The coloured pieces represent the wolves that you have tagged.
3. Mix all the wolves together. Now you are ready to ‘sample’ the population by capturing 10 more wolves.
4. Count the ‘tagged wolves in this group (i.e., count the coloured ones), and record this in the chart for group 1.

Group	Group size	Number of coloured pieces (tagged wolves)
1	10	
2	10	
3	10	
4	10	
Average		

5. Return Group 1 to the container, and mix all the wolves together again.
6. Repeat steps 3, 4 and 5 three more times, and record the numbers of tagged wolves as Groups 2, 3, and 4.
7. Calculate the average of the number of tagged wolves for Groups 1-4.
8. You can now estimate the total population using the equation

$$\text{Total number of wolves} = \frac{100}{\text{Average number of tagged wolves per group}}.$$

BLM 9: Let's Go Fishing

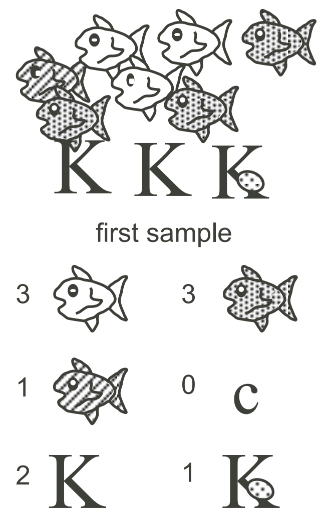
You will need a collection of several items such as different types of pasta in a container. The total number of items should be 100. Each type of pasta will represent a different kind of fish or crab or frog similar to the ones shown in the example below. You could also use a collection of items in different colours. In that case, each colour could represent a different type of water animal. You are going to use sampling and probability to predict the number of each to be found in a particular pond.

1. "Fish" up a sample of 10 items, and record the number of each type of pasta (i.e. of each type of animal). Then return the sample to the container.

2. Based on this sample, what might you predict about the percentage or fraction of each each type of animal in the total population?

EXAMPLE:
In this example, 10 animals were taken from a pond that contained 3 types of fish, 2 types of crabs, and 1 type of frog. From the sample shown, the biologist recorded:

- dark-coloured fish: 3 out of 10 or 30%
- light-coloured fish: 3 out of 10 or 30%
- striped fish: 1 out of 10 or 10%
- spotted crabs: 1 out of 10 or 10%
- non-spotted crabs: 2 out of 10 or 20%
- frogs: 0 out of 10 or 0%



Based on this sample, we might predict that there are no frogs in the pond. Would this be a reasonable conclusion? Explain.

3. Mix all the animals together, and fish up another sample of 10. Count the number of each type, and add these to the numbers already recorded. Then return the sample of 10 to the "pond".

4. Repeat step 3 eight more times, so that you have taken 10 samples altogether.

5. Your totals now provide an estimate of the number of each type of animal out of a total of 100 animals. Dump out the contents of the container, and check the accuracy of your estimates by counting all the items of each type. Were your estimates close? Why or why not?

6. Why is it necessary for wildlife analysts to take several samples? Which of the following should be considered in this type of sampling:
   - size of each sample?
   - time of year the sample is taken?
   - location of the samples (all in one area versus in different areas)?

7. What other factors might be considered in determining the population of wild animals? Explain.

BLM 10: Numbers at Random

1. Write a ‘1’ or a ‘0’ in each box across the page in the first row, then the second row, and so on. Try to write the numbers at random.

2. Count the number of zeros and ones in your chart.

	Tally	Total
Number of zeros		
Number of ones		

3. If the digits are truly random, approximately how many of each would you expect? Did this happen with your chart? Why or why not?

4. Count the number of two-digit numbers in your chart. Be careful to count them all. Write the missing two-digit numbers in the first column.

	Tally	Total
Number of 00		
Number of 01		
Number of		
Number of		

Here’s a hint for a way to help you keep track of the two-digit numbers

BLM 11: Generating Random Numbers

1. Use a regular die. Roll it 48 times. If you roll an even number, write '0' in the box; if you roll an odd number, write '1' in the box.

2. Test to see if this is a chart of random numbers by
   (a) counting the number of zeros and ones; about how many of each should there be?
   (b) counting the number of two-digit numbers in the chart; about how many of each should there be?

3. (a) Do these numbers seem to be random?
   (b) Are they more or less random than the numbers you wrote on BLM 10?
   (c) Is rolling a die a good way to generate random numbers? Explain.

4. For another method of generating possibly random numbers, use a page from a telephone book. Look at the last number of each telephone number. If the number is even, write '0' in the box; if the last digit is odd, write '1' in the box.

5. Test these numbers by counting the number of one- and two-digit numbers in the chart. Do these numbers appear to be random?

6. If you needed to generate random numbers for an experiment, which, of the three ways you have tested, would you use? Explain.

BLM 12: Simulations

Suppose Mr. Froth (see BLM 5) wants the same number of each letter under the bottle caps. He asks Ms. Cola how many bottles someone would need to buy in order to guarantee one of each letter.

Ms. Cola decides to use a random number table (see BLM 13).
First, she gives each letter of "Frothy" a numerical value. For example, F = 1, R = 2, O = 3, T = 4, H = 5, Y = 6.

1. However, the random number table also has zeros, sevens, eights, and nines. What do you think Ms. Cola might do about this? Why?

2. Ms. Cola decides to cross out the numbers she doesn't want.
   The necessary part of the first row of the random number table would start with:

   6 6 3 6 ▒ 4 ▒ ▒ 5 2 3 2 ▒ 6 ▒ 5 ▒ 5

   What would the second row look like?

3. Then Ms. Cola matches a letter to each remaining number in the first row. Complete her list of letters for the initial part of the first row:

   6 6 3 6 4 5 2 3 2 6 5 5
   Y Y O Y T _ _ _ _ _ _ _

4. Does Ms. Cola have at least one of each letter?
   How much more of the first row will she need to use to get at least one of each letter?
   How many numbers in the first row did she need altogether to be sure she would have at least one of each letter? (Don't count the zeros, 7s, 8s, or 9s.)

5. Now repeat the matching of letters to numbers using the second row. How many numbers would Ms. Cola need to be sure of having one of each letter using this row?

6. Ms. Cola tried several samples, and then calculated the average to estimate the number of bottles of Frothy a person would have to buy in order to be sure to win a prize.

