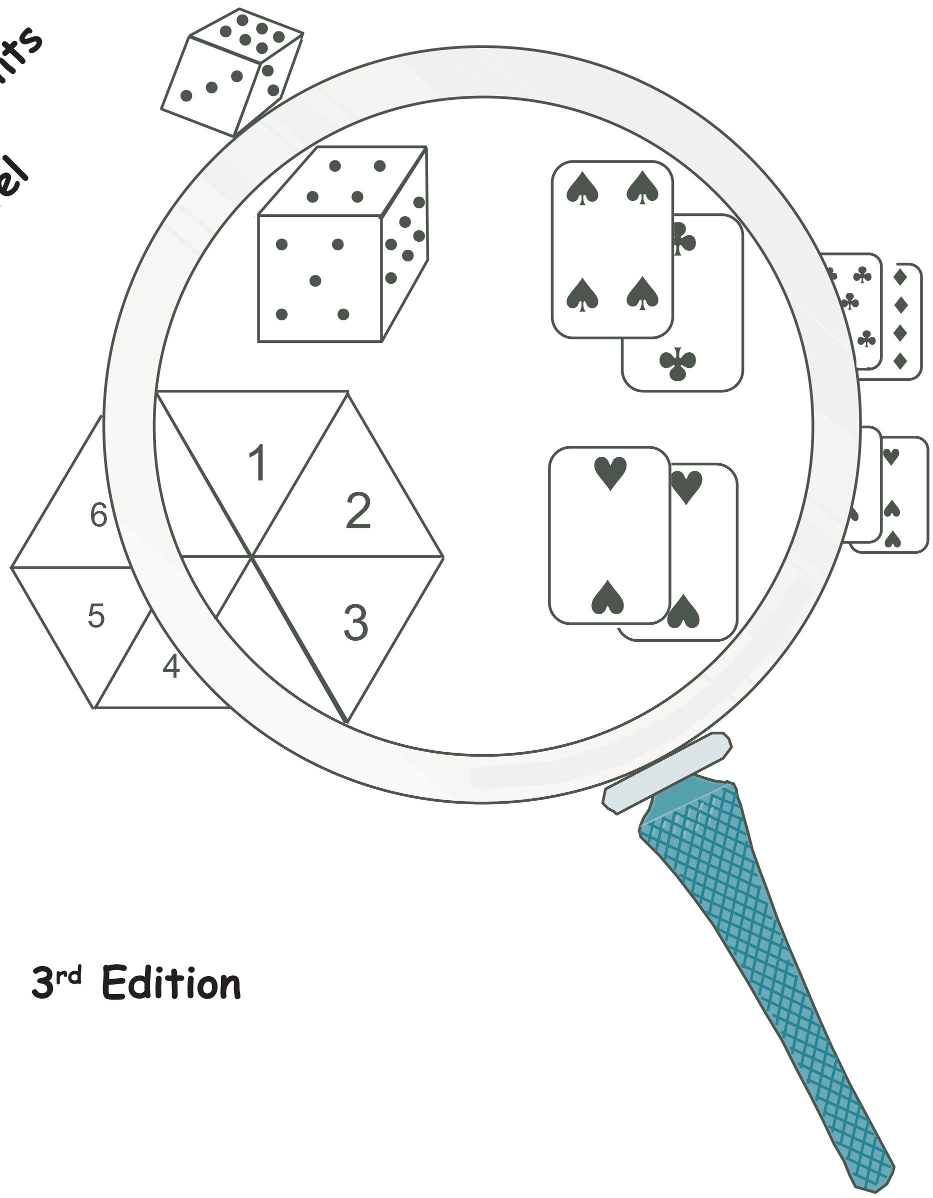
Invitations to Mathematics

Investigations in Probability

"Let's Play Fair"

Suggested for students at the Grade 5 level



3rd Edition

An activity of
The CENTRE for EDUCATION in MATHEMATICS and COMPUTING
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Preface

The Centre for Education in Mathematics and Computing at the University of Waterloo is dedicated to the development of materials and workshops that promote effective learning and teaching of mathematics. This unit is part of a project designed to assist teachers of Grades 4, 5, and 6 in stimulating interest, competence, and pleasure in mathematics, among their students. While the activities are appropriate for either individual or group work, the latter is a particular focus of this effort. Students will be engaged in collaborative activities which will allow them to construct their own meanings and understanding. This emphasis, plus the extensions and related activities included with individual activities/projects, provide ample scope for all students' interests and ability levels. Related "Family Activities" can be used to involve the students' parents/care givers.

Each unit consists of a sequence of activities intended to occupy about one week of daily classes; however, teachers may choose to take extra time to explore the activities and extensions in more depth. The units have been designed for specific grades, but need not be so restricted. Activities are related to the Ontario Curriculum but are easily adaptable to other locales.

"Investigations in Probability" is comprised of activities which introduce students to basic concepts of probability, techniques used to determine probability, and applications of probability. Everyday encounters with probability in weather forecasting, interpretation of polls, and commercials for various products and lotteries make it imperative that students acquire some basic knowledge of probability if they are to be able to interpret and evaluate such statements, and hence make well-informed decisions.

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Acknowledgements

Contributing Teachers

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Table of Contents

Overview

Common Beliefs

The activities in this booklet have been developed within the context of certain values and beliefs about mathematics generally, and about probability specifically. Some of these are described below.

Importance of Probability

Even a cursory glance at newspapers shows the extent to which the language of probability has become important. Individuals need a knowledge of probability to function in our society; consumer reports, cost of living indices, surveys, and samples are a part of everyday life. Nearly all endeavours in the working world require making decisions in uncertain conditions. The goal is to help students develop the critical thinking skills needed to reach sound conclusions based on appropriate data samples.

Instructional Considerations

"Classroom experiences should build on students' natural abilities to solve problems in everyday situations of uncertainty"

NCTM

For example, students learn to play games, and quickly develop a notion of "fairness" which is related to equally likely events. These and other activities develop essential skills for understanding probability — methods of organized counting, comparing results of experiments to theoretical probabilities, using the language of probability correctly — in the context of activities such as dice and spinner games which may be fair or unfair, decoding messages, designing a lottery, and sampling to determine population size.

Essential Content

The activities in this unit relate probability to experimental outcomes, explore how probability is related to fairness, and introduce probabilities as numerical ratios on a continuum from 0 (impossible) to 1(certain). In addition, there are Extensions in Mathematics, Cross-Curricular Activities and Family Activities. These may be used prior to or during the activity as well as following the activity. They are intended to suggest topics for extending the activity, assisting integration with other subjects, and involving the family in the learning process. During this unit the student will:

- constuct tree diagrams to determine all outcomes of an event;
- estimate the probability of various events on a continuum from 0 to 1;
- identify games involving dice, spinners, etc., as fair or unfair;
- invent fair and unfair games using similar materials;
- explore the nature of random numbers;
- use the language of probability correctly;
- justify opinions with coherent arguments;
- collaborate with other members of a group.

Notes

Notes

On the inside of the back cover of this booklet, you will find a chart connecting each activity to Ontario's curriculum expectations.

Overview

Curriculum Expectations

The material in this unit is directly related to Ontario curriculum expectations for Mathematics outlined below. By the end of Grade 5, students will:

- use tree diagrams to record the results of simple probability experiments;
- predict probability in simple experiments;
- use fractions to describe probabilities;
- connect real-life statements with probability concepts;
- pose and solve simple problems involving probability concepts.

Assessment

Assessment may be described as the process of gathering evidence about a student's knowledge, skills, and values, and of making inferences based on that evidence for a variety of purposes. These purposes include making instructional decisions, monitoring student progress, evaluating student achievement in terms of defined criteria, and evaluating programs.

To meet these aims, it is necessary to use a variety of assessment techniques in order to:

- assess what students know and how they think and feel about mathematics;
- focus on a broad range of mathematical tasks and taking a holistic view of mathematics;
- assess student performance in a variety of ways, including written and oral, and demonstrations;
- assess the process as well as the product.

Tests are one way of determining what students have learned, but mathematical competence involves such characteristics as communicative ability, problem-solving ability, higher-order thinking ability, creativity, persistence, and curiosity. Because of the nature of the activities it is suggested that a variety of assessment strategies be used. Suggestions include:

- observing students as they work to see if they are applying various concepts; to see if they are working cooperatively; to observe their committment to the tasks;
- assessing the completed project to see if instructions have been followed; to see if concepts have been applied correctly; to see if the language of mathematics has been used correctly;
- assessing the students' descriptions of their completed work to see if mathematical language is used correctly; to see if students understand the concepts used;
- providing opportunities for student self-assessment (Have students write explanations of their understanding, opinion, or feelings about an activity. One technique is to have them write under the headings What I Did, What I Learned, and How I Felt About It. Students could be asked to write a review of one day's activities or of the whole unit's work.);
- selecting an exemplary piece of work to be included in a portfolio for assessment purposes or for sharing with parents.

Overview

Prerequisites

Before beginning this unit, students should be able to

- divide a whole number by 10;
- calculate the average of 10 numbers;
- write equivalent fractions (e.g., '3 out of 5' or '3/5' is equivalent to '6 out of 10' or '6/10');
- add fractions with like denominators;
- identify fractional parts of a circle.

Logos

The following logos, which are located in the margins, identify segments related to, respectively:

Problem Solving Communication Assessment

Overview

Materials

ACTIVITY	MATERIALS
Activity 1 **Counting Dice**	• Copies of BLM 1 for all students • Acetate copy of BLM 1 for use with overhead projector • Standard dice/number cubes or spinners as on BLM 14 • Copies of BLM 14 (optional) • Copies of BLMs 2, 15 (optional)
Activity 2 **Probability from Zero to One**	• Copies of BLMs 3 and 4 • Copies of BLM 14 (optional) • Acetate copies of BLMs 3 and 4 for overhead projector
Activity 3 **Fair and Unfair**	• Copies of BLMs 5 and 6 • Acetate copy of BLM 3 for overhead projector
Activity 4 **Who's the Winner?**	• Copies of BLM 7 • Coins of 4 different types of two-colour counters • Standard dice or number cubes or spinners from Activity 1 • Copies of BLMs 8, 9, 10, 15 (optional)
Activity 5 **A Random Walk**	• Copies of BLM 11 • Acetate copy of BLM 11 for overhead projector • Playing pieces (counters, bottle tops, pen caps, ...) • Copies of BLMs 12, 13 and 14 (optional) • Copies of BLM 16 (optional)

Overview

Letter to Parents

SCHOOL LETTERHEAD

DATE

Dear Parent(s)/Guardian(s):

For the next week or so students in our classroom will be participating in a unit titled "Let's Play Fair". The classroom activities will focus on identifying games as fair or unfair, and exploring concepts such as the probability of a "sure thing".

You can assist your child in understanding the relevant concepts by working together to look for situations where probability occurs in everyday life, and to help in devising games that are either fair or unfair.

Various family activities have been planned for use throughout this unit. Helping your child with the completion of these will enhance his/her understanding of the concepts involved.

If you work with probability in your daily work or hobbies, please encourage your child to learn about this so that he/she can describe these activities to his/her classmates. If you would be willing to visit our classroom and share your experience with the class, please contact me.

Sincerely,

Teacher's Signature

A Note to the Teacher:

If you make use of the suggested Family Activities, it is important to schedule class time for sharing and discussion of results.

Activity 1: Counting Dice

Focus of Activity:

- identifying possible outcomes from an experiment
- expressing probabilities as ratios

What to Assess:

- accuracy of data collection/recording
- ability to identify probability as one or more out of all possible outcomes (e.g., the probability of rolling a 5 is 1 outcome out of 6 possible outcomes or '1 out of 6').
- collaboration between students in a pair (or small group)

Preparation:

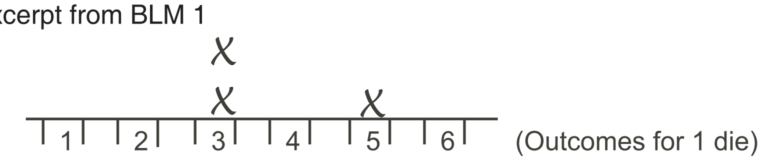
- see the table on page 4 for materials
- make copies of BLM 1 for each pair or small group
- make a copy of BLM 1 for the overhead projector
- provide two standard dice (or cubes numbered 1 to 6) for each pair of students. If dice are not available, provide copies of BLM 14 and have students construct spinners.
- make copies of BLM 2 (optional)
- make copies of BLM 15 (optional)

Activity:

Provide each pair of students with a copy of BLM 1 and one die or spinner. Show students how to record the results of individual rolls of the die as a line plot.

The sample below shows two rolls of 3 and one roll of 5.

Excerpt from BLM 1



Before beginning to roll the dice have students predict the results. You may wish to help with such questions as:

"What number do you think will turn up most often? Why?"

Some students may recognize that each number should turn up equally often. Record some predictions for reference later.

Have each pair record the results of 36 rolls. Ask students to draw conclusions from their results. Sample conclusions:

"We got all numbers about the same."

"We got 6 more than we got any other number."

Notes

In our discussion of probabilities in these activities, we are assuming perfectly balanced dice, coins, and spinners. In actuality, most dice, coins, and spinners are not perfectly balanced.

'BLM' refers to the Black Line Masters, which follow these Activity Notes.

For more on line plots, see Investigations in Data Management: "How About Our School?!", order form available at the end of this booklet.

Activity 1: Counting Dice

Notes

Refer back to the predictions made earlier and compare them with the results. Because the student pairs rolled the dice only 36 times, it is quite likely that the outcomes are not evenly distributed. That is, each number will not have been rolled equally often.

Optional: You may wish to tally the number of student pairs whose die 'favoured' each number. That is, how many student pairs rolled 6 more than any other number? How many pairs rolled 5 more often?

Collect all data orally and record on the overhead copy of BLM 1, #1(c).

You may wish to compare individual results with class results orally or you may wish to have students think about this in pairs and write their opinions (See 1(d) on BLM 1).

Students should find that the class results give a more even distribution of outcomes. That is, each number is rolled approximately the same number of times. Ask students why this might be so. Ask what they think might happen if they recorded 1000 rolls of a die.

