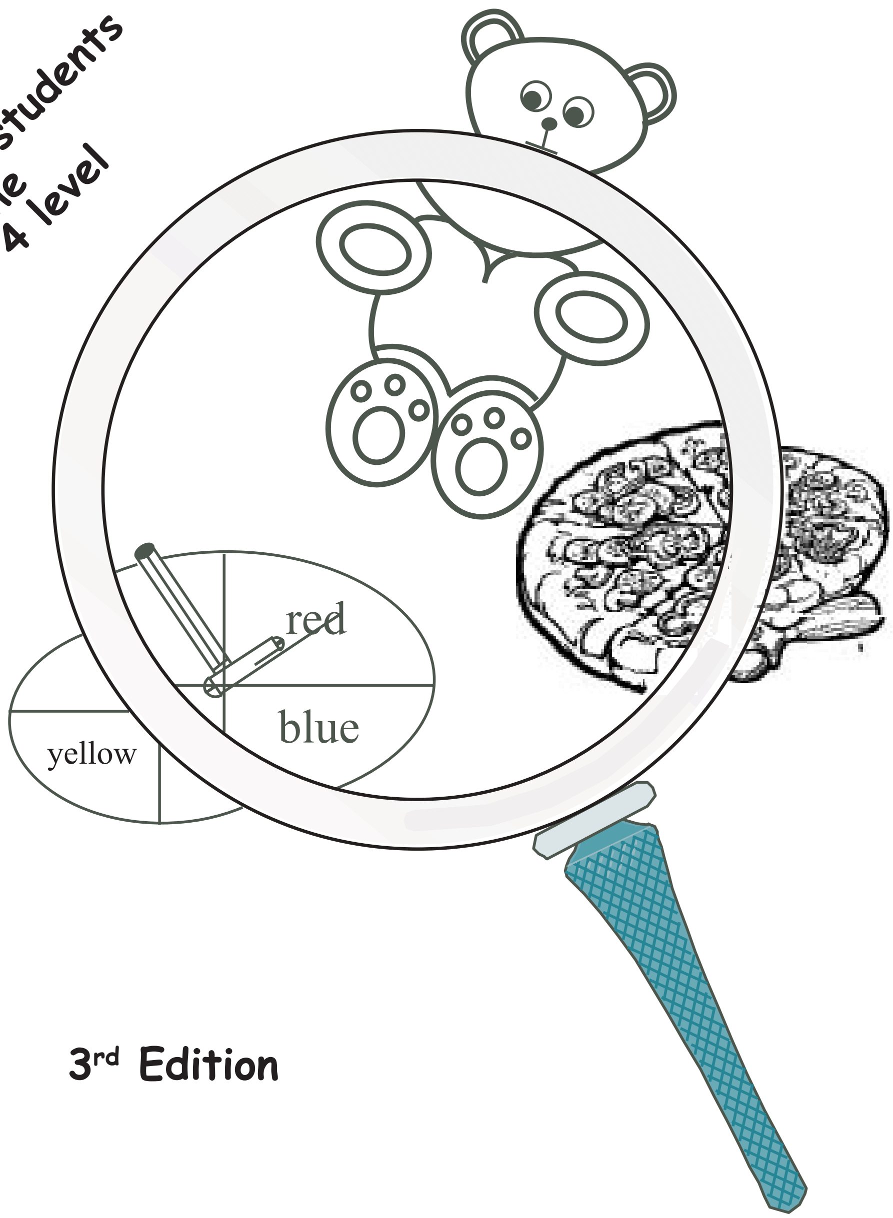
Invitations to Mathematics

Investigations in Probability

"Counting on Probability"

Suggested for students at the Grade 4 level



3rd Edition

An activity of

The CENTRE for EDUCATION in MATHEMATICS and COMPUTING

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Preface

The Centre for Education in Mathematics and Computing at the University of Waterloo is dedicated to the development of materials and workshops that promote effective learning and teaching of mathematics. This unit is part of a project designed to assist teachers of Grades 4, 5, and 6 in stimulating interest, competence, and pleasure in mathematics, among their students. While the activities are appropriate for either individual or group work, the latter is a particular focus of this effort. Students will be engaged in collaborative activities which will allow them to construct their own meanings and understanding. This emphasis, plus the extensions and related activities included with individual activities/projects, provide ample scope for all students' interests and ability levels. Related "Family Activities" can be used to involve the students' parents/care givers.

Each unit consists of a sequence of activities intended to occupy about one week of daily classes; however, teachers may choose to take extra time to explore the activities and extensions in more depth. The units have been designed for specific grades, but need not be so restricted. Activities are related to the Ontario Curriculum but are easily adaptable to other locales.

"Investigations in Probability" is comprised of activities which introduce students to basic concepts of probability, techniques used to determine probability, and applications of probability. Everyday encounters with probability in weather forecasting, interpretation of polls, and commercials for various products and lotteries make it imperative that students acquire some basic knowledge of probability if they are to be able to interpret and evaluate such statements, and hence make well-informed decisions.

Acknowledgements

Contributing Teachers

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Overview

Common Beliefs

The activities in this booklet have been developed within the context of certain values and beliefs about mathematics generally, and about probability specifically. Some of these are described below.

Importance of Probability

Even a cursory glance at newspapers shows the extent to which the language of probability has become important. Individuals need a knowledge of probability to function in our society; consumer reports, cost of living indices, surveys, and samples are a part of everyday life. Nearly all endeavours in the working world require making decisions in uncertain conditions. The goal is to help students develop the critical thinking skills needed to reach sound conclusions based on appropriate data samples.

Instructional Considerations

"Classroom experiences should build on students' natural abilities to solve problems in everyday situations of uncertainty"

NCTM

For example, students learn to play games, and quickly develop a notion of "fairness" which is related to equally likely events. These and other activities develop essential skills for understanding probability — methods of organized counting, comparing results of experiments to theoretical probabilities, using the language of probability correctly — in the context of activities such as dice and spinner games which may be fair or unfair, decoding messages, designing a lottery, and sampling to determine population size.

Essential Content

The activities in this unit introduce basic probability concepts and language, specifically, the meanings of more/less/equally likely, and predicting probabilities from the frequencies of outcomes in simple experiments. In addition, there are Extensions in Mathematics, Cross-Curricular Activities and Family Activities. These may be used prior to or during the activity as well as following the activity. They are intended to suggest topics for extending the activity, assisting integration with other subjects, and involving the family in the learning process.

During this unit the student will:

- construct tree diagrams to determine all outcomes of an event;
- identify all possible combinations of two or more events;
- identify spinners as fair or unfair;
- use data collected from simple experiments to calculate frequencies and predict probabilities;
- apply probability principles to simple arithmetic games;
- use the language of probability correctly;
- justify opinions with coherent arguments;
- collaborate with other members of a group.

Notes

On the inside of the back cover of this booklet, you will find a chart connecting each activity to Ontario's curriculum expectations.

Overview

Curriculum Expectations

The material in this unit is directly related to Ontario curriculum expectations for Mathematics outlined below. By the end of Grade 4, students will:

- demonstrate an understanding of probability, and use language appropriate to situations involving probability experiments;
- compare experimental results with predicted results;
- conduct simple probability experiments and use the results to make decisions;
- use tree diagrams to organize data according to several criteria;
- use a knowledge of probability to pose and solve simple problems (e.g., compare the probabilities of two events using the expressions more/less/ equally probable)

Assessment

Assessment may be described as the process of gathering evidence about a student's knowledge, skills, and values, and of making inferences based on that evidence for a variety of purposes. These purposes include making instructional decisions, monitoring student progress, evaluating student achievement in terms of defined criteria, and evaluating programs.

To meet these aims, it is necessary to use a variety of assessment techniques in order to:

- assess what students know and how they think and feel about mathematics;
- focus on a broad range of mathematical tasks and taking a holistic view of mathematics;
- assess student performance in a variety of ways, including written and oral, and demonstrations;
- assess the process as well as the product.

Tests are one way of determining what students have learned, but mathematical competence involves such characteristics as communicative ability, problem-solving ability, higher-order thinking ability, creativity, persistence, and curiosity. Because of the nature of the activities it is suggested that a variety of assessment strategies be used. Suggestions include:

- observing students as they work to see if they are applying various concepts; to see if they are working cooperatively; to observe their committment to the tasks;
- assessing the completed project to see if instructions have been followed; to see if concepts have been applied correctly; to see if the language of mathematics has been used correctly;
- assessing the students' descriptions of their completed work to see if mathematical language is used correctly; to see if students understand the concepts used;

Overview

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- providing opportunities for student self-assessment (Have students write explanations of their understanding, opinion, or feelings about an activity. One technique is to have them write under the headings What I Did, What I Learned, and How I Felt About It. Students could be asked to write a review of one day's activities or of the whole unit's work.);
- selecting an exemplary piece of work to be included in a portfolio for assessment purposes or for sharing with parents.

Prerequisites

Students need very little previous knowledge to deal with these activities, but the use of a tally when counting outcomes will be helpful, as will an understanding of division (in particular, what is meant by 'remainder').

Logos

The following logos, which are located in the margins, identify segments related to, respectively:

Problem Solving **Communication** **Assessment**

Notes

Overview

Materials

ACTIVITY	MATERIALS
Activity 1 **Fair Spinners**	• Copies of BLM 1 for all students • Copies of BLM 2 for all students • Acetate copies of BLMs 1 and 2 (optional) • Copies of BLM 17 (optional) • Acetate spinners for use on the overhead projector • Four acetate copies of BLM 3 (optional) • Copies of BLM 4 for all students (optional)
Activity 2 **Tree Diagrams**	• Copies of BLM 5 for all students • Copies of BLM 4 if not used in Activity 1 (optional) • Copies of BLM 6 (optional) • Copies of BLM 18 (optional)
Activity 3 **Frequencies**	• Copies of BLM 7 • Pages from a telephone book • Large chart for class data • Acetate copy of BLM 8 for overhead projector
Activity 4 **Probability**	• Copies of BLM 9, 10, 11, 12 • Copies of BLM 13 (optional) • Paper cups with flat bottoms • Marbles/beads/buttons or discs as on BLM 13 • Number cards as on BLM 13
Activity 5 **Probability in Games**	• Copies of BLMs 14, 15 • Copies of BLMs 2, 16 (optional) • Standard dice or number cubes in two colours • Number cards as on BLM 13

Overview

Letter to Parents

SCHOOL LETTERHEAD

DATE

Dear Parent(s)/Guardian(s):

For the next week or so students in our classroom will be participating in a unit titled “Counting on Probability”. The classroom activities will focus on organized counting, calculating simple probabilities using dice and spinners, and identifying games as fair or unfair.