   Take ten samples and record the results. Then calculate the average number of bottles of Frothy a person would need to buy to win. Does this sound reasonable? Explain.

   Share your results with other groups. Using all the trials from the class, calculate the average number of bottles purchased to win.

BLM 13: Random Number Table

The numbers are divided into groups of five to make reading easier.

66360	47852	32769	59586	00133	72584	26480	00245	48371	37526
22043	77224	26075	68778	87332	83287	54373	96391	82132	89338
78519	43251	18412	30777	14380	13550	37902	46169	27785	10488
58454	13026	26618	18537	44015	73261	42001	06096	21918	94440
00666	78245	32662	03375	54485	89848	90606	55556	49481	35329
80043	26080	72508	53576	49390	35273	86769	07108	66688	24636
53787	10007	66163	88811	21977	92078	95503	43655	57975	25768
88907	42653	05541	13459	89731	89459	98306	55222	32363	68675
76654	24020	67332	62362	65014	18061	92185	08657	92167	47793
11675	96819	10965	31214	39215	29883	34235	27113	22919	31278
90066	91253	59174	58312	84990	52539	64054	34864	00483	17913
29480	78114	48305	67868	85176	50048	62792	82816	52055	93273
93992	71132	91042	96303	11372	13817	15490	19452	08265	57612
79938	37498	27019	18573	88617	31245	60208	53962	52981	04301
20506	31384	51173	33453	93156	43166	33599	98112	09422	48744
43006	16020	49784	09917	50236	59837	18739	85767	49111	51512
45186	04205	76923	06181	81538	68226	73500	60779	65584	24305
49966	94867	62902	43090	37205	72584	78048	98669	83267	13303
62224	77713	14540	24003	20499	32752	42271	75891	45681	44445
73217	21643	46106	73942	02936	45948	74850	17297	44957	31068
11219	20296	59367	31426	31166	66247	54764	91861	83130	37507
02164	54666	21868	65824	97370	23627	39822	29285	31387	17045
73171	27920	41254	60089	00693	58712	88187	56810	92728	07894
48435	58944	61989	84538	67060	69031	28814	31405	82384	77694
45687	46494	61920	26751	54241	09903	71831	98113	33094	99925
64573	28270	63695	16900	25980	61906	38832	44327	01141	37889
36345	24793	88754	95921	99442	30336	07705	41314	53028	07381
37402	15236	64920	25909	25085	85456	00198	32419	54583	83635
27358	35142	91012	35570	50420	30509	44150	99868	77894	05250
17222	24172	26021	79527	44721	19041	04399	74266	15134	17952
48436	19800	03441	60218	83099	10869	27264	06777	70388	34992
08752	26430	45080	80472	35599	34343	90581	46482	13441	74151
79075	92335	12474	33423	72174	02953	37198	97172	98019	92623
73073	26360	19111	65852	87760	41988	77620	83328	24394	23932
48418	80642	09023	48310	25218	79006	12709	39456	02883	83600
01362	30222	93728	16044	23187	40562	71067	13330	11022	17378
38148	24320	87981	57518	37136	04182	67913	88235	61865	24638
27411	82008	23860	45246	03403	97639	28686	67623	00542	63666
48322	46340	31022	55657	58297	36244	25091	75297	14695	75932
38823	78043	75095	58043	95125	74783	24693	06360	66853	66663

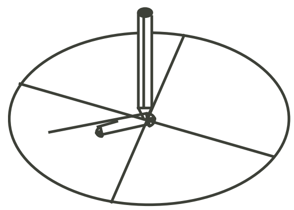
BLM 14: Constructing Spinners

To construct spinners, use the templates at the bottom of the pages. Paste the spinners on to bristol/cardboard.

Method 1:

For the spinner, straighten a paper clip as shown below.

Hold the spinner in place with a pen or pencil at the centre of the circle.



Flick the point of the paper clip with a finger.

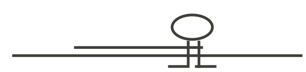
This is the simplest way to construct an acetate spinner for use with an overhead projector.

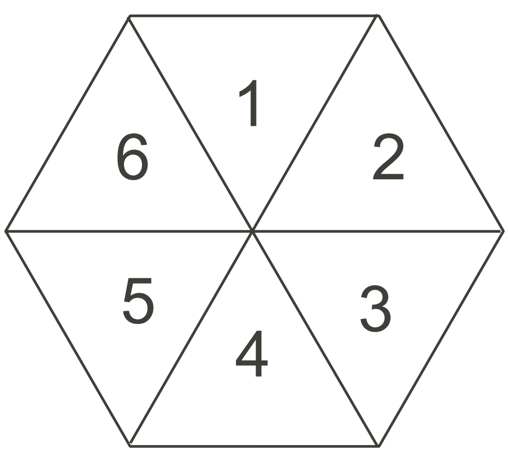
Method 2:

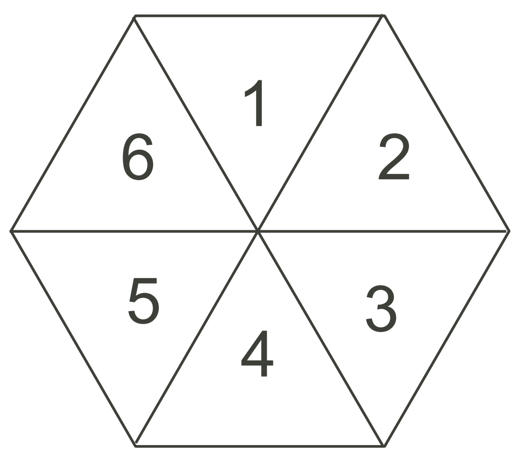
Cut arrows from bristol board or cardboard and punch a hole in one end.

Punch a hole in the centre of each spinner.
Use a paper fastener to fasten the two pieces together.

The connection should be tight enough so the arrow doesn't wobble, but loose enough so that it spins freely.