Students should come to the conclusion that each number should appear as often as each other number. The symmetry of the die suggests that the probability of each number is the same.

> "The probability of rolling a '2' is the same as the probability of rolling a '5'."

Students should be encouraged to use the word 'probability' at this stage.

Next, have students list all possible results when two dice are rolled and the results added (i.e., 2, 3, 4, 5, ..., 12).

These numbers should be recorded under the line given for #2 on BLM 1.

Ask students what they think the results of the experiment will be. Students frequently assume that each total will be rolled equally often.

Excerpt from BLM 1

(Outcomes for 2 dice)

Communication

Have students roll the two dice 36 times. and record the totals making a line plot for #2 on BLM 1.

Ask student pairs to describe their results.

> "Look at your line plot/graph. Are you surprised at the results? Why?"
> "What outcomes/results are possible? In how many ways can you roll each total?"

Activity 1: Counting Dice

You may wish to leave this last question until after students have completed #3 on BLM 1, in which students identify all possible combinations of the two dice for any given total.

Collect all the data and record. A chart such as the following may be helpful. Individual group results are recorded and their sum is calculated.

Total of Two Dice	Number of times Rolled
2	2 + 3 + 1 + 0 + 1 ..... = ...
3	

Ask students to compare their own results with the class results. Ask students to identify differences and suggest reasons for these differences.

One way to illustrate why some sums of two dice occur more frequently than others is to use dice of different colours. If these are not available, roll one die with the right hand and one with the left, or use some other way to distinguish between them. One of the dice should always be recorded first. Students should understand this before they attempt to complete the chart in #3 on BLM 1. Students should realize that a roll of 5, 6 is different from a roll of 6, 5 and represents a different cell on the chart. It is not necessary for students to roll the dice in order to complete the chart, but some discussion beforehand is needed so that they understand precisely what they are to do.

An alternative way to list outcomes is by using a tree diagram. BLM 15 provides notes on this technique.

Have students complete the chart of possible outcomes (#3 on BLM 1).

As a class, make a list of the number of ways each total can be achieved — i.e., the number of 2s, of 3s, of 4s, etc. in the chart.

Because there are 36 equally likely cells on the chart, and six of them are 7s, we say that the probability of rolling a sum of 7 is '6 out of 36'. What is the probability of rolling a sum of 2? of 5? of 11?

Communication

Just as any fraction can be written in a number of ways (e.g.,), so a ratio such as '2 out of 36' can be written in a number of equivalent ways, such as '1 out of 18' or '4 out of 72' or '6 out of 108'. You may wish to explore this idea briefly with students as a way of reviewing the writing of equivalent fractions. Ask students such questions as:

> "If the probability of rolling a sum of 3 is 2 out of 36, how many sums of 3 do you think would occur in 72 rolls? in 144 rolls?"

Activity 1: Counting Dice

Since each pair rolled the dice 36 times, the total number of outcomes in the class data for rolling two dice will be a multiple of 36. This means that you can compare the theoretical probability to the actual results, and perhaps discuss why they may not be exactly the same.

Extensions in Mathematics:

1. See BLM 2 for "Who Goes First", an activity in which students list all possible outcomes and determine the probabilities of particular outcomes.
2. Make a chart similar to the one in #3 on BLM 1, but have students record the products of the two numbers instead of the sums. Have them compare the two charts. For example, ask students:

   "Is the probability of the same as the probability of 1 + 5? Why or why not?"

   "Is the probability of rolling a product of 5 the same as that for rolling a sum of 5? Why or why not?"

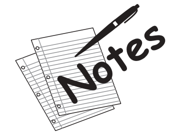
   If you use this activity, have students keep their completed charts to use when analyzing games on BLM 8.

Other Resources:

For additional ideas, see annotated "Other Resources" list on page 54, numbered as below.

8. "Making Sense of Data: Addenda Series, Grades K-6", Mary Lindquist
12. "What Are My Chances?", Creative Publications
17. "Racing to Understand Probability", Laura R. Van Zoest and Rebecca K. Walker
21. "Organizing Data and Dealing with Uncertainty", NCTM

Activity 2: Probability from Zero to One



Focus of Activity:

- relating the frequencies of outcomes to probabilities
- representing probabilities by numbers from 0 to 1

What to Assess:

- identification of all possible outcomes
- logical reasoning
- use of language of probability
- collaboration with others

'BLM' refers to the Black Line Masters, which follow these Activity Notes.

Preparation:

- prepare copies of BLMs 3 and 4 for each pair/group and for the overhead
- make copies of BLM 14 if you wish students to construct spinners (optional)

Activity:

Refer back to the dice experiments from Activity 1 and review the way of stating probability as the ratio of the number of favourable outcomes to the number of possible outcomes— e.g., The probability of rolling a 5 with a single die is 1 out of 6 because there are 6 possible outcomes of which one is the 'favourable outcome', which in this case is a '5'.

Communication

The word "favourable" refers to a specific outcome. For example, if you want to roll '5', then the favourable outcome would be '5'. If you want to roll an even number, then a favourable outcome would be '2', '4', or '6'.

Ask students to give probabilities based on the data collected during Activity 1. For example,
What is the probability of rolling a 4 with a single die? [Ans: 1 out of 6]
What is the probability of rolling an even number when rolling a single die? [Ans: 3 out of 6]
What is the probability of rolling a sum of 5 using two dice? [Ans: 4 out of 36]
What is the probability of rolling a total less than 6 using two dice? [Ans: 10 out of 36]

Review the source of the '6' with one die and the '36' with two dice as 'the number of possible outcomes.'

Ask students:

"What is the probability of getting a head when we toss a coin? Why?"

Make sure students understand that the probability is 1 out of 2 because there are two possible outcomes and only one of these is a head. (If students suggest that a coin can land on an edge, indicate that such outcomes are so rare that they can be ignored for this activity and that the activity will consider only heads or tails.)

Problem Solving

Assessment

Activity 2: Probability from Zero to One

Distribute copies of BLM 3. You may wish to read over the questions with the students to be sure they understand what they are to do. Then have students answer the questions either individually or in pairs.

This activity does not involve the actual flipping of coins or rolling of dice. If time permits, you may want to have students collect data about heads and tails by, for example, flipping coins, and then comparing actual results with the theoretical probabilities.

Discuss the responses with the class. It is important for students to realize that, in order to determine a probability, they must first identify all possible outcomes. That is the reason part (a) of #1, 2, and 3 asks for a listing of possible outcomes.

If students suggest that spinner C has only four outcomes (1, 2, 3, and 4), remind them how important it was to distinguish between two dice when completing the last chart on BLM 1. In the same way, each '2' or '3' on spinner C should be considered as a separate outcome. Thus, there are eight possible outcomes, not four.

You may wish to direct students' attention to the fact that spinner C is divided into eighths, and the probability of spinning any one of those eight sections is 1 out of 8 or $\frac{1}{8}$ *. Thus, the probability of spinning <u>each</u> '3' is* $\frac{1}{8}$*. So the probability of spinning <u>either</u> '3' is* $\frac{2}{8}$*. Visual learners may find this 'geometric probability' easier to understand or visualize than the more commonly used numerical expressions.*

Question 5(iii) on BLM 3 deals with some situations that cannot possibly happen. Spinners A and C have no numbers greater than 4 so the probability of spinning a number greater than 4 is '0 out of 4' for spinner A, and '0 out of 8' for spinner C. This is often given just as 'a probability of zero'.

Ask students what events in their lives might have probabilities of zero. (e.g., the probability of seeing a live dinosaur, the probability of getting younger each year)

Ask students what they think would be the probability of a 'sure thing'. For example, what is the probability of them having a birthday each year? If students have difficulty with this, return to the single die and ask what is the probability of rolling a number from 1 to 6. Since the probability of rolling each particular number is 1 out of 6 or $\frac{1}{6}$, the probability of rolling <u>any</u> number from one to six is 6 out of 6, i.e. $\frac{6}{6}$ or 1. Another way to think of this is that the probability of rolling <u>any</u> of the numbers 1, 2, 3, 4, 5, <u>or</u> 6 will be the sum of the probabilities of rolling each number — i.e., $\frac{1}{6}+\frac{1}{6}+\frac{1}{6}+\frac{1}{6}+\frac{1}{6}+\frac{1}{6}$, giving a total of or 1. That is, the probability of a sure thing is 1.

Activity 2: Probability from Zero to One

Similarly, when flipping one coin. The probability of getting a head is 1 out of 2 or $\frac{1}{2}$, and the probability of getting a tail is also $\frac{1}{2}$, so the probability of getting either a head or a tail is $\frac{1}{2}+\frac{1}{2}$ or 1. Or, you could refer to #2 (b) (iv) and (v) on BLM 3. Is there any outcome other than "6 or less or 7 or more"? Since these include all the outcomes, the sum of the two probabilities $\left(\frac{15}{36}+\frac{21}{36}\right)$ is 1.

Communication

Whether or not probabilities of zero and one are realistic is an interesting question and worth discussing. Students may wish to speak of probabilities very close to zero or one, rather than claiming something will never happen or that it is sure to happen.

Distribute copies of BLM 4 and have students complete it. If students complete this in pairs they will have opportunities to try to defend their choices, since a given event is not equally probable for all people.

Students should realize that the probability of such events often depends on the circumstances. For example, the probability of snow in the next three months depends on the time of year. The probabilities of other events will vary for individual students. For example, the eating of seafood depends to some extent on diet preference so this may have a low probability for the student who dislikes seafood but a high probability for someone who prefers seafood. It is also possible that a student can make this a sure thing by choosing to eat fish the next day. Thus, students may become aware that probabilities can be manipulated. The question then is, of course, whether we should speak of the 'probability' of a pre-determined event.

You may wish to point out that the thought processes for assigning probabilities to the complicated events (as on BLM 4) and coin/dice events (as on BLM 3) are different. In a dice event, we could assign a probability by intuition (similar to the process used for BLM 4), but we prefer to list equally likely outcomes and count them.

Assessment

When students have completed the activity, discuss the responses with them. Have them explain, in each case, why they selected a certain probability range for an event.

Notes

Activity 2: Probability from Zero to One

Family Activities:

1. Weather forecasters often use statements of probability. Students should discuss the meaning of such statements with their families and be prepared to report on family opinion the next day. Students could also collect such statements over the course of the week and discuss their accuracy.

Other Resources:

For additional ideas, see annotated "Other Resources" list on page 54, numbered as below.

8. "Making Sense of Data: Addenda Series, Grades K-6", Mary Lindquist
12. "What Are My Chances?", Creative Publications
17. "Racing to Understand Probability", Laura R. Van Zoest and Rebecca K. Walker
21. "Organizing Data and Dealing with Uncertainty", NCTM

Activity 3: Fair and Unfair (1)

Focus of Activity:

- relating probability to fairness or unfairness

What to Assess:

- identification of all possible outcomes for a given situation
- identification of favourable outcomes
- identification of 'fair' games as those in which each player has an equal chance of winning
- use of language of probability
- collaboration with partners

Preparation:

- make copies of BLMs 5 and 6 for each pair of students
- have available a copy of BLM 3 for the overhead projector

'BLM' refers to the Black Line Masters, which follow these Activity Notes.

Activity:

Display a copy of #5 on BLM 3 and introduce the idea of games that may be fair or unfair with questions such as the following:

> "Suppose you are playing a game in which you win if you spin a 1 or a 2. Otherwise you lose. Which spinner on BLM 3 would you choose? Why? Which spinner would you choose if you wanted to make this a fair game?"

Spinner C is the best choice for winning. The probability of spinning either a '1' or a '2' is 5 out of 8. For spinners A and B, the probabilities are, respectively, 2 out of 4 and 2 out of 6. Students may suggest that the probability of spinning either a '1' or a '2' is $\frac{1}{2}$ *for spinner A, less than* $\frac{1}{2}$ *for B and greater than* $\frac{1}{2}$ *for spinner C and that therefore spinner C is the preferred spinner. This is good reasoning. For a fair game, in which each player stands an equal chance of winning, use spinner A.*

> "Suppose you need a number less than 3 to win. Which spinner would you choose? Why? Would any spinner make this a fair game? Why?"

Since the only numbers on the spinners that are 'less than 3' are one and two, the solution to this problem is the same as the solution to the first problem given just above.

> "Suppose you need a 6 to win but otherwise you lose?"

> "Suppose you need an odd number to win but otherwise you lose?"

Activity 3: Fair and Unfair (1)

The rules are given as games for one player (or one group of players all on one side). Games on BLM 5 and 6 should be interpreted this way.

Ask students what they mean when they speak of a ‘fair’ game. In the field of probability this would mean each player has an equal chance of winning. Since all spinners on BLM 3 have equal probabilities of showing even or odd, any spinner would be ‘fair’ for ‘an odd number to win’. Tell students they are going to be examining other games, some of which are not fair. For these ‘unfair’ games, they are to choose the spinner or bag of marbles that gives them the best chance of winning.

Problem Solving

Distribute copies of BLMs 5 and 6 to the student pairs. You may wish to allow time for students to colour the spinners and marbles for easier “reading”. If you have dice or other cubes available, students could colour bits of masking tape to stick to the faces of the cubes to make the cubes described in #4 on BLM 6. This will help some of them toward solutions. Students could write their answers on the backs of the BLMs, or in math notebooks.

As with the spinners on BLM 3, some students may view spinners D and E on BLM 5 as having 3 outcomes, and may suggest that the probability of spinning red on spinner D is ‘1 out of 3’. Use the idea of “geometric probability” (see note in Activity 1) to show that the probability of spinning red is ‘1 out of 2’ $\frac{1}{2}$ *or , since half the circle is red. Alternately, have students draw in the ‘missing’ radii to divide each spinner into equal parts, as in Spinners B and C on BLM 3. Then the probability of red using spinner D is 2 out of 4 or* $\frac{2}{4}$ *(which can also be expressed as* $\frac{1}{2}$ *).*

If students do not complete the activity in the time allotted, assign as homework. Suggest that they ask family members for their solutions and reasons. This can then be discussed in the next class before proceeding with Activity 4.

If some students have finished but others are slower, suggest to those who have finished that they try one or more of the fair games they described in #6 on BLM 6. They should keep records to see if their ideas were good ones, and the games are fair. If they find the games are not fair, they should adjust their spinners or dice to make the games fair.