You can assist your child in understanding the relevant concepts by working together to perform simple experiments, and play games, and helping to locate everyday ways probabilities are used.

Various family activities have been planned for use throughout this unit. Helping your child with the completion of these will enhance his/her understanding of the concepts involved.

If you work with probability in your daily work or hobbies, please encourage your child to learn about this so that he/she can describe these activities to his/her classmates. If you would be willing to visit our classroom and share your experience with the class, please contact me.

Sincerely,

Teacher’s Signature

A Note to the Teacher:

If you make use of the suggested Family Activities, it is important to schedule class time for sharing and discussion of results.

Activity 1: Fair Spinners

In our discussions of probabilities in these activities, we are assuming perfectly balanced dice, coins, and spinners. In actuality, most dice, coins, and spinners are not perfectly balanced.

For more on line plots such as the one shown, see "Investigations in Data Management: Grade 4: Our Classroom Community". See ordering information at the end of this unit.

Focus of Activity:

- identifying spinners and games as fair or unfair

What to Assess:

- whether or not predictions, and justifications given, are reasonable
- ability to identify simple spinners and games as fair or unfair
- collaboration with others

Preparation:

- See the table on page 4 for materials
- construct spinners A, B, C, D as suggested on BLM 17 or collect materials and provide copies of BLM 17 for students to make them; templates for spinners are given on BLM 2
- construct an acetate spinner for use with the overhead projector
- make copies of BLM 1 for students and a copy for the overhead
- make 4 acetate copies of BLM 3 (optional)
- make copies of BLM 4 (optional)

Activity:

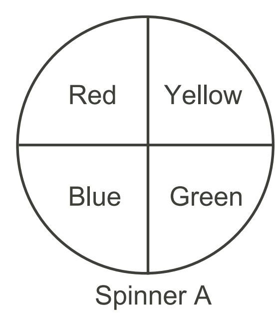
Using a display spinner of spinner A on BLM 1, ask students where it is likely to stop if it is spun many times. Will it stop on each colour the same number of times?

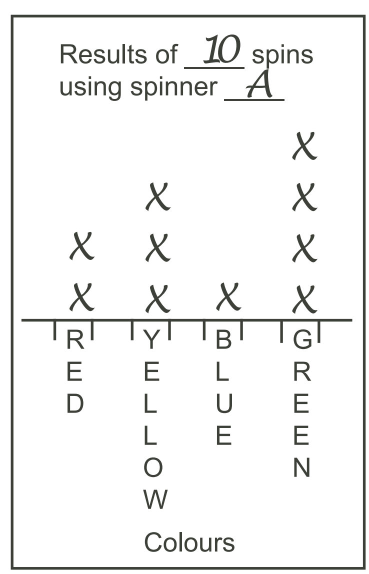
Spin 10 times and have students record the results. Compare results with the predictions. Ask what they think would happen if they counted more spins?

Distribute the spinners or the materials to make them.

Have students work in pairs or small groups. Each group should record the results of 10 spins, using spinner A.

Excerpt from BLM 1





Activity 1: Fair Spinners

Notes

Collect group results in a chart such as the one shown on acetate, blackboard, or chart paper.

If you make 4 copies of such a chart on acetate for use with an overhead projector, you can record results for all 4 spinners. (A master copy of a chart is given on BLM 3.)

To collect the data you may wish to have each group record their own results on the group chart as they complete the work with each spinner.

Excerpt from BLM 3

Results of ____ spins using spinner

Colour / Group	Red	Yellow	Blue	Green
1	2	3	1	4
2	3	1	3	3
3				
4				
5				
Total				

Ask students if the class results are similar to or different from each individual group's results. Ask why.

Communication

Students should see that the more often they spin the spinner, the more likely it is that each colour will occur one quarter of the time.

Because these are student-made spinners it is unlikely that each colour will occur for exactly one quarter of the spins.

Examine spinners B, C, and D with the students. Ask students what they think the results will be. Have each group record the results of 10 spins for each spinner.

Alternative: Assign spinner B to three or four pairs/groups, spinner C to three or four pairs/groups, and spinner D to three or four pairs/groups. Have each group record the results of 30 spins.

Record the results of all spins for each spinner on blackboard, overhead or chart paper.

Assessment

Ask students to compare their predictions (either orally or in writing) with the actual results. Use questions such as the following to guide the discussion/ writing:

"Did your prediction match the results? Why or why not?"
"Why do different spinners give different results?"

In discussing these issues, students should be able to say that, for example, they would expect to spin red twice as often as blue or yellow for spinner C.

Activity 1: Fair Spinners

Ask students:

"Suppose you were using spinner A for a game in which you won if you spun red, but otherwise you lost. Is this a fair game?"

Students may have different interpretations of the word "fair". What is meant in this case is: Is your chance of winning the same as your chance of losing?

Problem Solving

Ask students:

"Which spinner(s) would give you an equal chance of winning or losing?"

Describe other games:

"Suppose you win if you spin blue. Otherwise your opponent wins. Would any of the spinners give you both an equal chance of winning? Explain."

Students should be quick to see that using Spinner D would give both players an equal chance.

Give a challenge:

"Suppose you win, and add 1 point to your score, if you spin blue. If your opponent spins red, he/she adds 2 points to his/her score. Will any of the spinners make this a fair game? Explain."

Assessment

Students will need time to consider this. Have them discuss this in pairs/groups and have each pair/group write an explanation to read to the class.

Once again, Spinner D will make this a fair game. Since blue is twice as likely as red, then the 'blue' player will win 1 point twice (2 points) for every time the red player wins and gains 2 points.

Extensions in Mathematics:

1. *The activity described on BLM 4 can be used to lead into Activity 2, so you may wish to assign it at this point. Alternatively, it could be used as the introduction to Activity 2. It could also be used as a Family Activity.*

Problem Solving

Distribute copies of BLM 4 and go over the rules with the students. Allow them time to make predictions and to play the game. Discuss their conclusions. Collect data from the entire class and compare the aggregate data with individual group data. Ask students why it is a good idea to test a game a great number of times.

Activity 1: Fair Spinners

Students may or may not be surprised to discover that the game is fair. Each spinner has two even numbers and two odd numbers. To get an odd-numbered total, you need to spin one odd and one even number. Students may need to record all possible sums to see that 8 totals are even and 8 totals are odd. Activity 2 deals with an organized counting technique known as a tree diagram. You may wish to postpone analyzing the game on BLM 4 until after this technique has been introduced.

Family Activities:

1. Ask students to work with family members to invent and test a fair game that uses spinner C from BLM 1.
2. Ask students to examine the games they have at home to see if they are fair or unfair. Start a list on chart paper, black board or bulletin board of fair games. When time permits, a student could describe a game to the class and tell why it is thought to be fair.

Other Resources:

For additional ideas, see annotated Other Resources list on page 54, numbered as below.

3. "Linking Assessment and Instruction in Mathematics: Junior Years", OAME
8. "Making Sense of Data: Addenda Series, Grades K-6", Mary Lindquist et al.
11. "Measuring Up: Prototypes for Mathematics Assessment", MSEB & NCTM
20. "Cat and Mouse", Brian Lannen

Notes

Problem Solving

Activity 2: Tree Diagrams

Focus of Activity:

- organized counting using tree diagrams

What to Assess:

- identification of possible outcomes for given situations
- construction and interpretation of tree diagrams
- collaboration with others

Preparation:

- make copies of BLM 5
- make copies of BLM 6 (optional)
- make copies of BLM 4 (if not used for Activity 1)
- make copies of BLM 18 if you wish students to have notes on tree diagrams

Activity

Refer to Extension 1 of Activity 1. (See also BLM 4.)

If students have already tried this, start by asking how they decided if the game was fair or not. Lead into the idea that, to predict fairness, one should list all possible outcomes (i.e., all possible totals of numbers on the two spinners).

If students have not tried the game, you may simply wish to focus on the idea of listing all outcomes, rather than discussing the fairness or unfairness of the game.

Ask students how they could make such a list and be sure that they have listed all possible sums. Allow time for them to try some of the suggested techniques and discuss them. If no one suggests any version of a 'tree diagram', introduce it as a useful technique when dealing with two or three variables.

For example:

First, list the possible outcomes for the first spin (1, 2, 3, and 4).

Decide how many outcomes are possible for the second spin, and draw that number of 'branches' from each first-spin outcome. Since there are four outcomes for each spin, we need four branches from each first-spin outcome. See Figure 1.

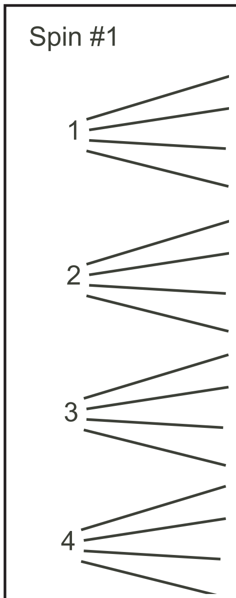
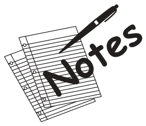


Fig. 1

List the four possible outcomes for the second spin, for each outcome of the first spin, as shown in Figure 2.

Calculate the sum of each branch as shown in Figure 3.