BLM 15: Tree Diagrams

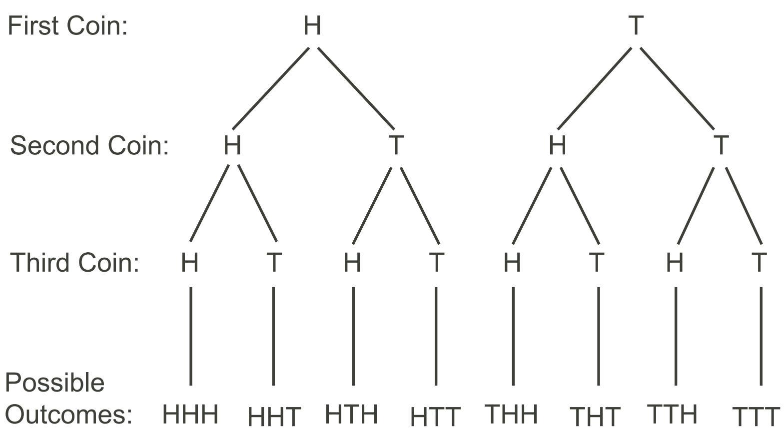
A tree diagram is a way of counting all possible outcomes for a simple experiment. For example, suppose we want to identify all possible outcomes for flipping three coins.

Step 1: List the possible outcomes for the first of the three coins.

Step 2: Draw 'branches' from each of these outcomes. The number of branches will be the number of possible outcomes for the second coin — that is, two.

Step 3: Draw branches for each of the possible outcomes of flipping the third coin.

Step 4: Read down the chart from the top to identify 8 different combinations — that is, the eight possible outcomes when three coins are flipped.



The number of possibilities for the first coin is 2.

For each of these, the number of possibilities for the second coin is 2.

For each of these, the number of possibilities for the third coin is 2.

Thus, there are 8 possible outcomes:

$$\mathbf{2 \times 2 \times 2 = 8}.$$

Tree diagrams are not useful if there are too many outcomes — for example, rolling three dice (216 outcomes) or even rolling two dice (36 outcomes).

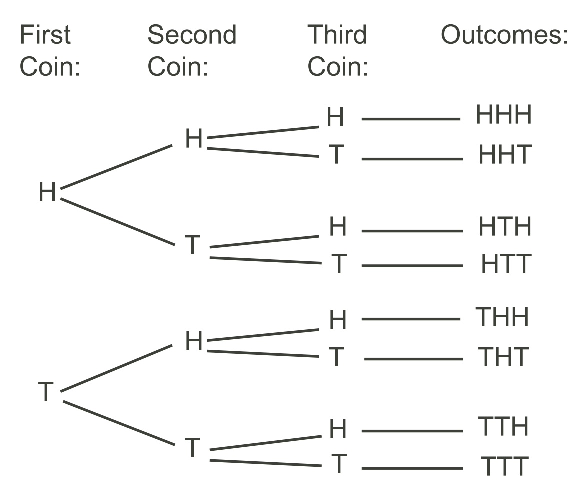
Tree diagrams are useful, however, for determing the outcomes of experiments like the following:

(a) flipping a coin and rolling a die;

(b) the number of outfits possible with rust, green, black, and cream t-shirts and brown, green, and orange shorts;

(c) spinning two or three spinners

Tree diagrams can also be drawn horizontally:



BLM 16: Random Numbers

Frequently, in studying probability, mathematicians use random numbers to simulate experiments. Random numbers (sets of single digits) have two characteristics:

1) Each digit has the same probability of occurring
2) Adjacent digits are independent of one another. That is, the occurrence of one digit has no effect on the occurrence of any other.

When we use the word "random" in everyday language we frequently mean only that something appears to happen haphazardly. The mathematical meaning is more precise and rigorous.

> Definitions of "random":
> 1) from Roget's Thesaurus: casual, aimless, by chance, accidental, unintended, purposeless.
> 2) from James and James "Mathematical Dictionary": A random sequence is a sequence that is irregular and non-repetitive. The probability of a particular digit being chosen is the same for each digit, and the choices at two different places are independent.

To illustrate that what we think is random often isn't, consider the following study. Students are being given a quick quiz in which they write only the answers. The first 5 questions are

i) How much is $3+3$? ii) How much is $5+5$?

iii) How much is $9+9$? iv) How much is $13+13$?

v) Name a whole number between 5 and 12.

One might expect that students' responses to the last item would range equally from 6 to 11. In actual fact, the number '7' was the overwhelming favourite. Reasons suggested included the following:

i) The other questions dealt with computation, so students did a computation using 5 and 12 $(12-5=7)$
ii) The previous questions dealt with odd numbers, so perhaps students automatically gave an odd number for question (v).

Whatever the reason, even though the students thought they were choosing a number at random, something was acting to produce '7' as a response. (For more on this see Other Resources # 17.)

Another study involved text-book writers. All writers felt that in dealing with basic facts (addition, say, at grade 2) they had managed to include all facts about equally often. That is, that each fact included in a practice page was chosen randomly. However, an analysis of the manuscripts showed otherwise. They had concentrated on the "more difficult" facts to the detriment of "adding 1", "adding 0" or "adding doubles'. It would appear that their knowledge of the practice students need was preventing them from choosing facts in a truly random manner even when they tried to do so.

If you choose to discuss these meanings of random with your students, you might wish to have them suggest situations which may, on the surface, appear to be random, but are probably not.

Solutions & Notes

Activity 1: Fair Spinners

BLM 2:

1. (a) Since there are 7 marbles, the probability of drawing the 1 red marble is ‘1 out of 7’ or ‘$\frac{1}{7}$’. Symbollically, we can write P(red) = $\frac{1}{7}$.

   (b) The probability of drawing the 1 blue marble is ‘1 out of 7’ or ‘$\frac{1}{7}$’. P(blue) = $\frac{1}{7}$.

   (c) There are 3 green marbles. Thus a green marble can be drawn in three ways (that is, there are 3 favourable outcomes: green #1, green #2, or green #3) and the probability of drawing a green marble is ‘3 out of 7’ or ‘$\frac{3}{7}$’. P(green) = $\frac{3}{7}$.

   (d) Since there are 2 yellow marbles, the probability of drawing a yellow marble is ‘2 out of 7’ or ‘$\frac{2}{7}$’. P(yellow) = $\frac{2}{7}$.

   (e) There are two favourable outcomes here - a red or a blue. Thus, the probability of drawing either a red or a blue is ‘2 out of 7’ or ‘$\frac{2}{7}$’. P(red or blue) = $\frac{2}{7}$.