Extensions in Mathematics:

1. Invent a fair game for each spinner or each bag of marbles on BLM 5.

Activity 3: Fair and Unfair (1)

Cross-Curricular Activities:

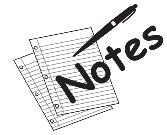
1. Discuss other games students play (e.g., video games, team sports, card games). Are they 'fair'? Does each person stand a equal chance of winning? What other factors might affect the winning or losing of a game?

   Students should realize that games we call 'fair' in everyday language may not meet the rigorous definition of mathematics. Usually a 'fair' game just means that certain conditions apply to all players, but that a player's ability also affects the winning or losing of the game.

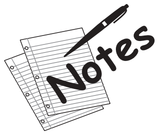
Other Resources:

For additional ideas, see annotated "Other Resources" list on page 54, numbered as below.

11. "Dealing with Data and Chance: Addenda Series, Grades 5-8", Judith S. Zawojewski et al.
17. "Racing to Understand Probability", Laura R. Van Zoest and Rebecca K. Walker
20. "Cat and Mouse", Brian Lannen



Communication



Activity 4: Who's the Winner?

Focus of Activity:

- fair and unfair games

What to Assess:

- identification of all possible outcomes
- logical reasons for identifying a game as fair or unfair
- collaboration with partners/other group members
- use of language of probability

Preparation:

- copies of BLM 7 for each student pair/group and one copy for the overhead
- provide 4 different coins or two-colour counters for each student pair/group. (Two-colour counters with one red and one yellow face are available from some math supply companies. Two-colour counters can be made by sticking two different colours of stickers back-to-back. This is not ideal since such "coins" will not be balanced, and are too light to flip well. Colouring or placing stickers on two sides of carboards disks is another way of producing two-sided "coins".)
- provide a standard die for each pair/group (or use a spinner as suggested for Activity 1)
- make copies of BLM 8 (optional)
- make copies of BLMs 9 and 10 (optional)
- make copies of BLM 15 (optional)

'BLM' refers to the Black Line Masters, which follow these Activity Notes.

Activity:

Discuss student suggestions for fair games they devised for #6 on BLM 6 from Activity 3. The activities below extend these ideas as students play the games and try to analyze the probabilities to decide if a game is fair or not.

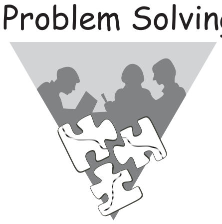
Distribute copies of BLM 7 to students pairs/groups. In order to be sure all students understand what is required of them, you may choose to play (and analyze) the first game as a whole class. Have individuals toss the three coins, while one student records the wins on the blackboard or an acetate copy of BLM 7 on the overhead projector.

If students use two or three different coins, it is easier to distinguish the outcomes. For example H (dime) and T (penny) is different from H (penny) and T (dime). Students need to be aware of this when listing all possible outcomes.

Notice that, after students have had some experience with analyzing games (#1 and #2), they are asked to predict the results of games (#3 and #4) before playing. The results of playing the games should then be compared with the predictions and differences explained. The predictions themselves should be based on an awareness of all possible outcomes, and all favourable outcomes.

The general rubric in "Suggested Assessment Strategies" (following Solutions and Notes) has been adapted for this particular problem, as a sample to assist you in assessing your students.

Problem Solving



Activity 4: Who's the Winner?

There is space on the BLM to record the results of playing the games. Responses to the questions could be part of a math journal, or students could be asked to write on the back of BLM 7.

Notes

If you wish to use tree diagrams as a method of identifying outcomes, see BLM 15, and Solutions and Notes for Activity 4.

It is possible for students to play each game alone since only the "you" in each case tosses the dice or the coins. Thus, if students do not complete the activity during the allotted time they could complete it at home. They could, of course, also play with a family member, and discuss probability with other family members at the same time.

Assessment

In discussion with students after they have completed the activity, you may find that some students are using their own results to label the game as fair or unfair. That is, they ignore the probabilities of each toss of the coins or roll of the die. However, it is quite possible for the results of only 10 rounds to be misleading. You may wish to collect data from all the groups for each game and use that aggregate data to determine if the the game is fair or unfair. You might also wish to refer back to Activity 1 in which students found that class results were closer to theoretical probabilities than small group results.

Extensions in Mathematics:

1. Three variations on a dice game of odds and evens are described on BLM 8. Students may be surprised to discover that the third game (which seems very like the first two) is not fair, while the first two games are.

2. A game that will allow you to discuss "odds" with students, if you wish to, is given on BLM 10.

   Problem Solving

   As students analyze the game they realize that, since the probability of rolling 7 is just 6 out of 36, or $\frac{1}{6}$, this is not a fair game. The Opponent is more likely to win. You may wish to remind students of the chart they completed on BLM 1 (Activity 1) showing all possible sums for two dice.

   The odds against rolling a 7 are 30 to 6. The chart shows that there are 30 sums that are <u>not</u> 7 and only 6 that are 7.

   We could also say the odds in favour of rolling a 7 are '6 to 30', usually written '6:30'. This is the same as '1:5'. That is, over a large number of rolls, there should be one roll of 7 to every five rolls of 'not-7'. Students should see that, to make the game fair, the Opponent should give the Player five points whenever a seven is rolled.

Notes

Problem Solving

Activity 4: Who's the Winner?

Family Activities:

1. A dice game called "Hog" is described in "Measuring Up" (see #11 in the Other Resources list) and a very similar game called "Skunk" appears in the journal "Mathematics Teaching in the Middle School" under #15 in the Other Resources list. BLM 9 gives the basic rules from "Measuring Up" and asks some questions which should help students in analyzing the game, and in trying to develop a winning strategy. This naturally involves examining probabilities.

   The game uses several dice (or spinners). A player decides how many dice he/she wishes to roll and adds the numbers showing. However, when a '1' appears, the player's score goes to zero and he/she is out of the game.

Other Resources:

For additional ideas, see annotated "Other Resources" list on page 54, numbered as below.

10. "Dealing with Data and Chance: Addenda Series, Grades 5-8", Judith S. Zawojewski et al.
11. "Measuring Up: Prototypes for Mathematics Assessment", MSEB
15. "Choice and Chance in Life: The Game of 'Skunk'", Dan Brutlag

Activity 5: A Random Walk

Focus of Activity:
- the concept of randomness

What to Assess:
- correct use of the random number table
- accuracy in performing the random walk
- use of mathematical language
- collaboration with others

Preparation:

- make copies of BLM 11 for each student pair, and a copy for the overhead
- provide playing pieces (use counters, bottle caps, ...): one per student pair
- make copies of BLM 12 and 14 (optional)
- make copies of BLM 13 (optional)
- make copies of BLM 16 if you wish students to have notes on random numbers

'BLM' refers to the Black Line Masters, which follow these Activity Notes.

Activity:

A random number table can be generated by a fair spinner. If the spinner is fair, then the chance of spinning each digit is the same as the chance of spinning any other digit, and no spin affects any other spin. However, since it is difficult to construct a spinner to this degree of accuracy we often use a number table such as the one shown on BLM 11.

For more on random numbers see BLM 16.

To use the number table, pick a starting number anywhere in the table and read contiguous numbers to the left, or right, or up, or down.

For example, if you start with the '5' at the beginning of the fifth row, you could read 5, 3, 6, 1, 2, 5, 1 to the right; 5, 5, 2, 4, 5, 5 down; or 5, 1, 5, 6, 3 up.

It is not necessary to begin at the beginning of a row. However, it should be emphasized that a starting position and direction should be chosen without looking at the numbers in the table.

Excerpt from BLM 11

3	1	1	6			
6	4	3	5	6		
5	6	1	5	4	1	3
1	5	6	2	2	1	4
5	3	6	1	2	5	1
5	2	1	6	5	1	4
2	3	5	6	2	4	
4	6	4	1	6	4	6
5	4	4	6	4		
5	1	6	4	4		

Notes

Activity 5: A Random Walk

The random walk described below will give students the opportunity to read a random number table and to realize that patterns of walks cannot be predicted if the numbers are truly random.

Communication

Distribute copies of BLM 11 to all student pairs/groups. Using the overhead, illustrate one or two walks.

Students should take ten random walks and record their results in the chart given. After taking 10 'walks' students should try to draw conclusions from the results. Discuss some of these with the students, asking them to give reasons for their choices.

Then make a chart similar to the one below, and record the number of times a walk ended on each space or went off at either end, collecting data from the entire class.

Number of times a walk ended on each space or went off the chart

Off, North	N7	N6	N5	N4	N3	N2	N1	START	S1	S2	S3	S4	S5	S6	S7	Off, South
62	1	7	0	4												

Once the class data are available, ask students if they wish to change their conclusions. They should give reasons for any changes.

Optional

BLM 12 provides another random walk activity, this time using four directions. It will be necessary to use paper clips to complete the spinners to indicate the directions (See BLM 14). As before, the numbers from a random number table are used to give the lengths of the 'steps'. In lieu of a spinner, students could write the four directions on bits of paper and draw one from an envelope, replacing it after each 'step'.

Students are asked to predict the results before completing the activity, since the experience of Random Walk 1 has given them some data on which to base their predictions. Such predictions should be revisited after the game has been played a few times.

After the game has been played, compare students' predictions with the results. You may wish to collect all results on a large chart and explore the results. For example, is there any pattern obvious? Would you expect one? Do you think the results would be any different if, instead of a spinner for the direction, you used a table of random numbers? Why or why not?

Activity 5: A Random Walk

Extensions in Mathematics:

1. If you have access to a computer with Logo software, students can explore random walks using the command

   FD RANDOM 20 RT RANDOM 360.

   (The command RANDOM 20 outputs a random number from 0 to 19.)

   Students should estimate how long it will take the turtle to walk off the screen if they keep repeating the command. They can test their estimates by entering the command

   REPEAT 10000 [FD RANDOM 20 RT RANDOM 360]

   If they work in pairs, one student can watch the screen while the other watches the clock.

   Collect data from several student walks and compare individual results with class results.

2. BLM 13 describes a game using playing cards that is similar to the North-South random walk in that students add and subtract numbers to count more than 10 or less than –10. However, the game allows choice at each step, and the cards provide only four of each number, unlike a random number sequence. Students are asked to compare the activities. You may wish to ask them how they could change the game so that it was more random (e.g., place all the cards face down and draw one for each turn rather than selecting one from one's hand). You may also wish to ask them if they think this makes the game fairer or less fair.

Family Activities:

1. Take a walk with your family. At each corner flip a coin to decide if you turn right or go straight. How far from home do you think you will be after 4 flips?

Other Resources:

For additional ideas, see annotated "Other Resources" list on page 54, numbered as below.

8. "Making Sense of Data: Addenda Series, Grades k-6", Mary Lindquist et al.
14. "Exploring Random Numbers", William R. Speer
16. "Looking at Random Events with Logo Software", Thor Charischak and Robert Berkman.

BLM 1: Rolling Dice

1. (a) Roll one die 36 times and record the results on the line plot below.

1 2 3 4 5 6

(Outcomes for 1 die)

(b) What did you expect to happen when you rolled the die 36 times? Did this happen? Why or why not? Write your answers on the back of this sheet.

(c) Record class results in the chart on the right.

(d) Compare your results with the class results. Are they different? How? Why?

Number on die	Number of times rolled
1	
2	
3	
4	
5	
6	

2. Roll 2 dice 36 times and record the results below.

2 3 4 12

(Outcomes for 2 dice)

Number on first die

Number on second die

	1	**2**	**3**	**4**	**5**	**6**
1						
2						
3						
4						
5						
6						

BLM 2: Who Goes First?

As a way of deciding who goes first in a game, Marilyn suggested the following: Put some coloured tiles in a box or bag and draw out 2 of the tiles. If you are the first person to draw 2 of the same colour, you go first.

Marilyn and her friends tried this with three different sets of tiles:

Set 1: 2 blue, 1 red
Set 2: 2 blue, 2 red
Set 3: 3 blue, 1 red

1. (a) Use coloured squares of paper or cardboard to represent the tiles and take turns drawing two tiles. Record how many tries it took before someone drew two of the same colour from Set 1. Do this again. In all, run ten tests for Set 1. Record the number of turns it took each time before two tiles of one colour were drawn. Calculate the average number of turns needed.

	Test	1	2	3	4	5	6	7	8	9	10	Average
Set 1:	Number of turns to draw two of one colour											
Set 2:	Number of turns to draw two of one colour											
Set 3:	Number of turns to draw two of one colour											

(b) Did you expect this result? Why or why not?

(c) How long do you think it will take to draw two tiles of one colour from Set 2? from Set 3? Run ten trials for Set 2 and ten for Set 3. Record the results in the chart above. Calculate the average number of turns needed for each set.

2. (a) List all possible outcomes for a single draw from each set. For example, from Set 1 you could draw blue and blue or blue and red, but because there are two blues (blue one and blue two, or B_1 and B_2) there are 2 combinations of blue and red possible: B_1R and B_2R. Thus there are 3 possible outcomes. B_1B_2, B_1R, B_2R. What is the probability of drawing two of the same colour from Set 1?

(b) In the same way, determine the probability of drawing two tiles of the same colour from Set 2 and from Set 3.

(c) Which set would you choose to decide who goes first in a game? Why?

BLM 3: Exploring Probabilities

1. (a) What are the possible outcomes/results when rolling a single die?

   (b) What is the probability of rolling
   - i) a 1?
   - ii) an even number?
   - iii) a multiple of 3?
   - iv) a number greater than 4?

2. (a) What are the possible outcomes/results when rolling a pair of dice and adding the two numbers?

   (b) What is the probability of rolling
   - i) a sum of 12?
   - ii) a sum less than 5?
   - iii) a double?
   - iv) a sum of 6 or less?
   - v) a sum of 7 or more?

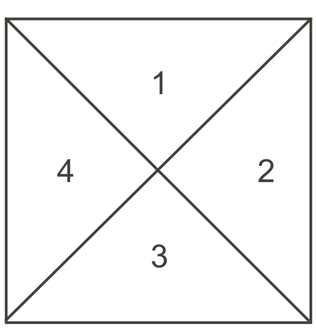
3. (a) What are the possible outcomes/results when flipping two coins?