Activity 2: Tree Diagrams

The sums can now be separated into even and odd numbers, showing an equal number of each. Thus we can say that the game is fair.

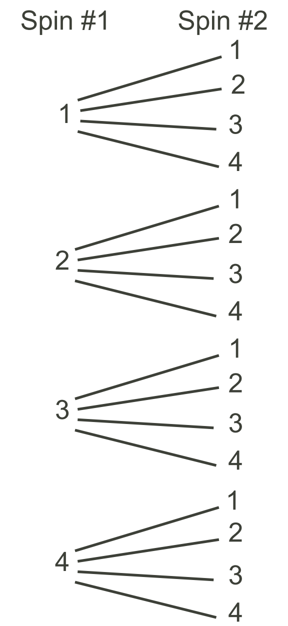


Fig. 2

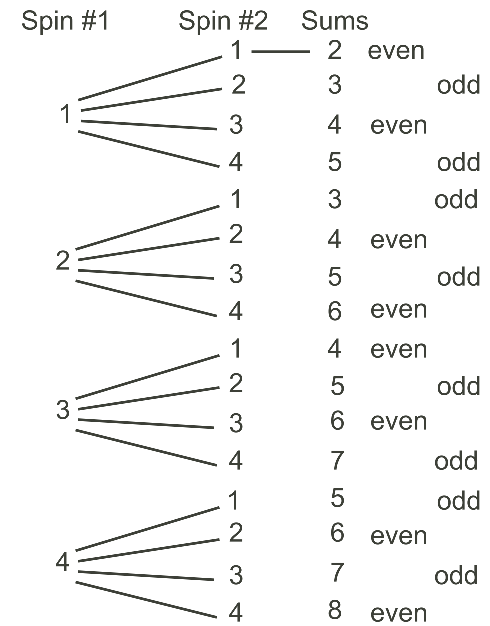
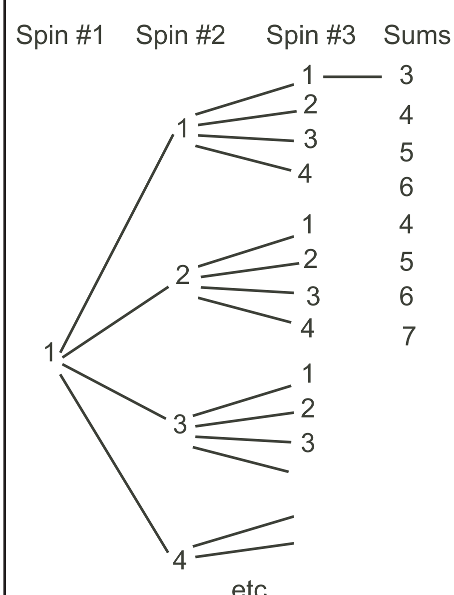


Fig. 3

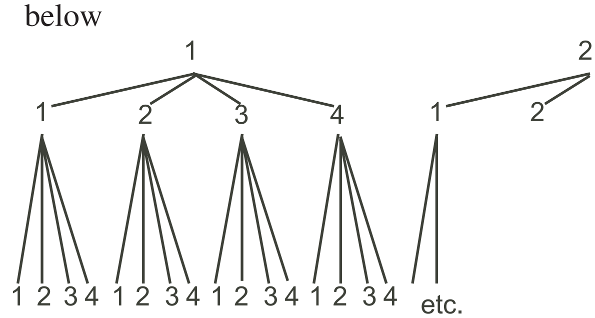
Assessment

Have students copy the tree diagram for practice. Most students need practice in spacing the outcomes, particularly in the first column. They often write these so close together that they cannot extend the branches in a legible manner. This is even more of a problem when there are three sets of outcomes to deal with, as in Extension 1 below. A partial tree diagram is shown. You may wish to discuss with students the importance of planning ahead when drawing a tree diagram in order to ensure that the branches are not crowded.

Partial Tree Diagram
for Extension 1



Tree diagrams may also be drawn as shown below



Students should use whichever format is easier for them.

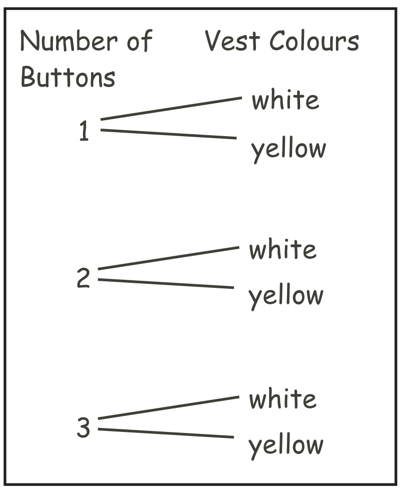
BLM 18 suggests some experiments that can be used to practice drawing tree diagrams.

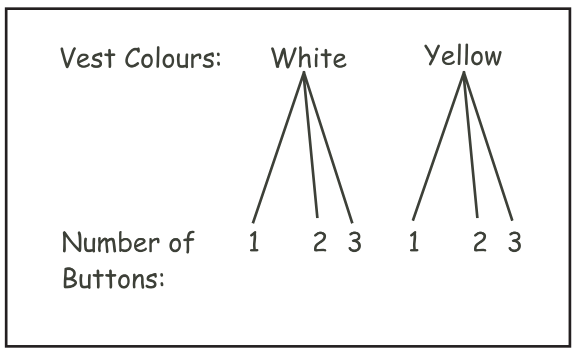
Activity 2: Tree Diagrams

Distribute copies of BLM 5. Make sure students understand the problem, and that they must try to justify their answers. Allow time for students to work on the problem. They should have markers or crayons available to colour the teddies as necessary.

Students may or may not choose to use a tree diagram at this point. Whether they do or not, the use of a tree diagram should be brought out in the discussion of the problem after students have tried to solve it.

One tree diagram will have three choices for the first item (numbers of buttons), but only two choices for the second (colour of vest). However, the colour of vest and the numbers of buttons could be dealt with in the opposite order. Two possible tree diagrams are given below.

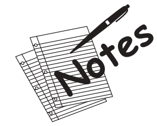




Students should see that either tree diagram gives the same six results. They should also be able to say with confidence that adding the red or blue stripes sinply turns each of the six results into two — one with red stripes and one with blue, and therefore there will be twelve possible outcomes.

You may want to lead students to the understanding that multiplying the number of choices at each stage will give the total number of possibilities.

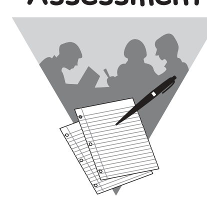
For example,
 the number of vest colours is 2;
 the number of choices for buttons is 3;
 the number of colours for stripes is 2.
Thus, there are 12 possible bears.
 2 x 2 x 3 = 12

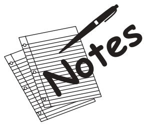


Problem Solving

The general rubric in "Suggested Assessment Strategies" (following Solutions and Notes) has been adapted for this particular problem, as a sample to assist you in assessing your students.

Assessment





Problem Solving

Activity 2: Tree Diagrams

You may, however, wish to use another example or two (as given below) before bringing this out, thus giving students a chance to 'discover' this for themselves.

> "Sandy is wrapping party favours. Each favour is to be wrapped using different colours of paper and ribbon. She has pink, gold, and green paper. She has blue, red, silver, and gold ribbon. How many differently wrapped party favours can she make?"

If you wish to present students with a challenge turn the problem around:

> "Sandy is wrapping party favours. Each favour is to be wrapped using different colours of paper and ribbon. She has pink, gold, and green paper. How many different colours of ribbon must she buy if she needs 12 differently wrapped party favours?"

The Extensions and Family Activities below provide further activities to give students practice in the use of tree diagrams. Some of these have been printed on BLM 6 so that copies can be given to the students. Students will find these problems easier if they use the technique of multiplying the numbers of choices, as shown on the previous page for the teddy bears.You might wish to give students a choice as to which problem they wish to work on.

Extensions in Mathematics:

1. A game similar to the one on BLM 4 uses three of the same spinners, rather than two. The other rules are the same. Ask students if they think this game is fair, and why. Allow time for them to play the game to test their predictions. At some point, help them to construct tree diagrams to list all possible outcomes, in order to see that half the outcomes are odd numbers and half the outcomes are even numbers. Therefore the game is fair.

For the next three activities, you might wish to add a question, asking if a tree diagram would be useful in solving this problem, and, if so, how it could be used.

2. See "Terry's T-shirts", BLM 6.

Family Activities:

1. See "Pete's Pizza Parlour", BLM 6.
2. See "License Plates", BLM 6.

Other Resources:

For additional ideas, see annotated Other Resources list on page 54, numbered as below.

13. "Truth or Coincidence", Daniel J. Brohier
20. "Cat and Mouse", Brian Lannen
22. "Calendar Mathematics", Lorna J. Morrow

Activity 3: Frequencies

Focus of Activity:

- use of tallies to record frequencies
- relating frequencies to probabilities

What to Assess:

- accuracy and efficiency of tallies
- use of language of probability (e.g., 'frequency', 'more likely')
- collaboration with others

Preparation:

- make copies of BLM 7
- provide telephone book pages
- prepare a large chart or an acetate copy of BLM 7 for class data
- make a copy of BLM 8 for the overhead, and copies for students (optional)

Activity:

This activity is designed to give students experience in collecting data and relating the frequencies of certain bits of data to the probability of their occurrence.

Give each student pair/group a copy of BLM 7 and a copy of a page from a telephone book. Have students select one column on the page and record the last digits of these numbers. If students have not used a tally before, you will need to show them how this is done.

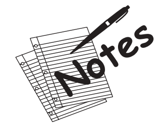
You might want to suggest that one student reads each final digit, while another completes the 'Tally' column. The final column, 'Frequency', simply means the number of times a digit occurs.