2. Parts (a) and (b) describe equally likely outcomes. The probability of drawing a red marble, or ‘P(red)’ equals the probability of drawing a blue marble, or ‘P(blue)’.
   or
   Parts (d) and (e) describe equally likely outcomes. P(yellow) = P(red or blue)

3. Answers will vary. Possible responses are given below

   (a) (i) The experiment is the rolling of the die.

   (ii) Two equally likely outcomes are rolling a 1 and rolling a 2. That is, P(1) = P(2) = $\frac{1}{6}$.

   (b) (i) The experiment could be drawing marbles from a bag.

   (ii) P(red or blue) = P(yellow) = $\frac{2}{5}$.

   (c) (i) The experiment could be spinning the spinner.

   (ii) P(2) = P(6) = $\frac{1}{4}$.

   (d) (i) The experiment could be flipping the two coins

   (ii) P(2 heads) = P(2 tails) = $\frac{1}{4}$.

Solutions & Notes

4. (a) Two outcomes that are not equally likely are the probability of rolling an odd number and the probability of rolling a 2. P(odd number) $\neq$ P(2). P(odd number) is 3 out of 6, and P(2) is 1 out of 6.

   (b) P(red) $\neq$ P(yellow). P(red) = $\frac{1}{5}$ and P(yellow) = $\frac{2}{5}$.

   (c) P(even number) $\neq$ P(8). P(even number) = $\frac{4}{4}$ or 1 and P(8) = $\frac{1}{4}$.

   (d) P(2 heads) $\neq$ P(one head and one tail). The possible outcomes are heads on both coins, tails on both coins, heads on the first and tails on the second, or tails on the first and heads on the second.
   Thus there are four outcomes, so P(2 heads) = 1 out of 4
   P(1 head and 1 tail) = 2 out of 4.

5. (a) Since red, blue, yellow, and green are the only possible colours, drawing one of them from the bag would be a 'sure thing' with a probability of 5 out of 5 or 1.

   (b) Since all the numbers on the spinner are even, the probability of spinning an even number is 4 out of 4 or 1.

   (c) Similarly, all the numbers on a standard die are either odd or even, so the probability of rolling either odd or even is 6 out of 6 or 1.

Extensions in Mathematics

1. (a) Whether or not the modelling clay makes a difference may depend on where it is placed, how the coin is spun and how much clay is used. For example, in testing this suggestion, the authors found that if the clay was on the tail surface of the coin, and held so that it was facing the experimenter and had the clay at the top, the coin was slightly more likely to fall showing a head. The result of 100 unprejudiced spins was 45 heads and 55 tails with the clay on the tail surface.

   This situation reminds one of the belief that when a slice of buttered bread falls, it always lands buttered side down. Believe it or not, this situation was actually the focus of a scientific study a few years ago. It was discovered that things that are dropped tend to turn, and, if they are dropped from a particular height, they will turn through 360° and land right side up. Unfortunately, for those of us who may be a bit clumsy, the height from which bread is usually dropped (counter - or table-height) is just enough to allow the bread to rotate through 180° and land buttered side down. Of course this assumes a simple drop. As we try to catch the bread we may impede its steady turning.

Family Activities

Students should realize that the probability of someone winning a game of skill cannot be given a specific value but can only be described as 'more likely' or 'less likely' than the probability of someone else winning.

Solutions & Notes

Activity 2: Probability and Codes

BLM 3

The observed frequencies of all letters as given in Reference #19 are shown below in two charts. In the first, the letters are listed alphabetically. In the second, they are listed in order of frequency.

A	8%	J	0.5%	S	6%
B	1.5%	K	0.5%	T	9%
C	3%	L	3.5%	U	3%
D	4%	M	3%	V	1%
E	13%	N	7%	W	1.5%
F	2.5%	O	8%	X	0.5%
G	1.5%	P	2%	Y	2%
H	5.5%	Q	0.3%	Z	0.2%
I	6.5%	R	6.5%		

E	13%	D	4%	G	1.5%
T	9%	L	3.5%	W	1.5%
A	8%	C	3%	V	1%
O	8%	M	3%	J	0.5%
N	7%	U	3%	K	0.5%
I	6.5%	F	2.5%	X	0.5%
R	6.5%	P	2%	Q	0.3%
S	6%	Y	2%	Z	0.2%
H	5.5%	B	1.5%		

Extensions in Mathematics

3. Some famous mathematicians and the frequencies of their initials are given as samples.
   (i) Blaise Pascal: 1.5% and 2% or $\frac{1.5}{100}$ and $\frac{2}{100}$. Frequency of either is $\frac{3.5}{100}$. That is, in a normal passage in English, Pascal could expect to see one of his initials approximately 3.5 times for every hundred letters.

   (ii) Frederick Gauss: 2.5% and 1.5% or $\frac{2.5}{100}$ and $\frac{1.5}{100}$. Frequency of either is $\frac{4}{100}$.

   (iii) Sophie Germain: $\frac{7.5}{100}$.

   (iv) Emmy Noether: $\frac{20}{100}$.

BLM 4

The decoded message reads

MANY PEOPLE DO NOT KNOW THAT THE STUDY OF MATHEMATICS

CAN HELP THEM READ OR WRITE A CODE.

Solutions & Notes

Activity 3

BLM 5

1. (a) You need to know how many marbles of each colour are in the bag.

   (b) You must draw 19 marbles. For example, you could draw all 6 of each of the first three colours before drawing one of the fourth colour.

   (c) You must draw 24 marbles to be certain of getting one of each colour, because you could draw all 10 black, all 8 red, and all 5 green, before drawing a single orange.

   (d) Answers will vary. Students should point out that you could draw 16 marbles by drawing 10 black and 6 red or 10 black, 5 green, and 1 orange, or 8 black and 8 red or any other combination of 16 marbles that does not contain all four colours.

2. (a) There ***could*** be 100 winners.
   There could not be more than 100 since there are only 100 'Y' bottles.
   However, a customer could buy a 'Y' bottle without buying bottles with the other 5 letters, and then there would be fewer than 100 winners.

   (b) 30 million and 1. In order to be sure of winning, you would need to buy all the 'F' bottles, all the 'R' bottles, all the 'O' bottles, all the 'T' bottles, and all the 'H' bottles. Then you would need to buy only one of the 'Y' bottles. This is, of course, not likely to happen but it is the only way to be ***certain*** of winning.