   (b) What is the probability of getting
   - i) 2 heads?
   - ii) a head and a tail?
   - iii) 2 tails?

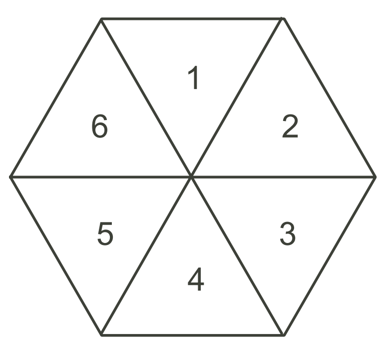
4. If you flip 3 coins, what is the probability of getting
   - i) 2 heads and 1 tail?
   - ii) 3 heads?
   - iii) 1 head and 2 tails?

5. For each of the spinners below, give the probability of spinning
   - i) a 3
   - ii) an odd number
   - iii) a number greater than 4

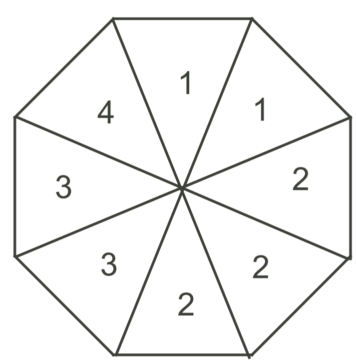
Spinner A



Spinner B



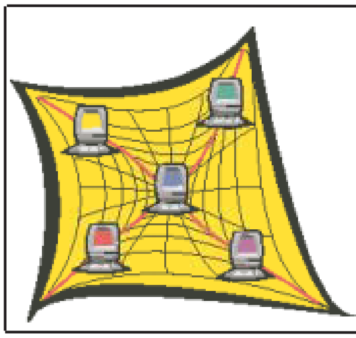
Spinner C



BLM 4: A Sure Thing

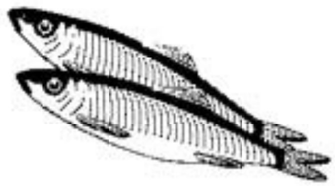
1. Probabilities range from zero to one. For each of the following, place the letter of the event somewhere on the number line from 0 to 1 to indicate what you think the probability should be.

Event A



You will spend time on the Internet this week.

Event B



You will eat fish or seafood before next Wednesday.

Event C

You will receive dozens of Valentines next month.

Event D

You will break your pencil doing your homework.

Event E

You will get a new pet within 6 months.

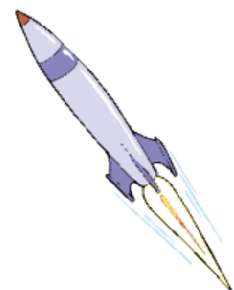
Event F

You will travel somewhere on your next holiday.

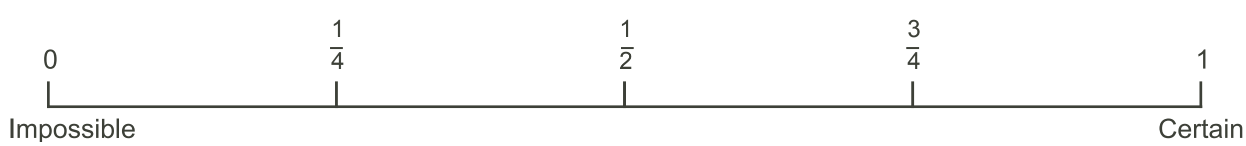
Event G

You will buy at least 1 new CD this month.

Event H



You will take a rocket to the moon before you are 50.



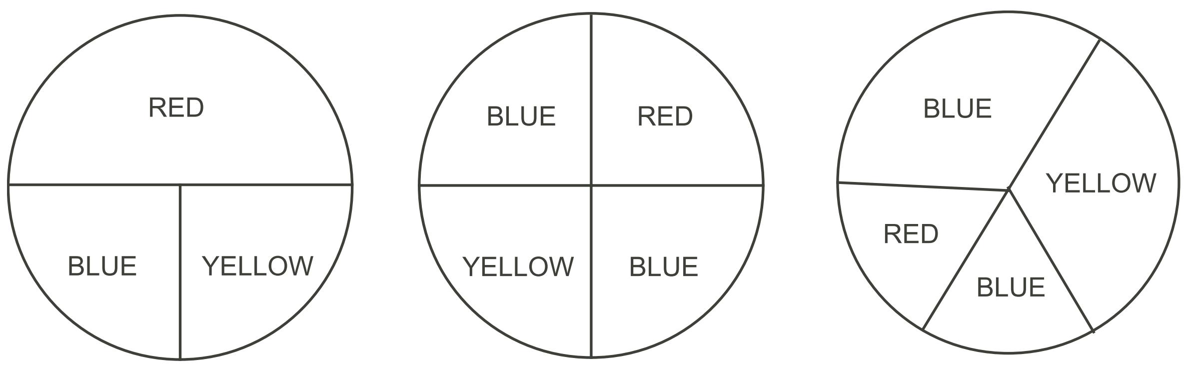
2. Describe an event that has:
   a) a probability close to zero
   b) a probability between 1/4 and 1/2
   c) a probability greater than 3/4
   d) a probability close to one.

BLM 5: Fair Games?

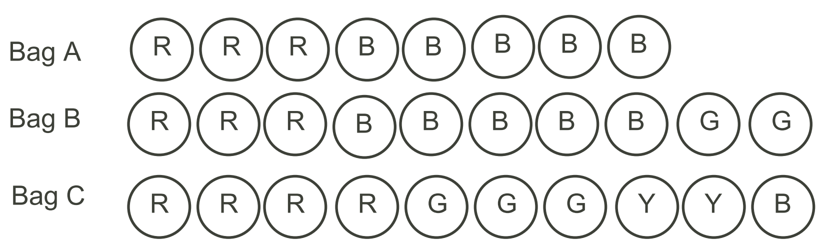
1. (a) Which of the spinners below would you choose if
   i) you need red to win; otherwise you lose;
   ii) you need blue to win; otherwise you lose;
   iii) you lose if you spin yellow; otherwise you win;
   iv) you win if you spin red or blue; otherwise you lose.

   (b) Give reasons for your choice in each case.

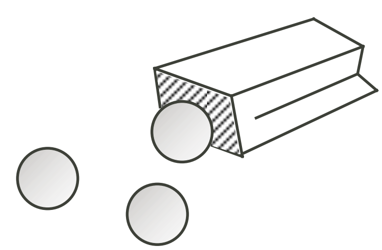
Spinner D Spinner E Spinner F



2. (a) Several marbles are placed in bags as indicated below, and drawn one at a time. Tell which is more likely for each bag and give reasons for your answers.
   i) drawing a red (R) or drawing a blue (B)
   ii) drawing a red (R) or drawing a green (G)
   iii) drawing a yellow (Y) or drawing a blue (B)



   (b) Tell which bag you would choose for a game if you lose unless
   i) you draw a blue
   ii) you draw a green
   iii) you draw a red.
   iv) you draw either a red or a green



   (c) Give reasons for each choice.

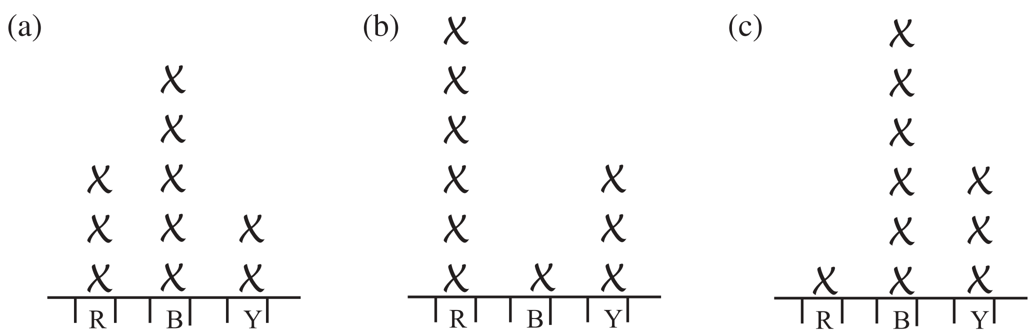
BLM 6: Fair Games (continued)

3. You have three cubes with different colours on the six faces as shown below.
   Cube A: 3 green faces, 2 red faces,1 blue face
   Cube B: 2 green faces, 1 red face, 2 blue faces, 1 white face
   Cube C: 1 red face, 1 blue face, 1 green face, 3 white faces

   (a) Which of the statements below would be true about each cube?
      i) The probability of rolling a blue is less than the probability of rolling a red
      ii) You are more likely to roll green than blue
      iii) The probability of rolling green is the same as the probability of rolling a different colour.

   (b) Give reasons for each choice.

4. Each graph below shows the results of spinning one of the spinners from #1 on BLM 5. Which of the spinners is most likely to have given the results shown in each line plot? Explain.



5. A Challenge:
   (a) Design at least 2 different spinners for each of the following games:
      i) a fair game in which you win if you spin red and your opponent wins if you spin blue.
      ii) a fair game in which you win if you spin an odd number and your opponent wins if you spin an even number.

   (b) Design at least 2 different dice for each of the following games:
      i) a fair game in which you win if you throw a multiple of 5 and your opponent wins if you throw a multiple of 3. (Ties are permitted.)
      ii) a fair game in which you win if you throw 3 or less and your opponent wins if you throw an even number. (Ties are permitted.)

BLM 7: Winning Strategies

1. Use coins or two-colour counters. Shake three of them. You win if two of them show the same face or colour. Otherwise, you lose. Try the game with your partner for 10 rounds. Did one of you win more often? Did you expect this result? What are the possible outcomes? What is the probability of getting two chips of the same colour? Is this a fair game? Why or why not?

Round / Name	1	2	3	4	5	6	7	8	9	10	Total number of wins

2. Use the same three counters. For this game you win if at least one head or red face shows. Otherwise your partner wins. Play this game 10 times. What were the results? Did you expect this? Is this a fair game? Why or why not?

Round / Name	1	2	3	4	5	6	7	8	9	10	Total number of wins

3. Toss a regular die. You win if an even number shows on any of the five faces you can see. Otherwise you lose. Predict the outcome of playing the game 10 times. Play with your partner. What happened? Did you expect this? Is this a fair game? Why or why not?

Round / Name	1	2	3	4	5	6	7	8	9	10	Total number of wins

4. Use coins or two-colour counters. Shake four of them. You win if at least two show the same face or colour. Otherwise your partner wins. Predict the outcome of playing this game. Play with your partner. What were the results? Did you explect this? Is this a fair game? Why or why not?

Round / Name	1	2	3	4	5	6	7	8	9	10	Total number of wins

BLM 8: Let's Play Fair

1. Suppose you have two regular dice and are playing a two-person game. You win if you roll a total that is an odd number. Your opponent wins if you roll a total that is an even number.

   (a) Do you think this is a fair game?

   (b) Why or why not?

   (c) Play the game for several rounds, and record the results in the chart below.

Round	Sample	Sample	1	2	3	4	5	6	7	8	9	10	11	12
Even/Odd?	O	E												
Win?Lose?	W	L												

   (d) Compare the results with your predictions. Is the game fair?

2. Suppose you are playing the same game but with three dice this time.

   (a) Will the game be fair or unfair? Why?

   (b) Play the game and record the results for several rounds in the chart below.

Round	Sample	Sample	1	2	3	4	5	6	7	8	9	10	11	12
Even/Odd?	O	E												
Win?Lose?	W	L												

   (c) Compare the results with your predictions. Is the game fair?

3. Suppose you roll two dice but multiply the numbers that turn up instead of adding them. An odd product is a win for you; an even product is a win for your opponent.

   (a) What effect, if any, will this have on the game? Why?

   (b) Test your prediction, by playing the game. Record the results in the chart below

Round	Sample	Sample	1	2	3	4	5	6	7	8	9	10	11	12
Even/Odd?	O	E												
Win?Lose?	W	L												

(c) Is the game fair?

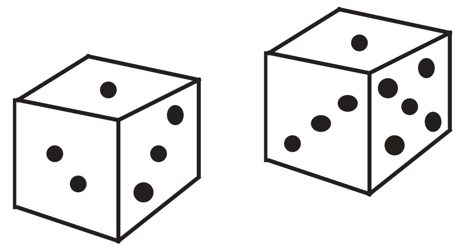
BLM 9: The Game of HOG

The game of "Hog" uses several dice. The rules are simple.

1. Choose the number of dice you wish to roll in your turn.
2. Roll them and add. This is your score, unless...
3. If you roll a '1' on any of the dice, your score is ZERO!

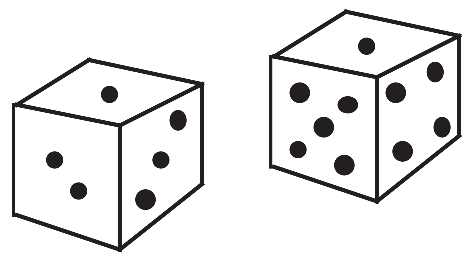
The questions below are about the game and will help you to analyze it.

1. Isaac wanted to roll 9 dice. He got 2, 3, 6, 4, 5, 1, 3, 4, 5. What was his score? Why?



2. Rachel chose to roll 7 dice. She got 3, 3, 5, 4, 2, 6, 2. What was her score? Why?

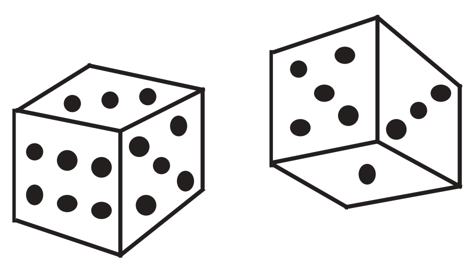
3. Ari rolled two dice. His score was 6. In how many ways could he have gotten 6 with two dice?



4. Michaela rolled 8 dice. Her score was 19.
   (a) Could all of the dice have been 3s? Why or why not?
   (b) Could three of the dice have been 3s? Why or why not?
   (c) Could one of the dice have been 6? Why or why not?