Excerpt from BLM 7:

Last digit in telephone number	Tally	Frequency
0	𝍸 𝍸 ||	12
1	𝍸 ||||	9
2		

Stop students when they have completed the first chart. Ask several pairs for their findings. Make a large chart on the blackboard/overhead/chart paper and record the findings from the whole class.

Activity 3: Frequencies



Excerpt from BLM 7:

Frequencies from pairs/groups

Last digit	Group 1	Group 2	Group 3	...	Overall frequency
0	9	13	10	...	
1	13	14	7	...	
2	25	8	6	...	
3	4	12	19	...	

Ask students to compare their individual results with the whole class results. They should see that the frequency of each digit is close to the frequency of any other digit when enough data has been collected.

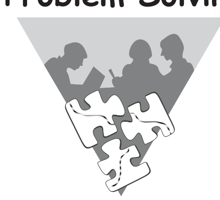
Alternatively, we could say that the "relative frequency" of one digit is close to the relative frequency of each other digit.

Suppose in a total sample of 800 phone numbers, there were 78 ones. The relative frequency of "1" would be 78 out of 800.

Communication

Before students begin with #3 on BLM 7, ask for some predictions. Discuss this briefly, having students give reasons for their predictions. This will help them in trying to put their reasons into words for #3 (b) and (c). Allow them time to complete all parts of #3.

Problem Solving



Students may assume that, since the frequencies of individual digits are reasonably even, the same will happen with sums, and they may be quite confused when this does not happen. If they are having real difficulty trying to explain why this happens, use the chart on BLM 8.

Complete the chart with the students to show all possible combinations of the last two digits.

Students should quickly see that some sums occur more often than others.

For example, there is only one combination that gives a sum of '2', but there are several that give a sum of, say, '9'.

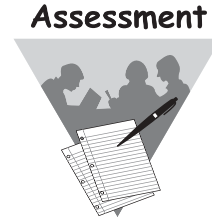
Excerpt from BLM 8:

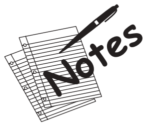
last digit

second last digit

+	0	1	2	3	4	5	6	7	8	9
0										
1										
2										
3										
4										
5										
6										
7										
8										
9										

Assessment





Activity 3: Frequencies

Therefore, the frequencies of the sums should not be equal. Students should incorporate this idea in their reasoning when responding to #3(c).

Ask students if they would expect similar results if they used the first two digits of seven-digit telephone numbers. They should see that there is far less variety in the first two digits. If the telephone book page is from a small town, all numbers may have the same first two digits.

Communication

Ask students why a chart is better than a tree diagram for determining the frequencies of all the sums. They should see that, although it is possible to use a tree diagram to determine not only all the possible sums but also their frequencies, it would be awkward because of the number of branches needed.

Family Activities:

1. Students could ask family members for predictions about the frequencies of both final digits and sums of last two digits of telephone numbers, and report these predictions the next day. Encourage them to use the opportunity to "teach" their families some ideas about organized counting and probability.

Other Resources

For additional ideas, see annotated Other Resources list on page 54, numbered as below.

12. "What Are My Chances?", Creative Publications
18. "Mathematics Teaching in the Middle School, Focus Issue on Data and Chance", NCTM
21. "Organizing Data and Dealing with Uncertainty", NCTM
22. "Calendar Math", Lorna J. Morrow

Activity 4: Probability Experiments

Focus of Activity:

- predicting and determining probabilities from observed frequencies

What to Assess:

- accurate collection of data
- use of data to predict frequencies
- use of language of probability
- collaboration with others

Preparation:

- make copies of BLMs 9, 10, 11 and 12
- make copies of BLM 13 (optional)
- have available paper cups: use the ones that are flat on the bottom
- have available marbles or other materials in colours as needed for Experiment 2 on BLM 10. Use colour discs on BLM 13 if other materials are not available.
- provide Number Cards as given on BLM 13 for use with Experiment 4 on BLM 12

Activity:

Refer to Activity 3 as an "experiment" in which students made predictions about telephone numbers and tested them by collecting data. Further experiments are outlined below.

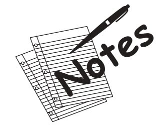
It is not essential that every student try every experiment. However, the number of students trying an experiment should be sufficient to provide enough data to draw reasonable conclusions.

Distribute copies of BLMs 9, 10, 11, and 12, along with materials needed, to pairs of students. If you do not wish all students to do all experiments distribute different experiments to different groups. Make sure there are at least four groups or pairs performing each experiment so that students can collect data from each other as directed in each Experiment. Alternatively, you may wish to extend this activity over two or more days. One or more of the experiments could also be used as Family Activities.

Communication

BLM 13 provides colour discs and Number Tiles for Experiments 2 and 4 respectively. Students can cut and paste these on cardboard and draw them from bags or envelopes to carry out the experiments.

Students should need very little explanation of the activities, but should be reminded to be accurate in their data collection. Their written answers could be done on the BLMs or in math journals or notebooks, but students should be expected to write something.



Activity 4: Probability Experiments

Once students have collected sufficient data from two or three experiments, use this to discuss the probability of each of the outcomes. For example, since there are two outcomes for coin flipping (heads and tails), and they are equally likely, then the probability of flipping a head is ‘1 out of 2’. Similarly, the probability of flipping a tail is ‘1 out of 2’. Although this is sometimes written as ‘’, speaking of probability as ‘***x*** out of ***y***’ has more meaning for students.

Problem Solving

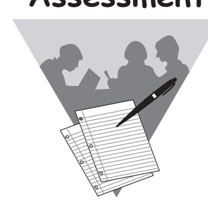
For Experiment 2 (BLM 10), there are 12 possible outcomes, although some are identical. Since there are 12 beads and three of them are red, the theoretical probability of drawing a red is ‘3 out of 12’. The theoretical probability of drawing a blue is 2 out of 12, and so on. Therefore, if a large enough sample of data is collected, we would expect to see that red is drawn more frequently than blue, and that blue and green are drawn about the same number of times. Students should compare the theoretical probabilities with their experimental results.

Some students may say that Experiment 2 has only 4 outcomes: drawing red, drawing blue, drawing green, or drawing yellow. They should understand that since there are 3 red beads, a red bead can be drawn in three different ways, each of which is a separate outcome. To clarify this, you may wish to label the red beads as ‘Red 1’, ‘Red 2’ and ‘Red 3’ or ‘R1’, ‘R2’ and ‘R3’. Other colours can be similarly labelled. See Extension 1 for a similar activity.

Experiment 3 (BLM 11) is a case where experimental data is needed to predict a probability of the cup landing in one of three ways. Students often predict that the cup will frequently land upside down.

In Experiment 4 (BLM 12) the theoretical probability of drawing any digit is 1 out of 10 since there are 10 Number Cards and each one is equally likely to be drawn. Since there are 10 outcomes for this Experiment, it is necessary to collect data from several groups before the experimental probability is close to the theoretical probability.

Assessment



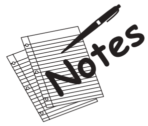
You might wish to expand Experiment 4 and, at the same time, review some terminology. Ask students for the probability of drawing, for example, an even number, a multiple of 3, a square number, a prime number.

Since Activity 5 will deal with Number Cards, keep the cards for use next day.

Extensions in Mathematics:

1. A bag of jelly beans contains 5 black, 3 red, 3 yellow, and 1 orange. If you draw out one jelly bean, what colour is it most likely to be? Why?

   Since there are 12 jelly beans, there are 12 outcomes. List them, starting with “black, black, ... ”.



Problem Solving

Communication

Activity 4: Probability Experiments

What is the probability of drawing a red? an orange?

Design a spinner that will give the same probabilities for each colour that the bag of jelly beans does.

Cross-curricular Activities:

1. Briefly discuss factors affecting readability - for example, word length, sentence length. Have students select a newspaper or magazine article. Then have them list, in order, the 3 word-lengths they expect will be most common, and have them tell why they think so. Then, have the students count the number of 1-letter words, 2-letter words, 3-letter words, ..., 10-letter words, and words of more than 10 letters.

An efficient way to count word-lengths is to have one student call out the number of letters in each word in sequence and another student record this in a tally such as the one shown.

Number of Letters	Number of Words
3	𝍸 𝍷𝍷
10	𝍷
More than 10	𝍷𝍷

Have them compare the actual counts with their expectations. You may wish to continue this exercise by comparing journalistic articles with fiction books or text books. Ask students if they would expect the same most common word-lengths in a Grade 1 story book and why. Have them test their predictions.

Family Activities:

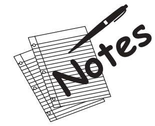
1. Have students and family members explore magazines, newspapers, and television to find instances of the use of the ideas of probability or chance.
   e.g., “There’s a 40% chance of rain tomorrow”.
   “Win six ways! Ten out of every hundred lottery tickets are winners!”

Other Resources:

For additional ideas, see annotated Other Resources list on page 54, numbered as below.

3. “Linking Assessment and Instruction in Mathematics: Junior Years”, OAME
12. “What Are My Chances?”, Creative Publications
15. “Choice and Chance in Life: The Game of ‘Skunk’”, Dan Brutlag
21. “Organizing Data and Dealing with Uncertainty”, NCTM

Activity 5: Probability in Games



Focus of Activity:

- identifying probabilities in number games
- using probabilities

What to Assess:

- ability to list possible outcomes
- ability to identify favourable outcomes
- use of the language of probability
- collaboration with others

Preparation:

- make copies of BLMs 14 and 15
- make copies of BLM 2 (optional)
- make copies of BLM 16 (optional)
- provide standard dice or number cubes in two colours
- make copies of BLM 13 or provide sets of number cards made earlier for Activity 4

Activity:

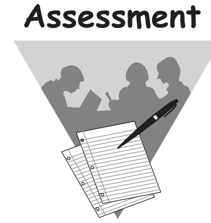
In this Activity students will be playing some simple number games and applying some concepts of probability from Activities 1 to 4. Students will be asked questions about favourable outcomes, likelihoods of specific outcomes, and winning strategies.