Extension

1. (a) Answers will vary but students may suggest having the same number of each letter; or having, say, 200 'S's and equal numbers of the other letters.

   (b) For a ***very*** easy contest, all the letters would be 'S's. Students should realize that the more 'S's there are, the more likely it is that a shopper will win.

   (c) Students should realize that the fewer the 'S's, the harder it will be to win the contest.

   (d) Yes. If the 'S's were returned to the drum and there were, say, 200 'S's, then shoppers would stand a good chance of winning throughout the contests.

   If students are interested in determining the number of three-letter combinations that are possible, suggest that they imagine a tree diagram. The first letter can be any one of 9. The second letter can be any one of 9, and so can the third. The first letter drawn is S or U or P or ... or O. Each of these letters will have 9 branches to indicate the second letter drawn. Students should see that this means there are 81 combinations of two letters. On a tree diagram, each of these 81 combinations would have 9 branches to indicate the third letter drawn. Thus there are , or 729 possible three-letter combinations.

Solutions & Notes

Activity 4

BLM 6

All possible combinations from a bag of 5 red, 3 green, and 1 blue are given below.
Statements C, H, I are interpreted as "There is at least one red/green/blue marble".
Statement E is interpreted as "There are at least 2 red marbles".

Colours ⟶	RRR	RRG	RRB	RGG	RGB	GGG	GGB
Statement A	T	F	F	F	F	T	F
Statement B	T	F	F	F	F	F	F
Statement C	T	T	T	T	T	F	F
Statement D	F	T	T	T	T	F	T
Statement E	T	T	T	F	F	F	F
Statement F	F	F	F	T	T	F	F
Statement G	T	T	F	T	F	T	F
Statement H	F	T	F	T	T	T	T
Statement I	F	F	T	F	T	F	T

BLM 7

Answers will vary: possible solutions are given below.

1. (a) The two pieces of pizza shown have about 4 and 5 pieces of pepperoni. This gives an average of .
If we assume that the pepperoni was scattered evenly across the pizza, then the 8 slices would contain about 36 pepperoni slices.

   (b) Using the same counting technique as in (a) above, this pizza had about 28 slices of pepperoni and 18 slices of tomato.

   (c) If the pepperoni was not spread evenly across the whole pizza, then both pizzas could have had the same number of pepperoni slices.

Solutions & Notes

2. (a) About 64-74 dandelions (b) About 30-38 dandelions

   (c) The estimates in (a) and (b) assume that the dandelions are scattered evenly over the whole lawn. However, this doesn't usually happen, so it is possible that both lawns have the same number of dandelions.

   (d) If each lawn has on it only the dandelions showing, then (a) would have 15 and (b) would have 12.

   (e) We need to know if the dandelions are scattered evenly (or nearly evenly) to have any confidence in our answers to (a) and (b).

BLM 8

Answers will vary.

BLM 9

Answers will vary.

Activity 5

BLMs 10 and 11

Answers will vary.

BLM 12

1. Answers will vary. Any logical suggestion should be accepted. The simplest is to cross out the unwanted numbers as stated in #2.

2. The first part of the second row will be

   2 2 ■ 4 3 ■ ■ 2 2 4 2 6 ■ ■ 5 6 ■ ■ ■ ■ ■ ■ 3 3 2

3. The letters will be Y Y O Y T H R O R Y H H

4. Ms. Cola still has no '1' and therefore no letter 'F'. She needs to continue in the first row until she reaches an '1'. Altogether, she will use 14 numbers. This is comparable to buying 14 bottles of Frothy to win.

5. Using row two, Ms. Cola needs 23 numbers. This is like buying 23 bottles of Frothy to win.

6. Answers will vary. Check to see that students are crossing out (or otherwise eliminating) the unwanted numbers. Ask students to devise a soft drink name that will make use of all the different digits in the random number table, and to estimate (by sampling) the average number of bottles purchased in order to win.

Suggested Assessment Strategies

Investigations

Investigations involve explorations of mathematical questions that may be related to other subject areas. Investigations deal with problem posing as well as problem solving. Investigations give information about a student's ability to:

- identify and define a problem;
- make a plan;
- create and interpret strategies;
- collect and record needed information;
- organize information and look for patterns;
- persist, looking for more information if needed;
- discuss, review, revise, and explain results.

Journals

A journal is a personal, written expression of thoughts. Students express ideas and feelings, ask questions, draw diagrams and graphs, explain processes used in solving problems, report on investigations, and respond to open-ended questions. When students record their ideas in math journals, they often:

- formulate, organize, internalize, and evaluate concepts about mathematics;
- clarify their thinking about mathematical concepts, processes, or questions;
- identify their own strengths, weaknesses, and interests in mathematics;
- reflect on new learning about mathematics;
- use the language of mathematics to describe their learning.

Observations

Research has consistently shown that the most reliable method of evaluation is the ongoing, in-class observation of students by teachers. Students should be observed as they work individually and in groups. Systematic, ongoing observation gives information about students':

- attitudes towards mathematics;
- feelings about themselves as learners of mathematics;
- specific areas of strength and weakness;
- preferred learning styles;
- areas of interest;
- work habits — individual and collaborative;
- social development;
- development of mathematics language and concepts.

In order to ensure that the observations are focused and systematic, a teacher may use checklists, a set of questions, and/or a journal as a guide. Teachers should develop a realistic plan for observing students. Such a plan might include opportunities to:

- observe a small number of students each day;
- focus on one or two aspects of development at a time.

Suggested Assessment Strategies

Student Self-Assessment

Student self-assessment promotes the development of metacognitive ability (the ability to reflect critically on one's own reasoning). It also assists students to take ownership of their learning, and become independent thinkers. Self-assessment can be done following a co-operative activity or project using a questionnaire which asks how well the group worked together. Students can evaluate comments about their work samples or daily journal writing. Teachers can use student self-assessments to determine whether:

- there is change and growth in the student's attitudes, mathematics understanding, and achievement;
- a student's beliefs about his or her performance correspond to his/her actual performance;
- the student and the teacher have similar expectations and criteria for evaluation.

Resources for Assessment

"For additional ideas, see annotated Other Resources list on page 59, numbered as below."