5. Chris decided that 100 dice should be rolled to get the greatest possible score. Do you agree with this? Why or why not?

6. Mel's strategy was to use two dice every time. Sandy's was to use just one die each time.
   (a) Do you think either of these is a good strategy? Why or why not?
   (b) Play the game with a partner. One of you should use Mel's strategy and the other should use Sandy's strategy. Is either of the strategies a good one? Why or why not?



7. Suppose you want to find the best strategy.
   (a) Would you play with the same number of dice for each roll? Why or why not?
   (b) What do you think is the best strategy? Why do you think it is the best?
   (c) Play with a partner, each of you using what you think is the best strategy. Play several games to give your strategies a good try-out.
   (d) What happened? Do you want to revise your strategy? Why or why not?

BLM 10: What Are the Odds?

For this game you need two dice or spinners and 20 counters of some kind, or you can keep track of the score with paper and pencil.

Play with a partner. One of you is the "Player" and the other is the "Opponent".

Each of you starts with 10 counters/points.

The Player rolls the two dice and adds the numbers.

If the sum is 7, the Opponent gives the Player three counters/points.

If the sum is anything except 7, the Player gives one counter/point to the Opponent.

The game ends after 10 rolls of the dice.

The winner is the one with the most counters, BUT, ..

If either of you runs out of counters before you have rolled the dice ten times, then that person loses.

Is this a fair game? Why or why not?

The charts below may help you keep track of your points as you play the game a few times. Write "Y" for "yes" if the sum is seven and "N" for "no" if the sum is not seven. Then record how many counters both the Player and the Opponent have after that roll.

GAME 1	Roll	1	2	3	4	5	6	7	8	9	10
	Sum of 7?										
Counters for PLAYER	10										
Counters for OPPONENT	10										

GAME 2	Roll	1	2	3	4	5	6	7	8	9	10
	Sum of 7?										
Counters for PLAYER	10										
Counters for OPPONENT	10										

GAME 3	Roll	1	2	3	4	5	6	7	8	9	10
	Sum of 7?										
Counters for PLAYER	10										
Counters for OPPONENT	10										

BLM 11: Random Walk

1. Place your marker on "START" on the board to the right.

2. Select five adjacent numbers (horizontally or vertically) in the Random Number Table below.

3. An even number means you move your marker North the number of spaces given by the number in the table. An odd number means you move your marker South the number of spaces given by the number in the table.

4. Each 'random walk' is five steps long. After you have moved your marker according to the five adjacent numbers you chose, record your final position in the chart below. If a walk takes you off the board, record your final position as "off North" or "off South".

N7
N6
N5
N4
N3
N2
N1
START
S1
S2
S3
S4
S5
S6
S7

Random Number Table

3	1	1	6	3	3	4	2	6	6	5	4
6	4	3	5	6	1	5	4	4	2	4	3
5	6	1	5	4	1	3	2	2	6	1	1
1	5	6	2	2	1	4	4	1	5	3	2
5	3	6	1	2	5	1	2	6	3	3	1
5	2	1	6	5	1	4	2	3	1	5	3
2	3	5	6	2	4	4	2	5	2	3	2
4	6	4	1	6	4	6	4	4	4	3	6
5	4	4	6	4	2	4	1	2	5	2	1
5	1	6	4	4	3	1	6	5	2	3	1

5. Record the results of ten random walks.

Walk #	Sample	1	2	3	4	5	6	7	8	9	10
End of Walk											
Number of Steps											

6. Reasoning from your own results, what conclusions can you draw about the following questions?

   (a) Is a walk more likely to go off the board to the North or to the South?

   (b) How many steps will most walks take before going off the board?

   (c) How many times out of 100 walks will a walk end on the board?

   Give reasons for your choices.

BLM 12: Random Walk 2

For the activity, you will need a paper clip to complete the spinner shown. Place your marker on "START" in the grid below.

On each turn, select a set of six adjacent numbers in the Random Number Table on BLM 7. For each "step", spin the spinner to see what direction you should move.

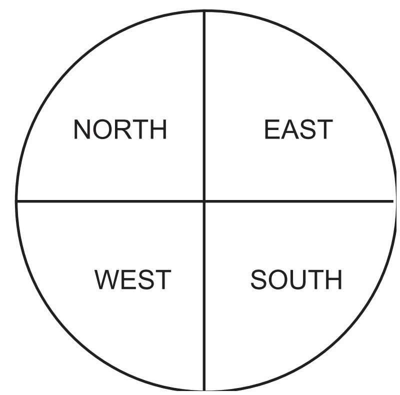
Before you begin, predict

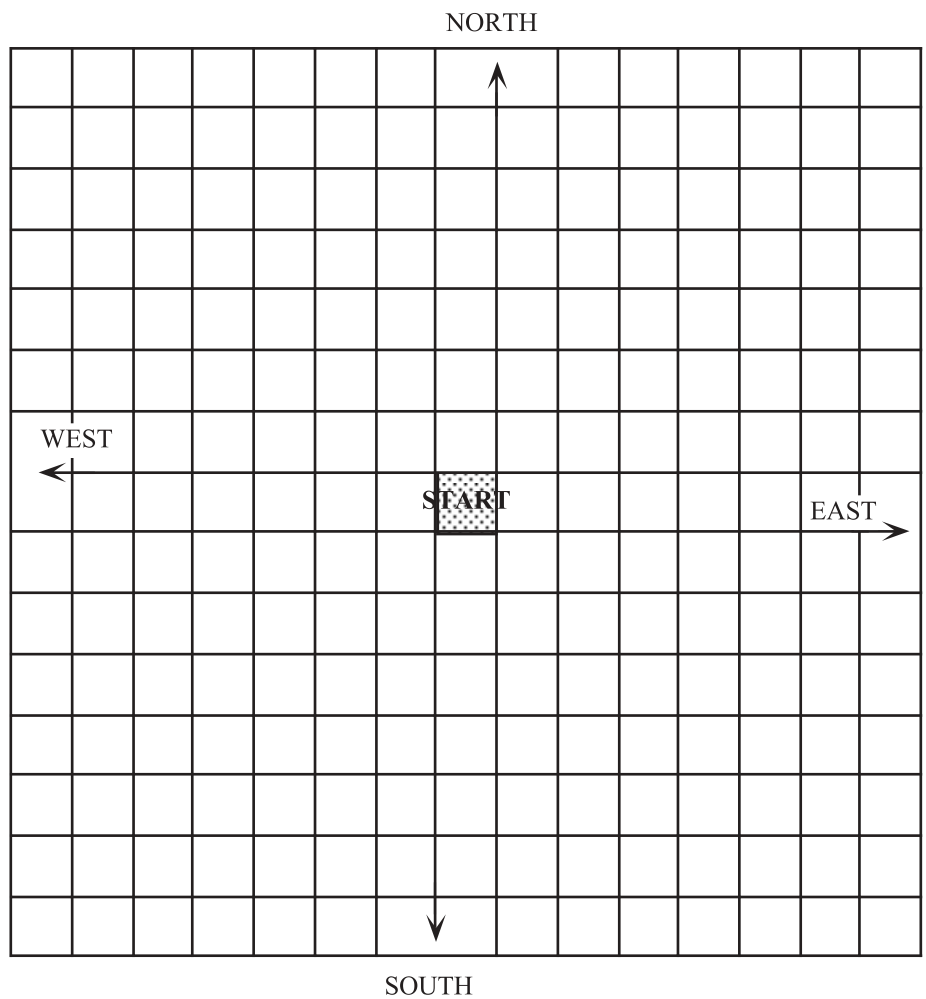
i) the number of times, in 10 walks, that you think you will go off the board

ii) the number of steps usually needed to go off the board

Take 10 walks of 6 steps each, and record your final position for each one.

Spinner

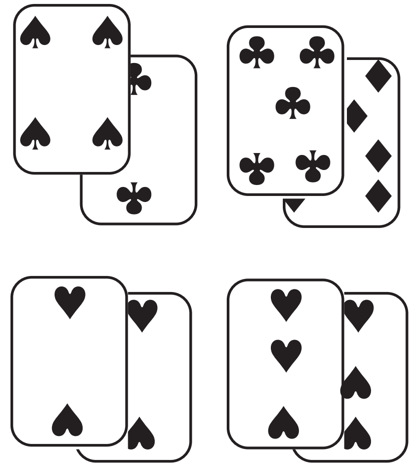




1. Compare your results with your predictions. Do you want to change any of your predictions?

2. Compare your results with other groups. Does this make you want to change your predictions? Why or why not?

BLM 13: How Random is Random?



1. Play this game with 3 or 4 others.
2. You will need a deck of cards, with the face cards removed.
3. Deal five cards to each player and place the rest of the deck face down in a pile.
4. After each card you play, draw one from the face-down pile so that you always have five cards in your hand.
5. Each player in turn plays a card face up, adding the values on black cards, and subtracting the values on the red cards.
6. Whenever a person makes the score go over 10 or below -10, that person scores one point. (See the example in the box below.)
7. Continue the play until all the cards are used.
8. The winner is the one with the most points.

You can use the random walk board below to help you decide the total after each card you play. Black cards move to the right and red cards to the left.

–10	–9	–8	–7	–6	–5	–4	–3	–2	–1	0	1	2	3	4	5	6	7	8	9	10

Example:

Sally plays a black three and says "3".
Chan plays a red 4 and says "-1" because he moves 4 places to the left of '3'.
Eli plays a black 7 and says "6" because he moves 7 spaces to the right of '-1'.
Dorrie plays a black six and says "I score one point because six moves to the right passes 10".
Sally starts at zero with her next move.

Questions to think about:

How is this game similar to the Random Walk on BLM 11? How is it different?

Are playing cards a good source of random numbers? Why or why not?

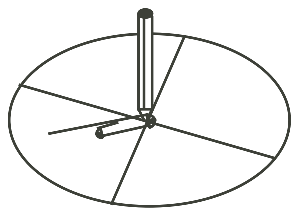
BLM 14: Constructing Spinners

To construct spinners, use the templates at the bottom of the pages. Paste the spinners on to bristol/cardboard.

Method 1:

For the spinner, straighten a paper clip as shown below.

Hold the spinner in place with a pen or pencil at the centre of the circle.



Flick the point of the paper clip with a finger.

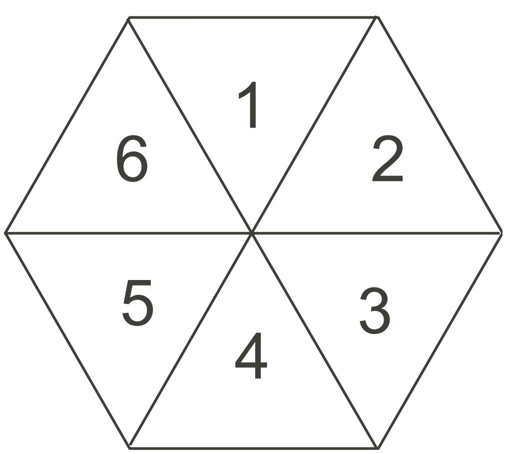
This is the simplest way to construct an acetate spinner for use with an overhead projector.

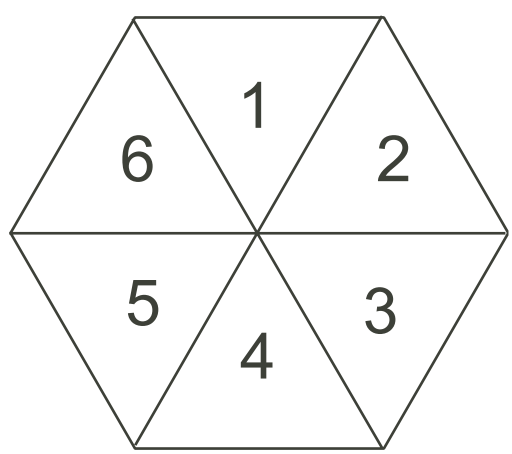
Method 2:

Cut arrows from bristol board or cardboard and punch a hole in one end.

Punch a hole in the centre of each spinner.
Use a paper fastener to fasten the two pieces together.

The connection should be tight enough so the arrow doesn't wobble,
but loose enough so that it spins freely.





BLM 15: Tree Diagrams

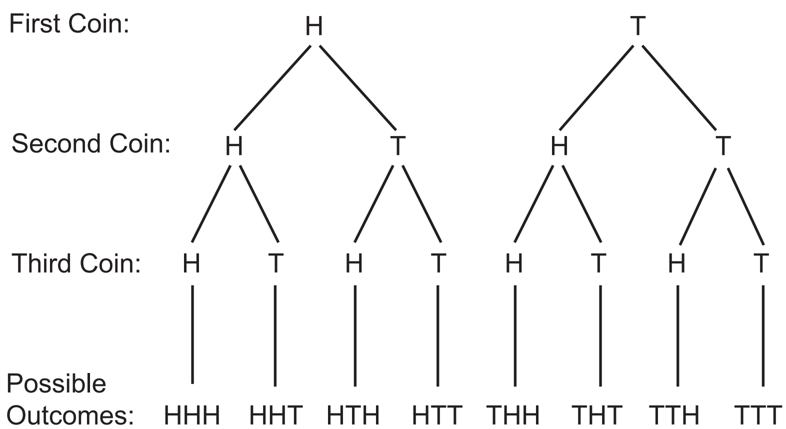
A tree diagram is a way of counting all possible outcomes for a simple experiment. For example, suppose we want to identify all possible outcomes for flipping three coins.

Step 1: List the possible outcomes for the first of the three coins.

Step 2: Draw 'branches' from each of these outcomes. The number of branches will be the number of possible outcomes for the second coin — that is, two.

Step 3: Draw branches for each of the possible outcomes of flipping the third coin.

Step 4: Read down the chart from the top to identify 8 different combinations — that is, the eight possible outcomes when three coins are flipped.



The number of possibilities for the first coin is 2.

For each of these, the number of possibilities for the second coin is 2.

For each of these, the number of possibilities for the third coin is 2.