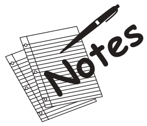
If dice are not available, construct spinners using the template on BLM 2 that has 6 equal sections, and label the sections of the spinner 1, 2, 3, 4, 5, 6. Instead of using dice of two colours, have the students distinguish between the first and second spin of the spinner.

An alternative to dice and spinners is the set of Number Cards from BLM 13. Students can paste the numbers on card or bristol board and place them face down. Two are drawn for each turn, indicating tens first, then ones. The cards for 0, 7, 8, and 9 can be ignored or can be used to expand the possibilities of the games. These cards will be needed for games on BLM 15.

Play Games 1 and 2 with the whole class since the games are quite simple. Students should then be able to consider both games and write answers to question #1. They should see that Game 2 provides more flexibility and that the probability of getting a higher score is greater than in Game 1.

Assessment





Communication

Problem Solving

Activity 5: Probability in Games

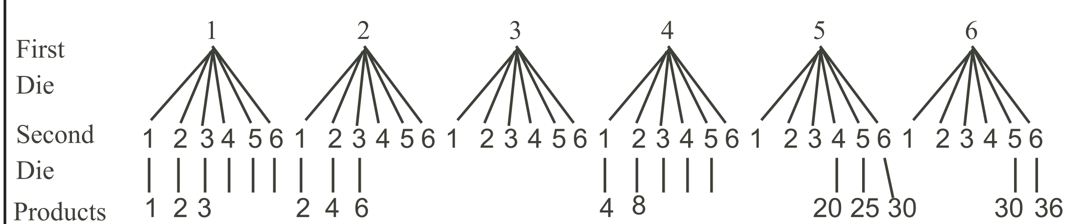
Outcomes for Game 1 could be listed using a chart or a tree diagram, and students asked to identify the favourable outcomes. Just which outcomes are 'favourable' will depend on the students. For example, they might consider any score in which the tens digit is greater than the ones digit to be favourable. Alternatively, only those with a tens digit of 5 or 6 might be considered favourable.

Have the students play Game 3 a few times before trying to answer the questions. Most students are not aware of certain properties of division and remainders that would help them decide which outcomes (rolls of the third die) are favourable. For example, many think that '1' is a good roll, until they realize that the remainder (and therefore their score) will always be zero when the divisor is '1'. They should come to realize that the number of possible remainders is one less than the divisor itself. That is, when dividing by 5, remainders can be 1, 2, 3, or 4 (four possibilities). If 'zero' is considered as a remainder, then dividing by five can have five possible remainders.

Students could use a tree diagram or a chart to list all possible outcomes from the multiplication step in Game 3:

For a chart see Activity 3, BLM 8. Change the chart to a multiplication chart. Use only the digits 1, 2, 3, 4, 5, and 6.

Partial Tree Diagram



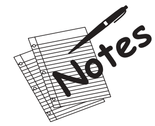
Once the products are listed, students can explore the results of dividing by the number on the third die. As already noted, division by one is not favourable. Since more than half the products are even numbers, division by 2 is also not favourable, since it will usually give a remainder of zero, and otherwise a remainder of only '1'.

It is not necessary to list all possible remainders for all possible divisors, but students should have good reasons for suggesting that certain rolls of the dice are more favourable than others.

Some students may be confused by the idea of dividing, say, a '3' by '5', but they should understand that "There are zero fives in three, with three remaining."

$$3 \div 5 = 0 \text{ R } 3$$

Activity 5: Probability in Games



Distribute BLM 15. Read the rules, and then play the game using Game Board 1 at least once with the whole class. Have each student sketch the Game Board on scrap paper. Draw a number card. Instruct the students to write that number in one of the six boxes. Continue until six numbers have been drawn. <u>Do not return</u> the cards to the face down collection. Instruct the students to calculate the total of the three two-digit numbers. The winner is the player with the greatest total. There will probably be more than one student with this total.

After playing the game once or twice, stop after the 4th draw and ask students:

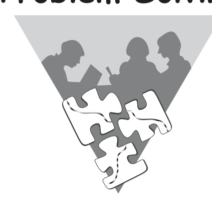
"What high numbers have been drawn?"

"How many low numbers have been drawn?"

"High" and "low" are somewhat arbitrary but most students interpret 'high' as 7,8,9 and 'low' as 0,1,2,3.

"Do you think the next number drawn will be a high number or a low number? Why?"

Problem Solving



Draw a fifth number card. Ask students:

"Which is more likely for the last number — that it will be high or that it will be low? Why?"

Draw the number and have students calculate their scores.

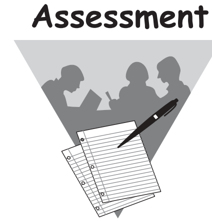
You may wish to continue with BLM 15 as a whole class activity since the games are easy to play and you can phrase questions to use the language of probability such as asking if a 'high' or 'low' number is more likely, or even such questions as "What is the probability that the next number will be a nine?"

Communication

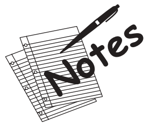
Students should have time to think about some of the questions, however, and question 2 on BLM 15 is a good one for a math journal.

You may wish to alter the rules for any of the games by stating that the winner will be the one with the lowest sum or difference or product. Game Board 4 could have as the winner the one who has the greatest/least quotient or the greatest/least remainder. Watch for students who are able to apply some of their conclusions from earlier games.

Assessment



The use of calculators for BLM 15 will allow students to concentrate on the game strategies rather than the computation.



Activity 5: Probability in Games

If this Activity cannot be completed in the time allotted, the games are good ones to take home and play with family members. Students can report on the experience the following day.

Extensions in Mathematics:

Problem Solving

1. BLM 16 provides another game in which students can identify favourable and unfavourable outcomes. Since the game depends a great deal on luck, a winning strategy is more difficult to define. However, knowing that some rolls of the dice (e.g., less than 80, more than 30) provide more choices, students may use this to identify good placements for their chips/counters.

   The game can be played in two ways: (i) do not allow more than one chip on any number, or (ii) do allow more than one chip on any number. Students could be asked how their winning strategies will change from (i) to (ii).

   If students correctly interpret the conditions in Question #4 on BLM 16, they will realize that this change gives them permission to cover any number on the board they want, and therefore the game becomes a kind of glorified tic-tac-toe.

Family Activities:

1. Have students take one or more of the games home and try to devise a winning strategy with other family members.

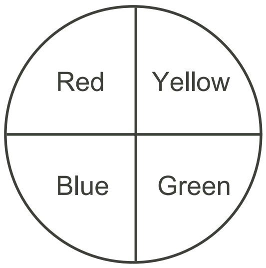
Other Resources:

For additional ideas, see annotated Other Resources list on page 54, numbered as below.

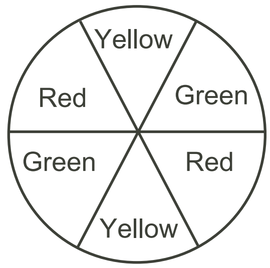
9. “Roll the Dice - an Introduction to Probability”, Andrew Freda
17. “Racing to Understand Probability”, Laura R. Van Zoest and Rebecca K. Walker
21. “Organizing Data and Dealing with Uncertainty”, NCTM

BLM 1: Fair Spinners

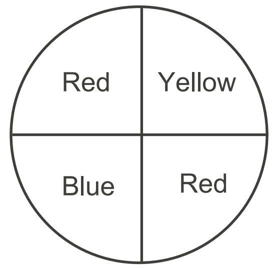
Make spinners as shown.



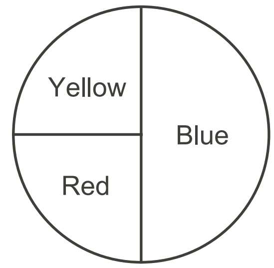
Spinner A



Spinner B



Spinner C



Spinner D

Spin each spinner 10 times and record the results in one of the boxes below.