1. The Ontario Curriculum, Grades 1-8: Mathematics.

2. *Assessment Standards for School Mathematics*, NCTM, 1995.

3. *Linking Assessment and Instruction in Mathematics: Junior Years*, Ontario Association of Mathematics Educators/OMCA/OAJE, Moore et al., 1996.

4. *Mathematics Assessment: Myths, Models, Good Questions, and Practical Suggestions*, by Jean Kerr Stenmark (Ed.), NCTM, 1991.

5. "Assessment", *Arithmetic Teacher* Focus Issue, February 1992, NCTM.

6. *How to Evaluate Progress in Problem Solving*, by Randall Charles et al., NCTM, 1987.

7. *Assessment in the Mathematics Classroom*, Yearbook, NCTM, 1993.

Suggested Assessment Strategies

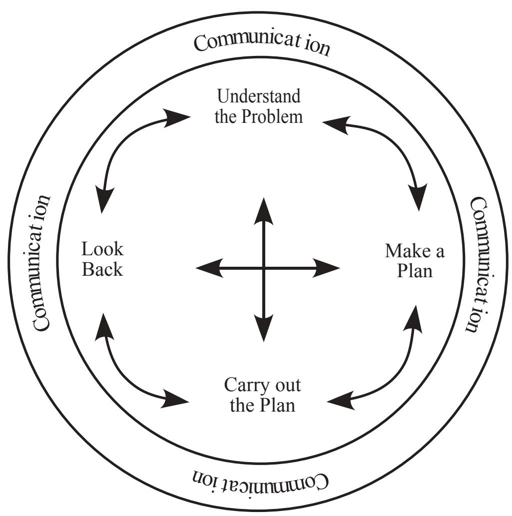
A GENERAL PROBLEM SOLVING RUBRIC

This problem solving rubric uses ideas taken from several sources. The relevant documents are listed at the end of this section.

"US and the 3 R's"

There are five criteria by which each response is judged:
Understanding of the problem,
Strategies chosen and used,
Reasoning during the process of solving the problem,
Reflection or looking back at both the solution and the solving, and
Relevance whereby the student shows how the problem may be applied to other problems, whether in mathematics, other subjects, or outside school.

Although these criteria can be described as if they were isolated from each other, in fact there are many overlaps. Just as communication skills of one sort or another occur during every step of problem solving, so also reflection does not occur only after the problem is solved, but at several points during the solution. Similarly, reasoning occurs from the selection and application of strategies to the analysis of the final solution. We have tried to construct the chart to indicate some overlap of the various criteria (shaded areas), but, in fact, a great deal more overlap occurs than can be shown. The circular diagram that follows (from OAJE/OAME/OMCA "Linking Assessment and Instruction in Mathematics", page 4) should be kept in mind at all times.



There are four levels of response considered:

Level 1: Limited identifies students who are in need of much assistance;

Level 2: Acceptable identifies students who are beginning to understand what is meant by 'problem solving', and who are learning to think about their own thinking but frequently need reminders or hints during the process.

Level 3: Capable students may occasionally need assistance, but show more confidence and can work well alone or in a group.

Level 4: Proficient students exhibit or exceed all the positive attributes of the **Capable** student; these are the students who work independently and may pose other problems similar to the one given, and solve or attempt to solve these others.

Suggested Assessment Strategies

LEVEL OF RESPONSE →

CRITERIA FOR ASSESSMENT ↓

	Level 1: Limited	Level 2: Acceptable	Level 3: Capable	Level 4: Proficient
UNDERSTANDING	• requires teacher assistance to interpret the problem • fails to recognize all essential elements of the task	• shows partial understanding of the problem but may need assistance in clarifying	• shows a complete understanding of the problem	• shows a complete understanding of the problem
STRATEGIES	• needs assistance to choose an appropriate strategy	• identifies an appropriate strategy	• identifies an appropriate strategy	• identifies more than one appropriate strategy
	• applies strategies randomly or incorrectly • does not show clear understanding of a strategy[1] • shows no evidence of attempting other strategies	• attempts an appropriate strategy, but may not complete it correctly[2] • tries alternate strateges with prompting	• uses strategies effectively • may attempt an inappropriate strategy, but eventually discards it and tries another without prompting	• chooses and uses strategies effectively[3] • recognizes an inappropriate strategy quickly and attempts others without prompting
REASONING	• makes major mathematical errors • uses faulty reasoning and draws incorrect conclusions • may not complete a solution	• may present a solution that is partially incorrect	• produces a correct and complete solution, possibly with minor errors	• produces a correct and complete solution, and may offer alternative methods of solution
	• describes[4] reasoning in a disorganized fashion, even with assistance • has difficulty justifying[5] reasoning even with assisstance	• partially describes[4] a solution and/or reasoning or explains fully with assistance • justification[5] of solution may be inaccurate, incomplete or incorrect	• is able to describe[4] clearly the steps in reasoning; may need assistance with mathematical language • can justify[5] reasoning if asked; may need assistance with language	• explains reasoning in clear and coherent mathematical language • justifies[5] reasoning using appropriate mathematical language
REFLECTION	• shows no evidence of reflection or checking of work • can judge the reasonableness of a solution only with assistance	• shows little evidence of reflection or checking of work • is able to decide whether or not a result is reasonable when prompted to do so	• shows some evidence of reflection and checking of work • indicates whether the result is reasonable, but not necessarily why	• shows ample evidence of reflection and thorough checking of work • tells whether or not a result is reasonable, and why
RELEVANCE	• unable to identify similar[6] problems	• unable to identify similar[6] problems	• identifies similar[6] problems with prompting	• identifies similar[6] problems, and may even do so before solving the problem
	• unlikely to identify extensions[7] or applications of the mathematical ideas in the given problem, even with assistance	• recognizes extensions[7] or applications with prompting	• can suggest at least one extension[7], variation, or application of the given problem if asked	• suggests extensions[7], variation, or applications of the given problem independently

Suggested Assessment Strategies

Notes on the Rubric

1. For example, diagrams, if used, tend to be inaccurate and/or incorrectly used.

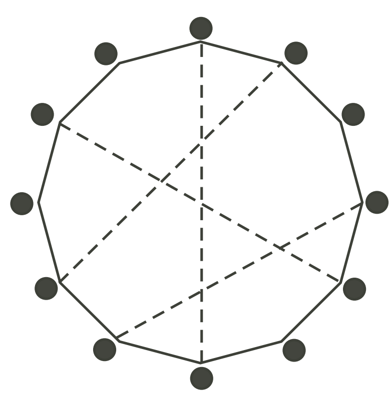
2. For example, diagrams or tables may be produced but not used in the solution.