Thus, there are 8 possible outcomes:

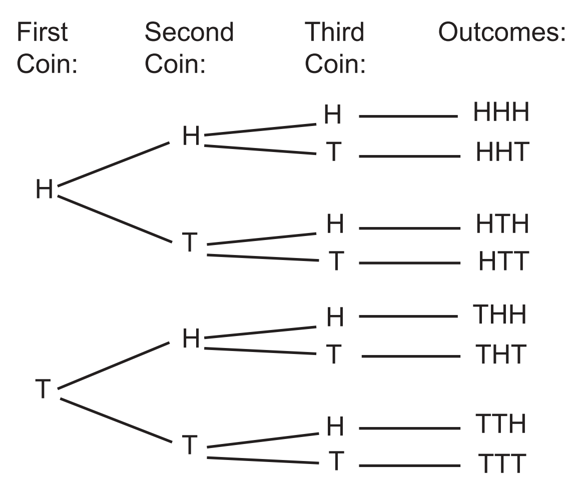
$$\mathbf{2 \times 2 \times 2 = 8}.$$

Tree diagrams are not useful if there are too many outcomes — for example, rolling three dice (216 outcomes) or even rolling two dice (36 outcomes).

Tree diagrams are useful, however, for determing the outcomes of experiments like the following:

(a) flipping a coin and rolling a die;
(b) the number of outfits possible with rust, green, black, and cream t-shirts and brown, green, and orange shorts;
(c) spinning two or three spinners like the ones on BLM 3.

Tree diagrams can also be drawn horizontally:



BLM 16: Random Numbers

Frequently, in studying probability, mathematicians use random numbers to simulate experiments. Random numbers (sets of single digits) have two characteristics:

1) Each digit has the same probability of occurring
2) Adjacent digits are independent of one another. That is, the occurrence of one digit has no effect on the occurrence of any other.

When we use the word "random" in everyday language we frequently mean only that something appears to happen haphazardly. The mathematical meaning is more precise and rigorous.

> Definitions of "random":
> 1) from Roget's Thesaurus: casual, aimless, by chance, accidental, unintended, purposeless.
> 2) from James and James "Mathematical Dictionary": A random sequence is a sequence that is irregular and non-repetitive. The probability of a particular digit being chosen is the same for each digit, and the choices at two different places are independent.

To illustrate that what we think is random often isn't, consider the following study. Students are being given a quick quiz in which they write only the answers. The first 5 questions are

i) How much is $3+3$? ii) How much is $5+5$?
iii) How much is $9+9$? iv) How much is $13+13$?
v) Name a whole number between 5 and 12.

One might expect that students' responses to the last item would range equally from 6 to 11. In actual fact, the number '7' was the overwhelming favourite. Reasons suggested included the following:

i) The other questions dealt with computation, so students did a computation using 5 and 12 $(12-5=7)$
ii) The previous questions dealt with odd numbers, so perhaps students automatically gave an odd number for question (v).

Whatever the reason, even though the students thought they were choosing a number at random, something was acting to produce '7' as a response. (For more on this see Other Resources # 17.)

Another study involved text-book writers. All writers felt that in dealing with basic facts (addition, say, at grade 2) they had managed to include all facts about equally often. That is, that each fact included in a practice page was chosen randomly. However, an analysis of the manuscripts showed otherwise. They had concentrated on the "more difficult" facts to the detriment of "adding 1", "adding 0" or "adding doubles'. It would appear that their knowledge of the practice students need was preventing them from choosing facts in a truly random manner even when they tried to do so.

If you choose to discuss these meanings of random with your students, you might wish to have them suggest situations which may, on the surface, appear to be random, but are probably not.

Solutions & Notes

Activity 1: Fair Spinners

BLM 1

The sums of the two numbers on the two dice are given in the chart below.

Number on second die \ Number on first die	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Extensions in Mathematics

BLM 2

1. Answers will vary.

2. (a) The possible outcomes for the three sets are given below:

Set 1: B_1B_2, B_1R, B_2R
Set 2: B_1B_2, B_1R_1, B_2R_1, B_1R_2, B_2R_2, R_1R_2
Set 3: B_1B_2, B_2B_3, B_1B_3, B_1R, B_2R, B_3R

(a) (b) The probability of drawing two of the same colour for
Set 1 is 1 out of 3
Set 2 is 2 out of 6 (or 1 out of 3)
Set 3 is 3 out of 6 (or 1 out of 2)

(c) The probability of drawing two that are not the same colour for
Set 1 is 2 out of 3
Set 2 is 4 out of 6 (or 2 out of 3)
Set 3 is 3 out of 6 (or 1 out of 2)

If students expect to be the first to draw tiles they will probably choose to use Set 1 or Set 2 because they would have a better chance of drawing 2 of the same colour.

However, Set 3 has the same probability for drawing two of the same colour as for drawing two that are not the same colour. Using Set 3 makes this a fairer way of choosing who goes first.

Solutions & Notes

Extension 2

The products of the numbers on the two dice are given in the chart below.

Number on second die \ Number on first die	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	8	10	12
3	3	6	9	12	15	18
4	4	8	12	16	20	24
5	5	10	15	20	25	30
6	6	12	18	24	30	36

Since the order of the dice is important, the probability of rolling '$1+5$' is the same as the probability of rolling '1×5', namely 1 out of 36.

However, the probability of rolling a sum of 5 and the probability of rolling a product of 5 are not the same. The first is 4 out of 36; the second is just 2 out of 36.

Students should compare the two charts for similarities and differences.

Activity 2

BLM 3

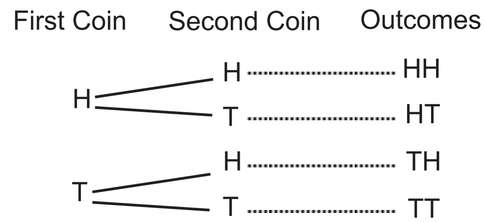
One way of symbolizing "the probability of event x" is P(x). This notation is common, but not universal.

1. (a) The outcomes are 1, 2, 3, 4, 5, 6
   (b) (i) The probability of rolling a 1, or P(1), is 1 out of 6 or $\frac{1}{6}$.
   (ii) P(even number) = 3 out of 6
   (iii) P(multiple of 3) = 2 out of 6
   (iv) P(greater than 4) = 2 out of 6

2. (a) The outcomes are 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 (Refer back to the chart in #3 on BLM 1)
   (b) (i) P(12) = 1 out of 36
   (ii) P(less than 5) = 6 out of 36
   (iii) P(a double) = 6 out of 36
   (iv) P(6 or less) = 15 out of 36
   (v) P(7 or more) = 21 out of 36

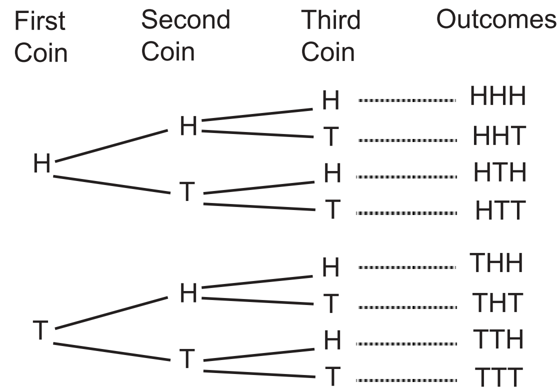
Solutions & Notes

3. (a) The outcomes are HH, HT, TH, TT
A tree diagram (see BLM 15) could be used here.



(b) (i) P(HH) = 1 out of 4
(ii) P(a head and a tail) = 2 out of 4
(iii) P(TT) = 1 out of 4

4. Students should list the outcomes first.
(See tree diagram)
(i) P(2H, 1T) = 3 out of 8
(ii) P(3H) = 1 out of 8
(iii) P(1H, 2T) = 3 out of 8



5.

		Spinner A	Spinner B	Spinner C
(i)	P(3)	1 out of 4	1 out of 6	2 out of 8
(ii)	P(odd)	2 out of 4	3 out of 6	4 out of 8
(iii)	P(greater than 4)	0	2 out of 6	0

Activity 3

BLM 5

1. (i) The spinner that gives the best chance of showing red is Spinner D.
P(red) = $\frac{1}{2}$ for spinner D but less than this for Spinners E and F
(ii) Both E and F have a probability of $\frac{1}{2}$ for spinning blue. Spinner D has P(blue) equal to $\frac{1}{4}$.
(iii) If you lose by spinning yellow, you want the spinner with the least chance of spinning yellow, and that is spinner A or B.
(iv) You win if you spin red or blue. The spinner that gives you the best chance of winning is either D or E since the total area of red and blue is greater in spinners D and E than in spinner F.

2. (a) (i) for bag A, the P(blue) is greater than P(red): P(blue) = $\frac{5}{8}$ and P(red) = $\frac{3}{8}$

for bag B, P(blue) is greater than P(red): P(blue) = $\frac{5}{10}$ and P(red) = $\frac{3}{10}$

for bag C, P(red) is greater than P(blue): P(red) = $\frac{4}{10}$ and P(blue) = $\frac{1}{10}$

Solutions & Notes

(ii) for bag A, P(red) is greater than P(green): P(red) = $\frac{3}{8}$ and P(green) = 0

for bag B, P(red) is greater than P(green): P(red) = $\frac{3}{10}$ and P(green) = $\frac{2}{10}$

for bag C, P(red) is greater than P(green): P(red) = $\frac{4}{10}$ and P(green) = $\frac{3}{10}$

(iii) for bag A, P(blue) is greater than P(yellow): P(blue) = $\frac{5}{8}$ and P(yellow) = 0

for bag B, P(blue) is greater than P(yellow): P(blue) = $\frac{5}{10}$ and P(yellow) = 0

for bag C, P(yellow) is greater than P(blue): P(yellow) = $\frac{2}{10}$ and P(blue) = $\frac{1}{10}$

(b) (c) (i) To give the best chance of drawing a blue choose bag A, since P(blue) = $\frac{5}{8}$ for bag A which is better than the probabilities for bags B and C

(ii) Bag C gives the best chance of drawing a green.

(iii) Bag A gives the best chance of drawing a red. (In order to answer this, students must compare $\frac{3}{8}$ and $\frac{4}{10}$)

(iv) Bag C gives the best chance. The probability of drawing either a red or a green from each bag is 3 out of 8 for bag A, 5 out of 10 for bag B, and 7 out of 10 for bag C.

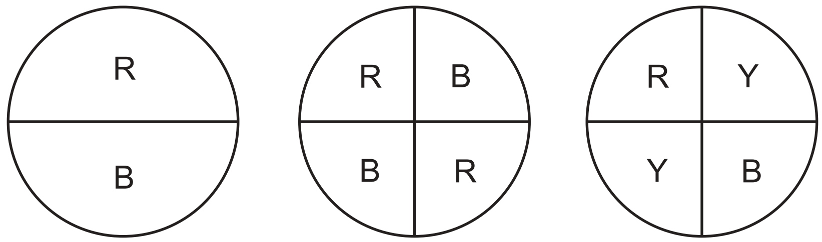
3.

	Cube A		**Cube B Cube C**
(i)	true	false	false
	P(B) = 1 out of 6	P(B) = 2 out of 6	P(B) = 1 out of 6
	P(R) = 2 out of 6	P(R) = 1 out of 6	P(R) = 1 out of 6
(ii)	true	false	false
	P(G) = 3 out of 6	P(G) = 2 out of 6	P(G) = 1 out of 6
	P(B) = 1 out of 6	P(B) = 2 out of 6	P(B) = 1 out of 6
(iii)	false	true	true
	P(G) = 3 out of 6	P(G) = 2 out of 6	P(G) = 1 out of 6
	P(B) = 1 out of 6	P(B) = 2 out of 6	P(R) = 1 out of 6
	P(R) = 2 out of 6		P(B) = 1 out of 6

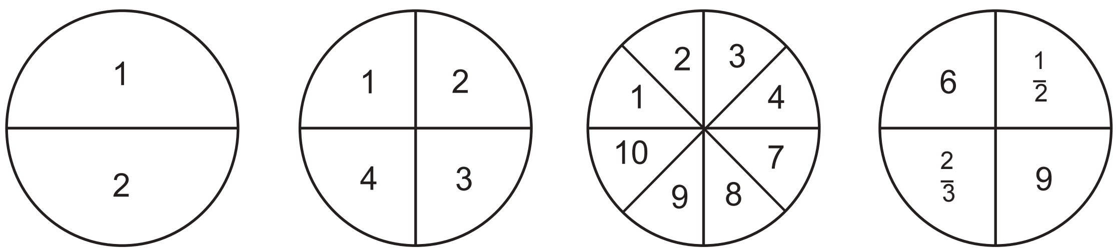
4. Answers may vary
Graph (a) is most likely but not necessarily from Spinner E because it shows about twice as many blue spins as red or yellow.
Graph (b) is most likely but not necessarily from Spinner D because it shows more red than either blue or yellow.
Graph (c) is most likely but not necessarily from Spinner F because it shows more blue and yellow than red, but more blue than yellow.

Solutions & Notes

5. Answers will vary
   (a) (i) The simplest spinner would be half red and half blue, but other colours could be added as long as red and blue are equally likely.

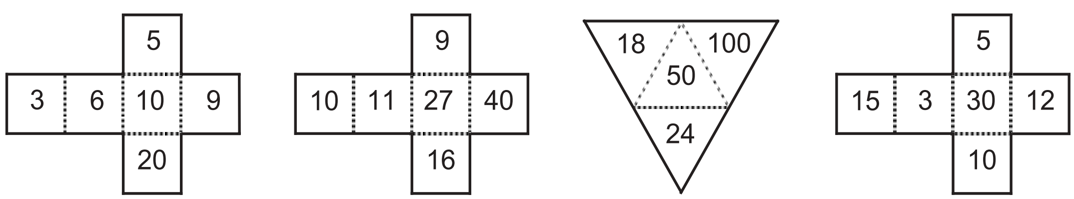


   (ii) Reasoning as for (i), the simplest spinner would have one odd number and one even. If other numbers are added, there should be the same number of odd numbers as of even numbers.



   Some students may reason that a spinner with one odd and one even number, plus numbers that are neither even or odd will work. See the fourth spinner above.