Results of ____ spins using spinner ____

Red Yellow Blue Green
Colours

Results of ____ spins using spinner ____

Red Yellow Blue Green
Colours

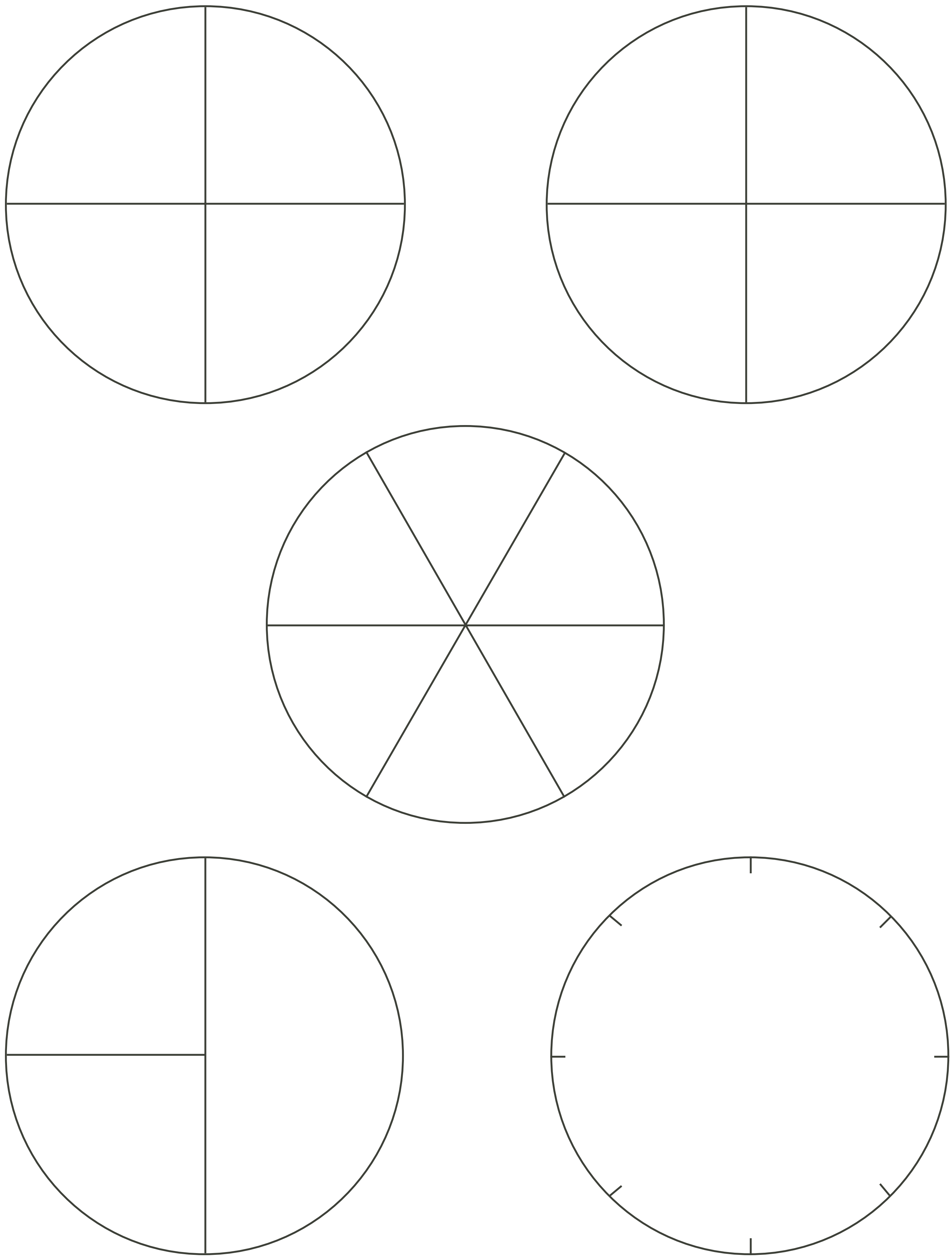
Results of ____ spins using spinner ____

Red Yellow Blue Green
Colours

Results of ____ spins using spinner ____

Red Yellow Blue Green
Colours

BLM 2: Spinner Templates



BLM 3: Data Collections

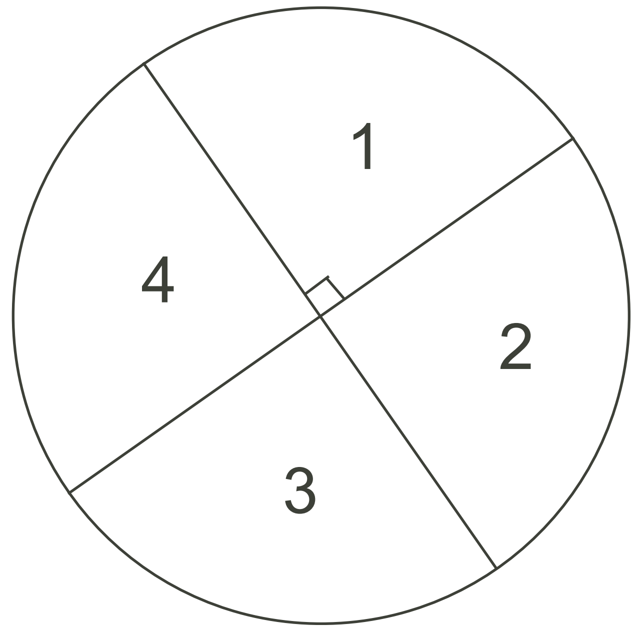
Results of ____ spins using spinner ____

Colour Group	Red	Yellow	Blue	Green
Total				

BLM 4: A Fair Game?

This is an activity for two people or two teams. One is Odd and one is Even. Decide who will be 'Even' and who will be 'Odd' before reading further.

Use the spinner shown. Make a pointer for the spinner the way you did for BLM 1.



In each turn, spin the spinner twice and add the two numbers. It doesn't matter who spins the spinner.

One player or team wins if the sum is even; the other player or team wins if the sum is odd.

1. Before you start the game tell whether or not you think this is a fair game. Why or why not?

Take ten turns and record your result in the chart below.

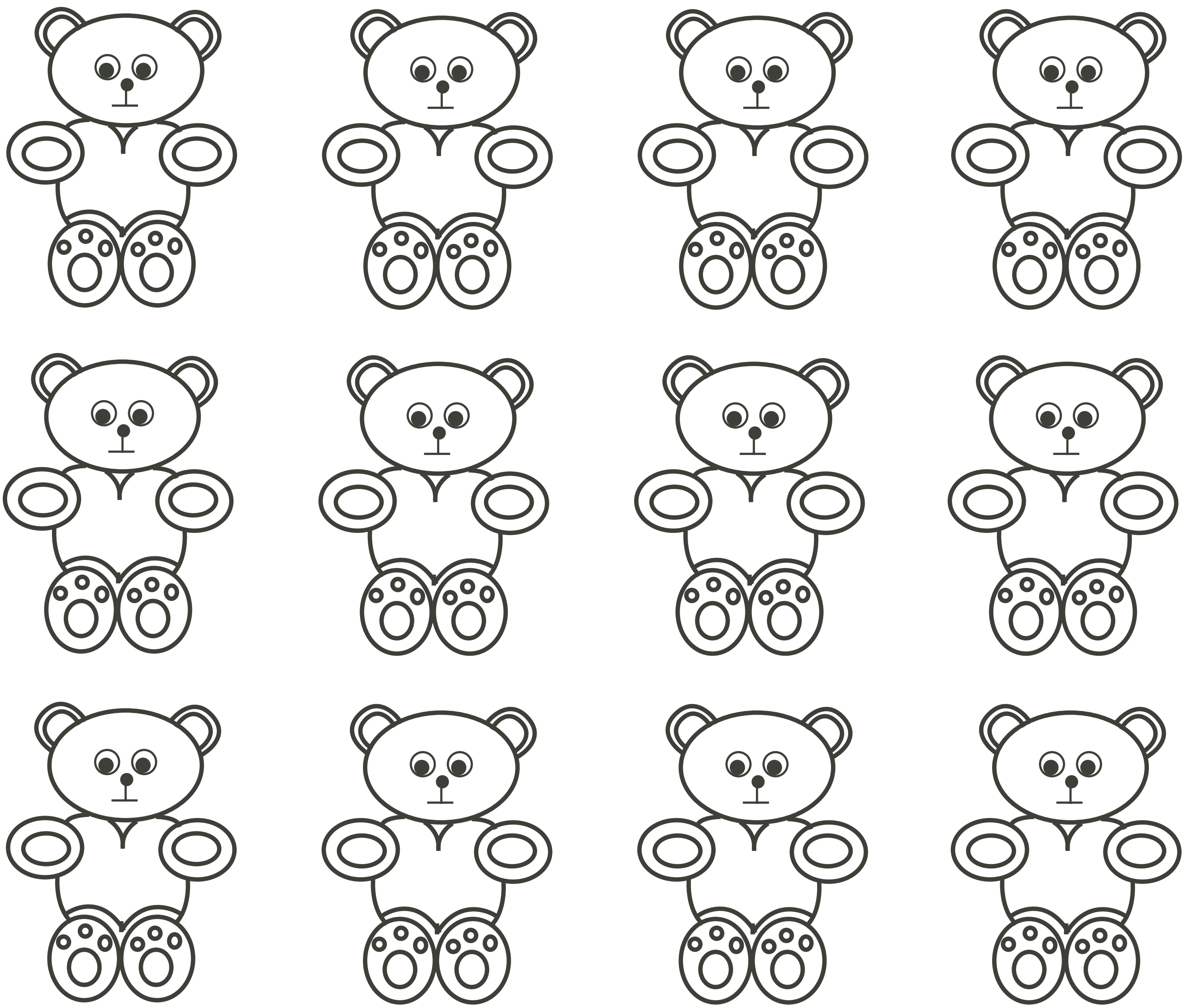
Turns	Sample	1	2	3	4	5	6	7	8	9	10
Sum of the two spins	6										
Player Odd: Win or Lose?	L										
Player Even: Win or Lose?	W										

2. Was your prediction accurate? Is this a fair game? Explain.

BLM 5: How Many Teddies?

Each teddy should have 1 or 2 or 3 buttons on his vest.
His vest is either white or yellow.

1. How many different teddies are possible? How do you know?



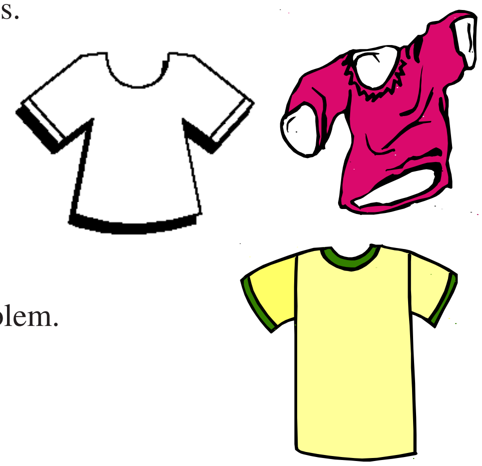
2. Suppose the teddies could have either red or blue stripes on their vests. How many teddies could there be now?

3. How can you be sure you have counted all the possibilities?

BLM 6: Extensions and Family Activities

Terry's T-shirts

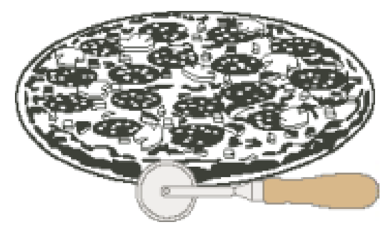
The manager of Terry's T-shirts wanted to order more stock. One factory could supply Terry with t-shirts in different colours and different sizes, and with different neck styles.



1. Make up a problem based on this information.
2. Write the problem on one side of a sheet of paper.
3. Write a full solution on the other side of the paper.
4. Trade problems with another group and try to solve their problem.
5. Compare your solution with theirs.