3. For example, diagrams, if used, will be accurate models of the problem.

4. To *describe* a solution is to tell *what* was done.

5. To *justify* a solution is to tell *why* certain things were done.

6. *Similar* problems are those that have similar structures, mathematically, and hence could be solved using the same techniques.
   For example, of the three problems shown below right, the better problem solver will recognize the similarity in structure between Problems 1 and 3. One way to illustrate this is to show how both of these could be modelled with the same diagram:



Problem 1: There were 8 people at a party. If each person shook hands once with each other person, how many handshakes would there be? How many handshakes would there be with 12 people? With 50?

Problem 2: Luis invited 8 people to his party. He wanted to have 3 cookies for each person present. How many cookies did he need?

Problem 3: How many diagonals does a 12-sided polygon have?

   Each dot represents one of 12 people and each dotted line represents either a handshake between two people (Problem 1, second question) or a diagonal (Problem 3).

   The weaker problem solver is likely to suggest that Problems 1 and 2 are similar since both discuss parties and mention 8 people. In fact, these problems are alike only in the most superficial sense.

7. One type of extension or variation is a "what if...?" problem, such as "What if the question were reversed?", "What if we had other data?", "What if we were to show the data on a different type of graph?".

Suggested Assessment Strategies

Suggested Adapted Rubric for Activity 4, BLM 6

The rubric below has been adapted for the problem on BLM 6 of Activity 4: Do You Have All Your Marbles? This rubric considers the understanding of the problem, the selection and application of strategies, and reflection on the results.

Level 1: Limited	Level 2: Acceptable	Level 3: Capable	Level 4: Proficient
• is not quite sure what to do with the marbles drawn; needs help getting started	• is not quite sure what the purpose of the activity is, but tries hard to interpret the given statements as true or false for any given draw	• may not initially identify some statements as ambiguous, but can deal with this if it arises in discussion	• recognizes that some statements can be interpreted differently, and chooses to apply one particular interpretation throughout the experiment
• does not interpret the statements in the same way for all samples; may not even be aware of doing this	• may not interpret the statements in the same way for all samples	• consistently uses the same interpretations for the statements for each set of 3 marbles	• applies consistent interpretations of the statements
• does not understand how the results of the experiment can be used to predict the marbles that are in the bag.	• has some difficulty applying the results to determining the contents of the bag, but shows evidence of using some statements at least	• may have trouble using the results to identify the contents of the bag, but can identify each statement as probably true or false as far as the total contents are concerned	• realizes that the truth or falsity of the statements is directly related to the whole contents of the bag
• conclusions about the contents of the bag depend on only a few results of the experiment and will be limited (e.g., There is more than one colour)	• conclusions about the contents of the bag are general and may be given without justification	• draws definite conclusions about the contents of the bag, but these may be general statements such as “There are at least three colours”	• is willing to make definitive statements about the contents in the bag in some detail and with justification (e.g., Since we never drew more than one blue, but often drew green and red, we think there are 3 colours and more red and green than blue)
• when given the number of marbles in the bag, can see no reason to change his/her conclusions	• does not change his/her conclusions when given the number of marbles in the bag	• considers the number of marbles in the bag, but does not change his/her statements because they are too general	• considers the total number of marbles and may change specific conclusions; will justify the changes

Other Resources

1. The Ontario Curriculum, Grades 1-8: Mathematics.

2. *Assessment Standards for School Mathematics*, NCTM, 1995.
This comprehensive document examines the purposes of assessment and the nature of various types of assessment. Actual experiences in the classroom are used to clarify and augment suggestions made.

3. *Linking Assessment and Instruction in Mathematics: Junior Years*, Ontario Association of Mathematics Educators/OMCA/OAJE, Moore et al., 1996.
The document provides a selection of open-ended problems (including Fair Games activities) tested at the Junior Years (grades 4 to 6) level. Performance Rubrics are used to assess student responses (which are included) at four different levels.

4. *Mathematics Assessment: Myths, Models, Good Questions, and Practical Suggestions*, by Jean Kerr Stenmark (Ed.), NCTM, 1991.
Included are notes on observation techniques, interviews, conferences, portfolios. Sample student work is included with assessment to show how to apply the suggestions in the book. Includes an annotated bibliography.

5. "Assessment", *Arithmetic Teacher* Focus Issue, February 1992, NCTM.
This issue contains articles on linking assessment with instruction, alternative forms of assessment, self-evaluation, using manipulatives on tests, and suggestions for assessing cooperative problem solving.

6. *How to Evaluate Progress in Problem Solving*, by Randall Charles et al., NCTM, 1987.
Chapter headings include "What are you trying to evaluate?", "What are some evaluation techniques?" and "How do you organize and manage an evaluation program?" Sample 'test' items are included that show how to assess students' understanding; holistic scoring scales and student opinion surveys suggest ways of assessing progress and affective domain.

7. *Assessment in the Mathematics Classroom*, Yearbook, NCTM, 1993.
The chapters in this book deal with a wide range of forms of assessment, techniques for implementing different forms, and examples from the classroom showing how observation, interview, journal writing, and tests, among others, can be used to make a meaningful assessment of a student and his/her work.

8. *Making Sense of Data: Addenda Series, Grades K-6*, Mary Lindquist et al., NCTM, 1992
Students explore different spinners in Grade 3 and determine which outcomes are more likely than others. Older students could design spinners to give specified outcomes. A grade 5 activity uses a "Random walk" to explore the nature of random numbers:: Logo is used to explore this idea further with a computer.

Other Resources

9. "Roll the Dice — an introduction to Probability", Andrew Freda, in *Mathematics Teaching in the Middle School*, NCTM, October, 1998, pp. 85-89.

This describes how the introduction of an unfair game that appears, at first, to be fair, is used to intersect students in exploring how probabilities depend on relative frequencies of outcomes. Computer simulation leads to the conclusion that, the greater the sample, the closer the experimental probability is to theoretical probability. Suitable for Grades 5 or 6.