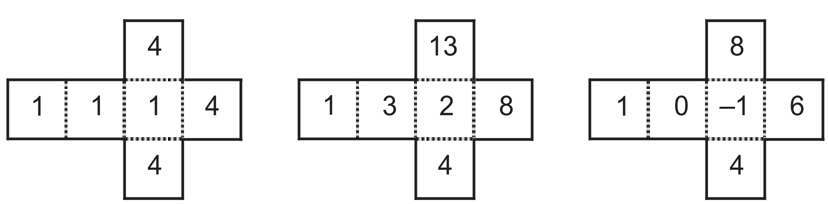
   (b) (i) If a cube has 5 on three faces and 3 on the other three faces, this die will match the conditions of the problem. Other possibilities are given as nets below.



   In the second example, numbers that are multiples of neither 3 nor 5 are included. Students may not consider using such numbers. You may wish to drop a hint to get a variety of spinners. You will notice from the third example that the shape of the required die need not be restricted to a cube.

   On the fourth net there are two numbers that are multiples of both 3 and 5. With such a die it is possible that, with one roll (say, 30), both players win and we have a tie.

   (ii) Some possible nets are shown.



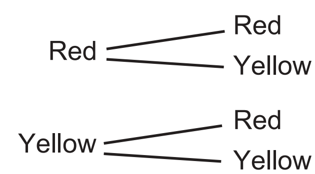
Solutions & Notes

The first net shows one of the simplest examples, with no overlap between 'your' winning numbers and your opponent's.
The second includes a '2' which is in both winning categories and 9 and 13 which are in neither winning category.
The third shows '0' and '–1' which are certainly less than 3 but may not be considered by many students. Some students may use fractions $\left(\text{e.g.}, \frac{1}{2}, \frac{3}{4}, 2\frac{1}{5}\right)$ as numbers less than 3.

Activity 4: Probability Experiments

BLM 7

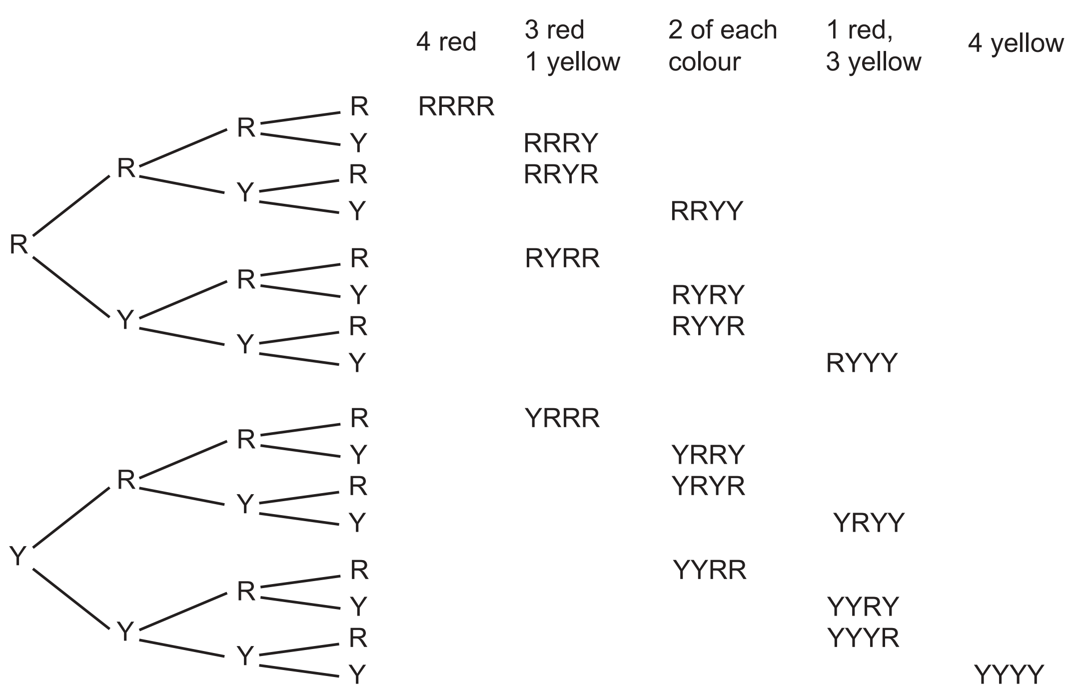
1. The outcomes can be determined using a tree diagram. There are 4 outcomes. Two of these have both chips showing the same colour: RR and YY. Thus, the probability of having two chips of the same colour is 2 out 4 (or 1 out of 2 or $\frac{1}{2}$). The probability of having two chips showing different colours is also 2 out of 4. This is a fair game.



2. The outcomes, as determined for #1 are RR, RY, YR, YY. Three of these have at least one red face. Only one does not. This is not a fair game.

3. There are 3 even numbers on a die. Since only 1 number is hidden on each roll (i.e. the number on the bottom), the player will always be able to see an even number. This game is not fair.

4. The tree diagram below shows all possible outcomes.



Solutions & Notes

(The outcomes are spread out only to indicate the groupings: all-red, 3-red-one-yellow, 2-red-2-yellow, 1-red-3-yellow, all yellow)
It should be clear that all outcomes have at least two of the same colour. Thus the "partner" never wins. This is not a fair game.

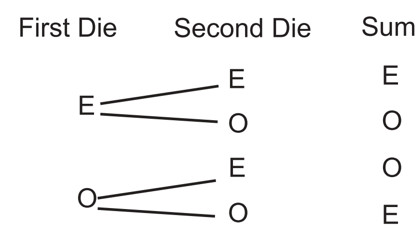
Students may simply list all possible colour combinations as listed in parentheses above. From this list it is obvious that all possibilities have at least two faces of the same colour. It is not necessary to know how many of each exist.

BLM 8

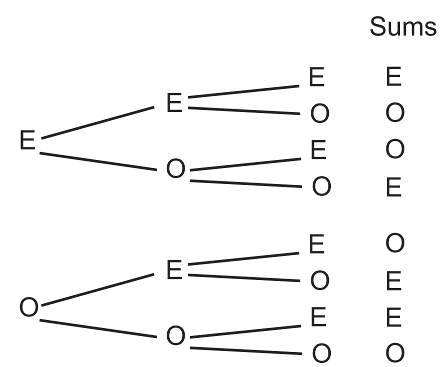
1. One way to solve this problem is to use the chart in #3 on BLM 1. Counting the odd and even numbers shows that they are equally likely and that this is a fair game.

   A simpler way is to consider the Even-Odd combinations. For example,

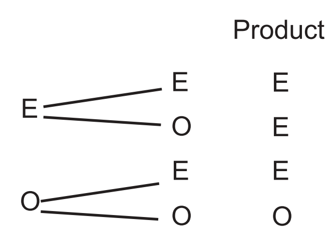
   This can be done because each die has an equal chance of being odd or even.



2. Using a tree digram of Evens and Odds shows that the sums are equally divided between even and odd. Thus the game is fair.



3. If students realize that a single even number as a factor gives a product that is even, this is a fairly simple analysis.
   The game is unfair.



Solutions & Notes

BLM 9

1. Isaac's score was 0 because he rolled a '1'.
2. Rachel's score was 25 because she didn't roll a '1'.
3. Ari might have rolled 3, 3 or 2, 4 or 4, 2.
   Students may list 1, 5 or 5, 1, forgetting about the '1 = 0' rule.
4. (a) If all 8 dice had shown 3, her score would have been 24. Thus, all the dice could have been '3'.
   (b) If 3 dice showed '3', she would score 9 with those dice.
   This leaves 10 to be scored with the other five dice. If each one showed '2', then she would have a total of 19.
   Yes, three of the dice could have been '3'.
   (c) If one die showed 6 and all the others showed 2 which is the lowest they could be, her score would be twenty. If one of the others showed 1, her score would not be 19, but would be zero.
   $$6+2+2+2+2+2+2+2=20$$
   $6+2+2+2+2+2+2+1=19$ but her score would be zero.

Answers to further questions on BLM 9 will vary.
Look for logical reasoning, an understanding of the possibilities and probabilities.
For example, the probability of rolling 1 is 1 out of 6. Therefore there should be several 1's if 100 dice are rolled.

BLM 10

See notes under "Extensions in Mathematics", Activity 4.

Activity 5

BLM 11

Results are given here for the authors' sample of 100 random walks:

42 went off to the North
10 went off to the South
The average number of "steps" to go off was about 3.
The most common ending points on the walk way were S3, S1, N2, and N6.

BLMs 12 and 13

Results will vary

Suggested Assessment Strategies

Investigations

Investigations involve explorations of mathematical questions that may be related to other subject areas. Investigations deal with problem posing as well as problem solving. Investigations give information about a student's ability to:

- identify and define a problem;
- make a plan;
- create and interpret strategies;
- collect and record needed information;
- organize information and look for patterns;
- persist, looking for more information if needed;
- discuss, review, revise, and explain results.

Journals

A journal is a personal, written expression of thoughts. Students express ideas and feelings, ask questions, draw diagrams and graphs, explain processes used in solving problems, report on investigations, and respond to open-ended questions. When students record their ideas in math journals, they often:

- formulate, organize, internalize, and evaluate concepts about mathematics;
- clarify their thinking about mathematical concepts, processes, or questions;
- identify their own strengths, weaknesses, and interests in mathematics;
- reflect on new learning about mathematics;
- use the language of mathematics to describe their learning.

Observations

Research has consistently shown that the most reliable method of evaluation is the ongoing, in-class observation of students by teachers. Students should be observed as they work individually and in groups. Systematic, ongoing observation gives information about students':

- attitudes towards mathematics;
- feelings about themselves as learners of mathematics;
- specific areas of strength and weakness;
- preferred learning styles;
- areas of interest;
- work habits — individual and collaborative;
- social development;
- development of mathematics language and concepts.

In order to ensure that the observations are focused and systematic, a teacher may use checklists, a set of questions, and/or a journal as a guide. Teachers should develop a realistic plan for observing students. Such a plan might include opportunities to:

- observe a small number of students each day;
- focus on one or two aspects of development at a time.

Suggested Assessment Strategies

Student Self-Assessment

Student self-assessment promotes the development of metacognitive ability (the ability to reflect critically on one's own reasoning). It also assists students to take ownership of their learning, and become independent thinkers. Self-assessment can be done following a co-operative activity or project using a questionnaire which asks how well the group worked together. Students can evaluate comments about their work samples or daily journal writing. Teachers can use student self-assessments to determine whether:

- there is change and growth in the student's attitudes, mathematics understanding, and achievement;
- a student's beliefs about his or her performance correspond to his/her actual performance;
- the student and the teacher have similar expectations and criteria for evaluation.

Resources for Assessment

"For additional ideas, see annotated Other Resources list on page 53, numbered as below."

1. The Ontario Curriculum, Grades 1-8: Mathematics.

2. *Assessment Standards for School Mathematics*, NCTM, 1995.

3. *Linking Assessment and Instruction in Mathematics: Junior Years*, Ontario Association of Mathematics Educators/OMCA/OAJE, Moore et al., 1996.

4. *Mathematics Assessment: Myths, Models, Good Questions, and Practical Suggestions*, by Jean Kerr Stenmark (Ed.), NCTM, 1991.

5. "Assessment", *Arithmetic Teacher* Focus Issue, February 1992, NCTM.

6. *How to Evaluate Progress in Problem Solving*, by Randall Charles et al., NCTM, 1987.

7. *Assessment in the Mathematics Classroom*, Yearbook, NCTM, 1993.

Suggested Assessment Strategies

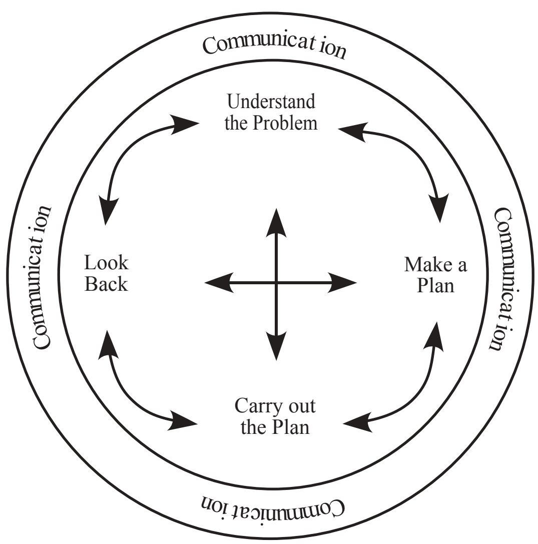
A GENERAL PROBLEM SOLVING RUBRIC

This problem solving rubric uses ideas taken from several sources. The relevant documents are listed at the end of this section.

"US and the 3 R's"

There are five criteria by which each response is judged:
Understanding of the problem,
Strategies chosen and used,
Reasoning during the process of solving the problem,
Reflection or looking back at both the solution and the solving, and
Relevance whereby the student shows how the problem may be applied to other problems, whether in mathematics, other subjects, or outside school.

Although these criteria can be described as if they were isolated from each other, in fact there are many overlaps. Just as communication skills of one sort or another occur during every step of problem solving, so also reflection does not occur only after the problem is solved, but at several points during the solution. Similarly, reasoning occurs from the selection and application of strategies through to the analysis of the final solution. We have tried to construct the chart to indicate some overlap of the various criteria (shaded areas), but, in fact, a great deal more overlap occurs than can be shown. The circular diagram that follows (from OAJE/OAME/OMCA "Linking Assessment and Instruction in Mathematics", page 4) should be kept in mind at all times.



There are four levels of response considered:

Level 1: Limited identifies students who are in need of much assistance;
Level 2: Acceptable identifies students who are beginning to understand what is meant by 'problem solving', and who are learning to think about their own thinking but frequently need reminders or hints during the process.
Level 3: Capable students may occasionally need assistance, but show more confidence and can work well alone or in a group.
Level 4: Proficient students exhibit or exceed all the positive attributes of the **Capable** student; these are the students who work independently and may pose other problems similar to the one given, and solve or attempt to solve these others.