Pete's Pizza Parlour

At Pete's Pizza Parlour, a basic pizza is $8.75. Each topping is an extra 50¢. Possible toppings are mushrooms, pepperoni, bacon, pineapple, and extra cheese.



1. How many different pizzas are possible with
   (i) 1 extra topping?
   (ii) 2 extra toppings?
   (iii) 3 extra toppings?
2. A Challenge: One of Chris' friends is allergic to pepperoni, so Chris decides not to include pepperoni on any pizza. How many choices does Chris have if he never orders more than two toppings?

License Plates

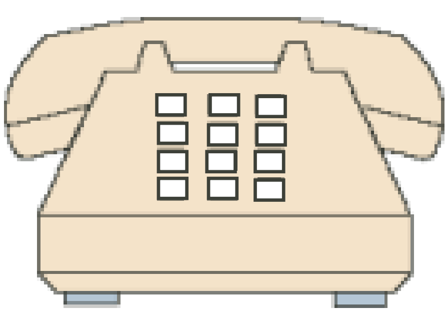
1. A standard Ontario car license plate has 3 letters and 3 digits.
   If all 10 digits (0,1,2,3,4,5,6,7,8,9) can appear in any one of the 3 spots, how many different digit combinations are possible?
2. If all possible letter combinations are allowed, how many combinations of 3 letters are there?
3. How many different licenses are possible under these conditions?
   [Hint: How many digit combinations are possible for each letter combination? How many letter combinations are there?]
4. Would this be enough license plates for everyone in Ontario? How do you know?

BLM 7: Frequencies

Last digit in telephone number	Tally	Frequency
0		
1		
2		
3		
4		
5		
6		
7		
8		
9		

1. Select one column of a page from a telephone book.
2. Record, in the chart, the number of times each digit appears as the last digit of the telephone number.
   Before you start, predict the results and explain why you think so.

3. In the chart at right you will be recording the sum of the last two digits of the telephone numbers. Use the same column of phone numbers as you did for questions 1 and 2.
   (a) What sums are possible? Record them in the first column of the chart.
   (b) How often do you think each sum will occur? Will some appear more often than others? Give reasons. Write your predictions on the back of this sheet.
   (c) Compare the results with your predictions. Were you close? Explain why. Write your comparison and explanation on the back of this sheet.



Sum of last two digits	Tally	Frequency
0		
1		

BLM 8: Addition Chart

Last Digit

Second Last Digit

+	0	1	2	3	4	5	6	7	8	9
0										
1										
2										
3										
4										
5										
6										
7										
8										
9										

BLM 9: Experiment 1: Coin Flipping

1. Flip a coin 10 times for the First Trial and record the number of heads or tails you got in the first row of the table.

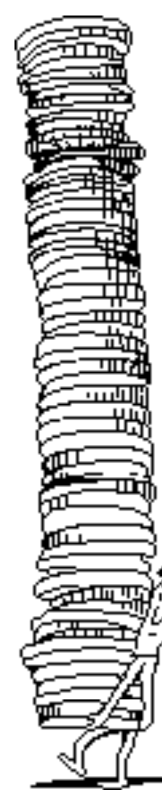
	Number of flips	Number of Heads	Number of Tails
First Trial	10		
Second Trial	10		
Third Trial	10		
Fourth Trial	10		
Fifth Trial	10		
TOTAL	50		

2. Did the results surprise you? Why?

3. Flip the coin 10 more times for the Second Trial and record the results.

4. Share results with other groups. Write their results as the Third, Fourth, and Fifth Trials. Calculate the totals. Did you expect this result? Why?

5. What do you think would happen if you flipped the coin 100 times? 1000 times? Why?



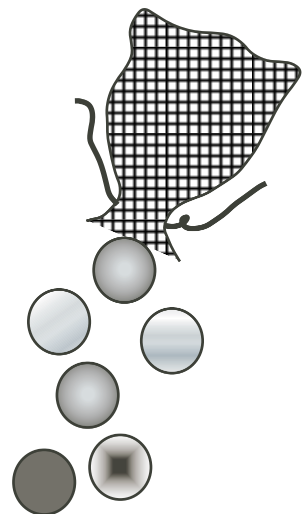
BLM 10: Experiment 2: It's In The Bag

Place 3 red, 2 blue, 2 green, and 5 yellow beads/cards/marbles in a bag or envelope. You will be drawing one at a time for this experiment and replacing the bead after each draw.

1. There are 12 beads in the bag. Do you have a better chance of drawing red or of drawing blue? Why? Which colour are you most likely to draw? Why?

2. Make 12 draws, remembering to return the bead to the bag after each draw. Record the results in the first row for First Trial.

	Number of draws	Number of red drawn	Number of blue drawn	Number of yellow drawn	Number of green drawn
First Trial	12				
Second Trial	12				
Third Trial	12				
Fourth Trial	12				
Fifth Trial	12				
TOTAL	60				



3. Do your results match your predictions? Why or why not?

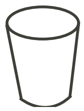
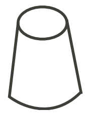
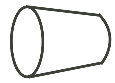
4. Make 12 more draws and record the results as the Second Trial. Do your combined results of the First and Second Trials match your predictions? Why or why not?

5. Share your results with other groups. Record their results as Third, Fourth and Fifth Trials. Calculate the totals. Did you expect this result? Why?

6. Compare the total results with your predictions. Are they alike or different? Why?

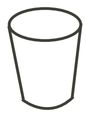
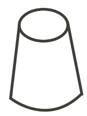
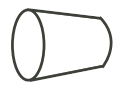
BLM 11: Experiment 3: Cup Tossing

1. If you toss a paper cup 10 times, how many times do you think it will land
   (i) right side up? (ii) upside down? (iii) on its side?

   Why do you think so?

2. Toss a cup 10 times and record the number of times it falls each way in the chart below.

	Number of Tosses			
First Trial	10			
Second Trial	10			
Third Trial	10			
Fourth Trial	10			
Fifth Trial	10			
TOTAL	50			

3. Compare your results with your predictions. Do they differ? If so, how? Why do you think this happened?

4. Share results with other groups. Write their results as the Second, Third, Fourth, and Fifth Trials. Calculate the totals. Did you expect these results? Why?

5. Predict the result of 100 tosses; of 1000 tosses. Give reasons for your predictions.

BLM 12: Experiment 4: Number Cards

1. Make a set of Number Cards from 0 to 9. Place a set in a bag or large envelope. You will be drawing one at a time and returning the Number card after each draw.

2. Make 10 draws, remembering to return the Number Card after each draw. Record your results as the First Trial in the chart below.

		Number Card Drawn									
	Number of Draws	0	1	2	3	4	5	6	7	8	9
First Trial	10										
Second Trial	10										
Third Trial	10										
Fourth Trial	10										
Fifth Trial	10										
TOTAL	50										

3. Compare your results with your predictions? Are they the same? different? Why?

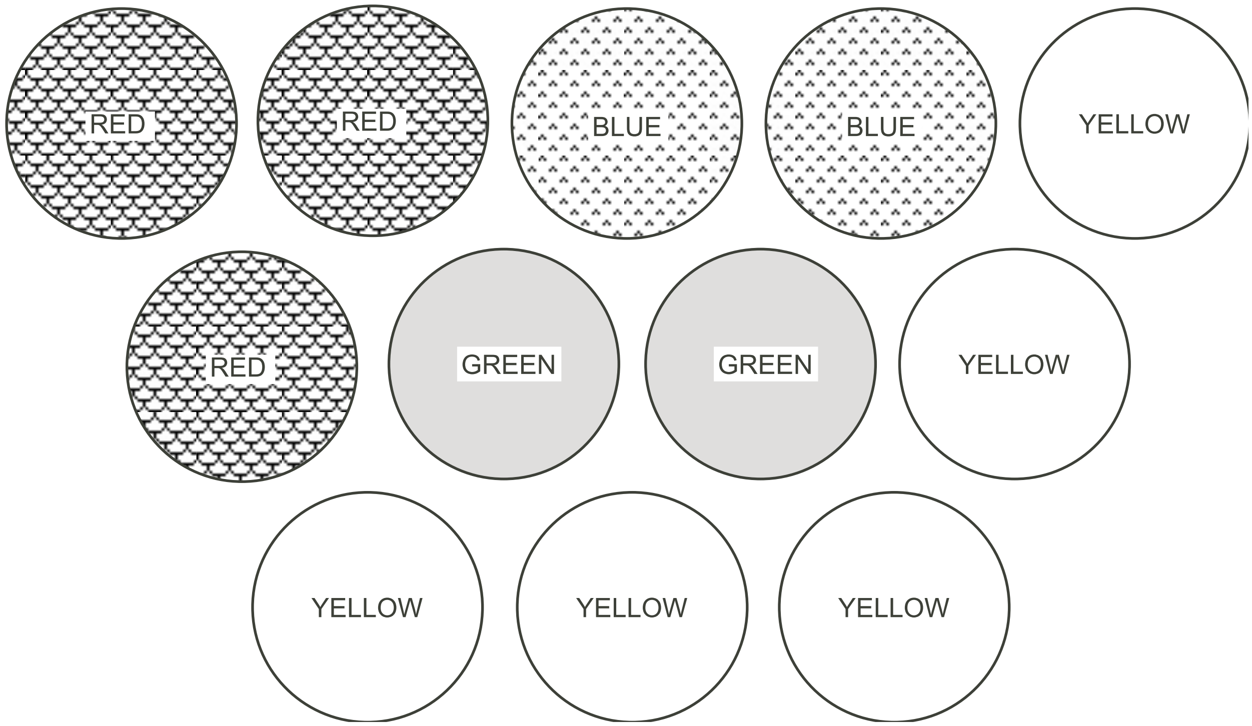
4. Make 10 more draws and record the results as Second Trial. Are your First Trial and Second Trial results similar? Why?