10. *Dealing with Data and Chance: Addenda Series, Grades 5-8*, Judith S. Zawojewski et al, NCTM, 1991.

One activity deals with determining the fairness or "unfairness" of games, another with determining how many purchases you must make to collect all the enclosed cards necessary to "win" a prize. The game of "Montana Red Dog" (p. 41) involves students in predicting probabilities.

11. *Measuring Up: Prototypes for Mathematics Assessment*, Mathematical Sciences Education Board and National Research Council, Washington, DC.

Performance assessment activities are given that deal with a number of mathematics topics. The "Hog" Game uses dice to encourage students to determine winning strategies while dealing with simple probability.

12. *What Are My Chances?, Book A,* and *What Are My Chances?, Book B*, Creative Publications, 1977.

These books provide a number of activities (black line masters) suitable for a wide range of activities. Students flip coins or two-colour chips, roll number cubes, draw marbles from closed containers (with and without replacement) and explore simple probability activities dealing with the weather and ice-cream flavour combinations. Book B also explores dependence and independence of events.

13. "Truth or Coincidence?", Daniel J. Brohier, *Teaching Children Mathematics*, December 1996, pp. 180-183, NCTM.

This activity for grades 3 to 6 involves children using heads and tails of coins to write "answer keys" for True-False tests. For example, if the test has 6 questions, how many possible combinations of True and False are possible? Applications to sports, social studies, and science are suggested.

14. "Exploring Random Numbers", William R. Speer, *Teaching Children Mathematics*, January 1997, pp. 242-245, NCTM.

Students are introduced to the idea of randomness by trying to write random numbers. Other simple activities reinforce the idea that randomness is not easily achieved.

15. "Choice and Chance in Life: The Game of 'Skunk'", Dan Brutlag, *Mathematics Teaching in the Middle School*, April, 1994, pp. 28-33, NCTM.

The game of "Skunk" uses "good" and "bad" rolls of the dice to create or eliminate a player's score. Analysis of the rules leads to a discussion of ways that "chance" enters our lives.

Other Resources

16. “Looking at Random Events with Logo Software”, Thor Charischak and Robert Berkman, *Mathematics Teaching in the Middle School*, January-March 1995, pp. 318-322, NCTM.

Students learn how to use Logo to simulate the rolling of up to 3 dice, and discuss the probabilities of different outcomes.

17. “Racing to Understand Probability”, Laura R. Van Zoest and Rebecca K. Walker, *Mathematics Teaching in the Middle School*, October 1997, pp. 162-170, NCTM.

Students play a game of racing sailboats that appears to be fair, but turns out to be skewed in favour of certain boats. Students make a chart of possible outcomes for 2 dice to try to explain why the game is unfair.

18. *Mathematics Teaching in the Middle School*, March 99, Focus Issue on “Data & Chance”, NCTM.

The entire journal is devoted to articles dealing with Data Management & Probability. One article explores probability through an even-odd game with dice of different shapes. Another uses animal crackers to simulate wild-life tagging and releasing to count animals in the wild.

19. “Mode Code”, David Spangler, *Mathematics Teaching in the Middle School*, April, 1999, pp. 464-466, NCTM.

Students explore the frequencies of letters of the alphabet in English. They apply this to decoding a message based on these frequencies, which are given as percents.

20. “Cat and Mouse”, Brian Lannen. *Mathematics Teaching in the Middle School*, April 1999, pp. 456-459, NCTM.

Students play a cooperative game to see whether or not the mouse finds the cheese. Aspects of probability include fairness or unfairness and tree diagrams to list outcomes. This activity is suitable for many different grades.

21. *Organizing Data & Dealing with Uncertainty*, NCTM, 1979

The first section of the book deals with collecting and interpreting data. The second part explores probability through dice, license plates, paper cups, tacks, and beads. Black line masters are provided.

22. “Calendar Mathematics”, Lorna J. Morrow, *Arithmetic Teacher*, March 1993, pp. 390-391.

Problems with dice, or t-shirt and shorts combinations involve organized counting. Children must determine the number of outcomes and show how they are certain that they have all outcomes.

23. *DIME Probability Packs A and B*, available from Spectrum Educational Publishing, Aurora, ON 1988

Each pack contains materials and activity cards for four different experiments. There are sufficient materials for 6 groups or pairs of students to work on the same experiment. The focus is on analyzing games as fair or unfair, and involves determining the probabilities of the outcomes in order to do so.

Curriculum Connections: Probability

ACTIVITY	DESCRIPTION OF THE ACTIVITY	CURRICULUM EXPECTATIONS
Activity 1 **Exploring Probability**	- determining probability of an event by identifying all possible outcomes and all favourable outcomes - comparing experimental probabilities with theoretical probabilities for simple experiments - identifying 'sure things' as those with probability of '1' and 'impossible things' as those with probability of '0'	• connect the possible events and the probability of a particular event • examine experimental probability results in the light of theoretical results • examine the concepts of possibility and probability
Activity 2 **Probability and Codes**	- comparing given relative frequencies of events with experimental data - applying relative frequencies of letters to the decoding of a given message	• compare experimental probability results with theoretical results • show an understanding of probability in making relevant decisions • use a knowledge of probability to pose and solve problems
Activity 3 **Bottle Caps and Lotteries**	- investigating the probability of winning a given lottery - using tree diagrams to identify all possible outcomes	• connect the possible events and the probability of a particular event • use tree diagrams to record the results of systematic counting • use a knowledge of probability to pose and solve problems
Activity 4 **Pizzas and Dandelion**	- estimating, using probability, the number of items of a particular type within a much larger set - explaining how sampling can be used to give such an estimate	• connect the possible events and the probability of a particular event • show an understanding of probability in making relevant decisions • use a knowledge of probability to pose and solve problems
Activity 5 **Random or Not?**	- exploring meanings of "random" - using a random number table to simulate real-life situations - testing methods of ways to generate random numbers - recognizing the difficulties involved when one tries to devise a list of random numbers - using a random number table to simulate a lottery, in order to determine the probability of a favourable outcome	• use tree diagrams to record the results of systemstic counting • shows an understanding of probability in making relevant decisions • use a knowledge of probability to pose and solve problems • connect the possible events and the probability of a particular event