Suggested Assessment Strategies

LEVEL OF RESPONSE →

CRITERIA FOR ASSESSMENT ↓

	Level 1: Limited	Level 2: Acceptable	Level 3: Capable	Level 4: Proficient
UNDERSTANDING	• requires teacher assistance to interpret the problem • fails to recognize all essential elements of the task	• shows partial understanding of the problem but may need assistance in clarifying	• shows a complete understanding of the problem	• shows a complete understanding of the problem
STRATEGIES	• needs assistance to choose an appropriate strategy	• identifies an appropriate strategy	• identifies an appropriate strategy	• identifies more than one appropriate strategy
	• applies strategies randomly or incorrectly • does not show clear understanding of a strategy[1] • shows no evidence of attempting other strategies	• attempts an appropriate strategy, but may not complete it correctly[2] • tries alternate strateges with prompting	• uses strategies effectively • may attempt an inappropriate strategy, but eventually discards it and tries another without prompting	• chooses and uses strategies effectively[3] • recognizes an inappropriate strategy quickly and attempts others without prompting
REASONING	• makes major mathematical errors • uses faulty reasoning and draws incorrect conclusions • may not complete a solution	• may present a solution that is partially incorrect	• produces a correct and complete solution, possibly with minor errors	• produces a correct and complete solution, and may offer alternative methods of solution
	• describes[4] reasoning in a disorganized fashion, even with assistance • has difficulty justifying[5] reasoning even with assisstance	• partially describes[4] a solution and/or reasoning or explains fully with assistance • justification[5] of solution may be inaccurate, incomplete or incorrect	• is able to describe[4] clearly the steps in reasoning; may need assistance with mathematical language • can justify[5] reasoning if asked; may need assistance with language	• explains reasoning in clear and coherent mathematical language • justifies[5] reasoning using appropriate mathematical language
REFLECTION	• shows no evidence of reflection or checking of work • can judge the reasonableness of a solution only with assistance	• shows little evidence of reflection or checking of work • is able to decide whether or not a result is reasonable when prompted to do so	• shows some evidence of reflection and checking of work • indicates whether the result is reasonable, but not necessarily why	• shows ample evidence of reflection and thorough checking of work • tells whether or not a result is reasonable, and why
RELEVANCE	• unable to identify similar[6] problems	• unable to identify similar[6] problems	• identifies similar[6] problems with prompting	• identifies similar[6] problems, and may even do so before solving the problem
	• unlikely to identify extensions[7] or applications of the mathematical ideas in the given problem, even with assistance	• recognizes extensions[7] or applications with prompting	• can suggest at least one extension[7], variation, or application of the given problem if asked	• suggests extensions[7], variation, or applications of the given problem independently

Suggested Assessment Strategies

Notes on the Rubric

1. For example, diagrams, if used, tend to be inaccurate and/or incorrectly used.

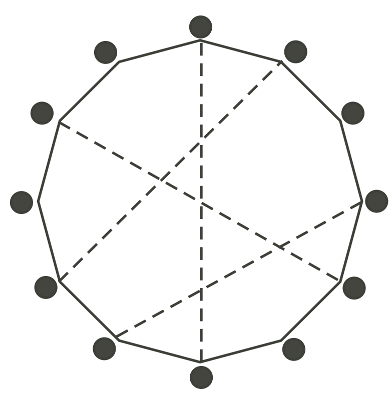
2. For example, diagrams or tables may be produced but not used in the solution.

3. For example, diagrams, if used, will be accurate models of the problem.

4. To *describe* a solution is to tell *what* was done.

5. To *justify* a solution is to tell *why* certain things were done.

6. *Similar* problems are those that have similar structures, mathematically, and hence could be solved using the same techniques.
   For example, of the three problems shown below right, the better problem solver will recognize the similarity in structure between Problems 1 and 3. One way to illustrate this is to show how both of these could be modelled with the same diagram:



Problem 1: There were 8 people at a party. If each person shook hands once with each other person, how many handshakes would there be? How many handshakes would there be with 12 people? With 50?

Problem 2: Luis invited 8 people to his party. He wanted to have 3 cookies for each person present. How many cookies did he need?

Problem 3: How many diagonals does a 12-sided polygon have?

   Each dot represents one of 12 people and each dotted line represents either a handshake between two people (Problem 1, second question) or a diagonal (Problem 3).

   The weaker problem solver is likely to suggest that Problems 1 and 2 are similar since both discuss parties and mention 8 people. In fact, these problems are alike only in the most superficial sense.

7. One type of extension or variation is a "what if...?" problem, such as "What if the question were reversed?", "What if we had other data?", "What if we were to show the data on a different type of graph?".

Suggested Assessment Strategies

Suggested Adapted Rubric for Activity 2

The rubric below has been adapted for BLM 7 of Activity 4: Winning Strategies. The rubric examines the understanding of the problems, the choice of strategies, the reasoning involved, and some reflection on the results.

Level 1: Limited	Level 2: Acceptable	Level 3: Capable	Level 4: Proficient
• interprets "fair" in a personal manner	• may need reminding that "fair" has to do with likelihood and not personal perception	• understands "fair" to mean "equally likely"	• understands "fair" and "unfair" as related to the probabilities of the outcomes
• can follow the instructions for playing the games but may not record the results carefully	• can follow the instructions and record the results accurately in the chart	• follows instructions for the first game or two, then shows evidence of considering all possible outcomes in order to predict results	• has no difficulty with the games and very soon relates the results to the probabilities of various outcomes
• does not refer to possible outcomes in deciding if a game is fair or not	• may analyze a game as fair or unfair based on the results of playing the game, but will relate fairness to probabilities of the outcomes if this is suggested	• uses probabilities of outcomes to decide if a game is fair or not	• uses tree diagrams or an organized list to determine outcomes for a game and uses the probabilities of these to identify a game as fair or unfair
• is not sure how to predict whether or not a game is fair without playing it	• can use possible outcomes to predict if a game is fair or not, but feels confident in conclusions only if playing the game seems to lead to the same conclusion	• is quite confident in using the possible outcomes to predict if a game is fair or unfair, without actually playing the game	• shows no need to play a game in order to predict its fairness

Other Resources

1. The Ontario Curriculum, Grades 1-8: Mathematics.

2. *Assessment Standards for School Mathematics*, NCTM, 1995.
   This comprehensive document examines the purposes of assessment and the nature of various types of assessment. Actual experiences in the classroom are used to clarify and augment suggestions made.

3. *Linking Assessment and Instruction in Mathematics: Junior Years*, Ontario Association of Mathematics Educators/OMCA/OAJE, Moore et al., 1996.
   The document provides a selection of open-ended problems (including Fair Games activities) tested at the Junior Years (grades 4 to 6) level. Performance Rubrics are used to assess student responses (which are included) at four different levels.

4. *Mathematics Assessment: Myths, Models, Good Questions, and Practical Suggestions*, by Jean Kerr Stenmark (Ed.), NCTM, 1991.
   Included are notes on observation techniques, interviews, conferences, portfolios. Sample student work is included with assessment to show how to apply the suggestions in the book. Includes an annotated bibliography.

5. "Assessment", *Arithmetic Teacher* Focus Issue, February 1992, NCTM.
   This issue contains articles on linking assessment with instruction, alternative forms of assessment, self-evaluation, using manipulatives on tests, and suggestions for assessing cooperative problem solving.

6. *How to Evaluate Progress in Problem Solving*, by Randall Charles et al., NCTM, 1987.
   Chapter headings include "What are you trying to evaluate?", "What are some evaluation techniques?" and "How do you organize and manage an evaluation program?" Sample 'test' items are included that show how to assess students' understanding; holistic scoring scales and student opinion surveys suggest ways of assessing progress and affective domain.

7. *Assessment in the Mathematics Classroom*, Yearbook, NCTM, 1993.
   The chapters in this book deal with a wide range of forms of assessment, techniques for implementing different forms, and examples from the classroom showing how observation, interview, journal writing, and tests, among others, can be used to make a meaningful assessment of a student and his/her work.

8. *Making Sense of Data: Addenda Series, Grades K-6*, Mary Lindquist et al., NCTM, 1992
   Students explore different spinners in Grade 3 and determine which outcomes are more likely than others. Older students could design spinners to give specified outcomes. A grade 5 activity uses a "Random walk" to explore the nature of random numbers:: Logo is used to explore this idea further with a computer.

Other Resources

9. "Roll the Dice — an introduction to Probability", Andrew Freda, in *Mathematics Teaching in the Middle School*, NCTM, October, 1998, pp. 85-89.

This describes how the introduction of an unfair game that appears, at first, to be fair, is used to intersect students in exploring how probabilities depend on relative frequencies of outcomes. Computer simulation leads to the conclusion that, the greater the sample, the closer the experimental probability is to theoretical probability. Suitable for Grades 5 or 6.

10. *Dealing with Data and Chance: Addenda Series, Grades 5-8*, Judith S. Zawojewski et al, NCTM, 1991.

One activity deals with determining the fairness or "unfairness" of games, another with determining how many purchases you must make to collect all the enclosed cards necessary to "win" a prize. The game of "Montana Red Dog" (p. 41) involves students in predicting probabilities.

11. *Measuring Up: Prototypes for Mathematics Assessment*, Mathematical Sciences Education Board and National Research Council, Washington, DC.

Performance assessment activities are given that deal with a number of mathematics topics. The "Hog" Game uses dice to encourage students to determine winning strategies while dealing with simple probability.

12. *What Are My Chances?, Book A,* and *What Are My Chances?, Book B*, Creative Publications, 1977.

These books provide a number of activities (black line masters) suitable for a wide range of activities. Students flip coins or two-colour chips, roll number cubes, draw marbles from closed containers (with and without replacement) and explore simple probability activities dealing with the weather and ice-cream flavour combinations. Book B also explores dependence and independence of events.

13. "Truth or Coincidence?", Daniel J. Brohier, *Teaching Children Mathematics*, December 1996, pp. 180-183, NCTM.

This activity for grades 3 to 6 involves children using heads and tails of coins to write "answer keys" for True-False tests. For example, if the test has 6 questions, how many possible combinations of True and False are possible? Applications to sports, social studies, and science are suggested.

14. "Exploring Random Numbers", William R. Speer, *Teaching Children Mathematics*, January 1997, pp. 242-245, NCTM.

Students are introduced to the idea of randomness by trying to write random numbers. Other simple activities reinforce the idea that randomness is not easily achieved.

15. "Choice and Chance in Life: The Game of 'Skunk'", Dan Brutlag, *Mathematics Teaching in the Middle School*, April, 1994, pp. 28-33, NCTM.

The game of "Skunk" uses "good" and "bad" rolls of the dice to create or eliminate a player's score. Analysis of the rules leads to a discussion of ways that "chance" enters our lives.

Other Resources

16. "Looking at Random Events with Logo Software", Thor Charischak and Robert Berkman, *Mathematics Teaching in the Middle School*, January-March 1995, pp. 318-322, NCTM.

    Students learn how to use Logo to simulate the rolling of up to 3 dice, and discuss the probabilities of different outcomes.

17. "Racing to Understand Probability", Laura R. Van Zoest and Rebecca K. Walker, *Mathematics Teaching in the Middle School*, October 1997, pp. 162-170, NCTM.

    Students play a game of racing sailboats that appears to be fair, but turns out to be skewed in favour of certain boats. Students make a chart of possible outcomes for 2 dice to try to explain why the game is unfair.

18. *Mathematics Teaching in the Middle School*, March 99, Focus Issue on "Data & Chance", NCTM.

    The entire journal is devoted to articles dealing with Data Management & Probability. One article explores probability through an even-odd game with dice of different shapes. Another uses animal crackers to simulate wild-life tagging and releasing to count animals in the wild.

19. "Mode Code", David Spangler, *Mathematics Teaching in the Middle School*, April, 1999, pp. 464-466, NCTM.

    Students explore the frequencies of letters of the alphabet in English. They apply this to decoding a message based on these frequencies, which are given as percents.

20. "Cat and Mouse", Brian Lannen. *Mathematics Teaching in the Middle School*, April 1999, pp. 456-459, NCTM.

    Students play a cooperative game to see whether or not the mouse finds the cheese. Aspects of probability include fairness or unfairness and tree diagrams to list outcomes. This activity is suitable for many different grades.

21. *Organizing Data & Dealing with Uncertainty*, NCTM, 1979

    The first section of the book deals with collecting and interpreting data. The second part explores probability through dice, license plates, paper cups, tacks, and beads. Black line masters are provided.

22. "Calendar Mathematics", Lorna J. Morrow, *Arithmetic Teacher*, March 1993, pp. 390-391.

    Problems with dice, or t-shirt and shorts combinations involve organized counting. Children must determine the number of outcomes and show how they are certain that they have all outcomes.

23. *DIME Probability Packs A and B*, available from Spectrum Educational Publishing, Aurora, ON 1988

    Each pack contains materials and activity cards for four different experiments. There are sufficient materials for 6 groups or pairs of students to work on the same experiment. The focus is on analyzing games as fair or unfair, and involves determining the probabilities of the outcomes in order to do so.

Curriculum Connections: Probability

ACTIVITY	DESCRIPTION OF THE ACTIVITY	CURRICULUM EXPECTATIONS
Activity 1 **Counting Dice**	- identifying all possible outcomes for a simple experiment such as tossing a die - identifying probability of an outcome as one or more out of a number of outcomes	• demonstrate an understanding of probability concepts and use mathematical symbols • use tree diagrams to record the results of simple probability experiments
Activity 2 **Probability from 0 to 1**	- identifying all possible outcomes of an event - identifying probability as related to the frequency of a favourable outcome and of all outcomes - describing events with probabilities of '1' and '0'	• demonstrate an understanding of probability concepts and use mathematical symbols • predict probability in simple experiments and use fractions to describe probability • connect real-life statements with probability concepts
Activity 3 **Fair and Unfair**	- identifying all possible outcomes - identifying favourable outcomes - identifying given games as fair/unfair, and devising spinners and dice for other fair games	• use a knowledge of probability to pose and solve simple problems • predict probability in simple experiments and use fractions to describe probability
Activity 4 **Who's the Winner?**	- identifying all possible outcomes for a simple experiment - identifying games as fair or unfair by examining probabilities of 'winning' moves	• predict probability in simple experiments and use fractions to describe probability • connect real-life statements with probability concepts
Activity 5 **A Random Walk**	- using a random number table - exploring the nature of random numbers	• demonstrate an understanding of probability concepts and use mathematical symbols • use a knowledge of probability to pose and solve simple problems