5. Share your results with other groups. Record their results as your Third, Fourth, and Fifth Trials. Calculate the totals. Did you expect this result? Why?

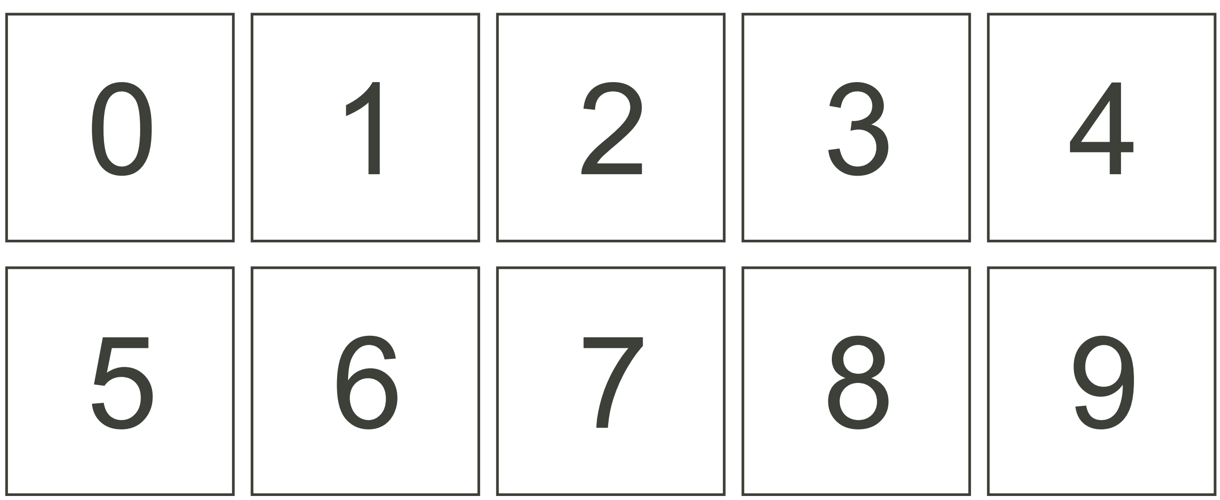
6. Compare the total results with your predictions. Are they alike or different? Why?

BLM 13: Number Cards and Colour Discs

For use with Experiment 2 on BLM 10:



For use with Experiment 4 on BLM 12 and with Activity 5



BLM 14: Number Games 1 – 3

Dice Game #1:

You need two dice of different colours. One die shows the number of tens, the other the number of ones.

For example, if you rolled a ‘2’ on the ‘tens’ die and a ‘3’ on the ‘ones’ die, your score would be ‘23’.

Take 5 turns each, recording your score each time.

Add your scores. A winner is anyone with a total greater than 200.

Dice Game #2:

Use two dice of any colours. One of the dice will be a ‘tens’ die and the other will be a ‘ones’ die as for Game 1, but you decide after each roll which will be which.

For example, if you roll ‘3’ and ‘4’, your score could be ‘34’ or ‘43’.

Take 5 turns each, recording your score each time.

Add your scores. A winner is anyone with a total greater than 200.

1. Which game gives you a better chance of winning — Game 1 or Game 2? Why?

Dice Game #3:

You need 3 dice, two of one colour and the third of another colour.

Roll the three dice. Multiply the numbers on the two dice of the same colour, and divide by the number on the other die.

Your score is the remainder from the division.

Take five turns each, recording your score each time.

The winner is the one with the greatest total score.

2. What remainders are possible in Game 3?

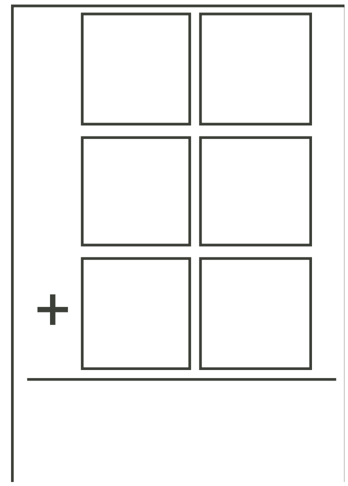
3. In how many ways is each remainder possible?

4. What is the best number to roll on the third die? Why?

BLM 15: Number Cards and Probability

To play these games you will need the Number Tiles from BLM 13.

Game Board 1



Turn the cards face down in front of you. Take turns drawing one. After each draw all players write that number in one of the six boxes in Game Board 1. Do <u>not</u> return that number card to the face-down collection. After six numbers have been drawn, add the three numbers on Game Board 1 together. The player with the greatest total wins.

Play the game two or three times, than answer the following questions.

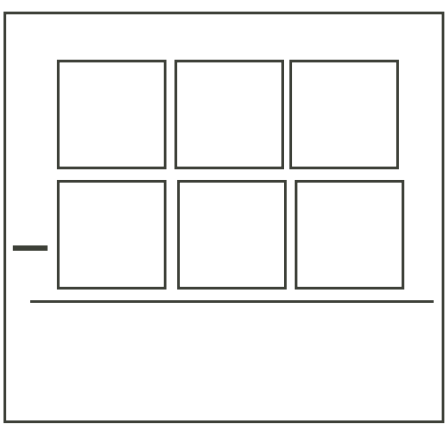
1. Suppose that the first three numbers drawn are 5, 2, and 4.
   (a) Where would you write these to have a good chance of winning?
   (b) Is it more likely that the next number will be a high number or a low number? Why?

2. Imagine you are teaching the game to someone else and you want to describe a winning strategy. What would you say?

3. How would your strategy change if the winner was the one with the least total?

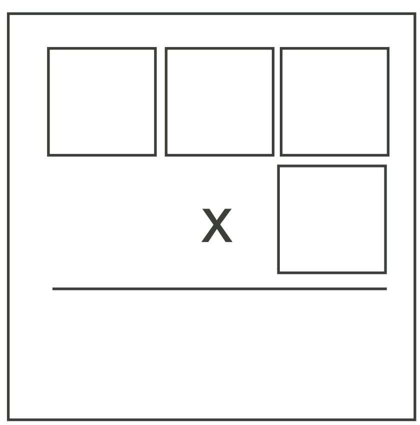
4. Choose one of the Game Boards below. Play the game.

Game Board 2



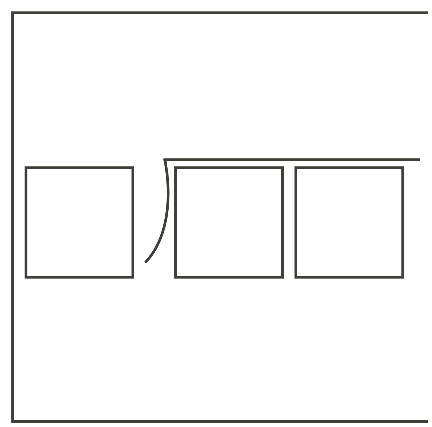
The winner is the one with the greatest difference.

Game Board 3



The winner is the one with greatest product.

Game Board 4



The winner is the one with the greatest remainder.

5. Answer this question about the game you chose in #4. Suppose you drew 9, 0, and 2 for the first three numbers. Where would you place them to have a good chance of winning? Explain.

BLM 16: Greater or Less

You will need chips/counters for each player and two dice (or two spinners with six sections each).
Mark the sides of one die (or the sectors of one spinner) with the numbers 20, 30, 40, 50, 70, 80.
Mark the other die/spinner "less than" on three faces/sectors and "more than" on the other three faces/sectors.
When it is your turn, roll the two dice (or spin the spinners).
If you roll, for example, "less than" and "30", find a number on the board that is less than 30 and put one of your chips on it.
Once you place a chip, you may not move it.
The first person to get 4 chips in a row is the winner.

59	17	34	92	42	81
73	22	64	53	75	89
33	47	98	43	86	72
38	97	11	24	51	62
15	26	31	68	29	41
66	53	76	14	84	90

Write answers on the back of this sheet.

1. (a) How many choices does each of these rolls give you?
   - (i) less than 70?
   - (ii) more than 50?
   - (iii) more than 30?
   - (iv) less than 20?
   - (v) more than 80?
2. What is one of the most favourable rolls? Why?
3. What is one of the least favourable rolls? Why?
4. Suppose you could cover a number either greater than or less than the number of the first die. How would this change the way you play the game?

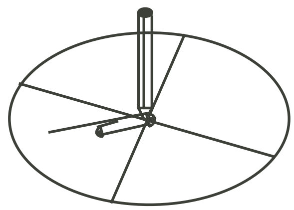
BLM 17: Constructing Spinners

To construct spinners, use the templates at the bottom of the pages. Paste the spinners on to bristol/cardboard.

Method 1:

For the spinner, straighten a paper clip as shown below.

Hold the spinner in place with a pen or pencil at the centre of the circle.



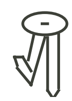
Flick the point of the paper clip with a finger.

This is the simplest way to construct an acetate spinner for use with an overhead projector.

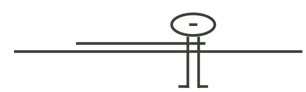
Method 2:

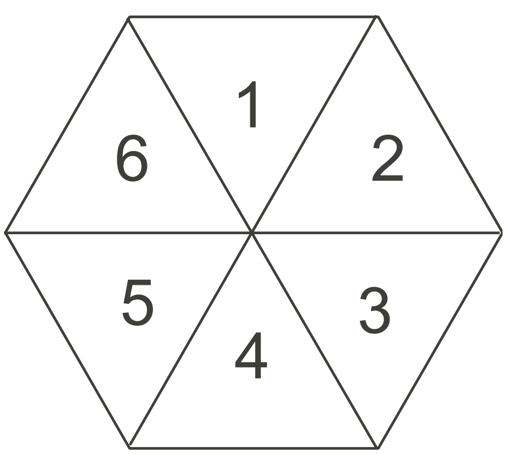
Cut arrows from bristol board or cardboard and punch a hole in one end.

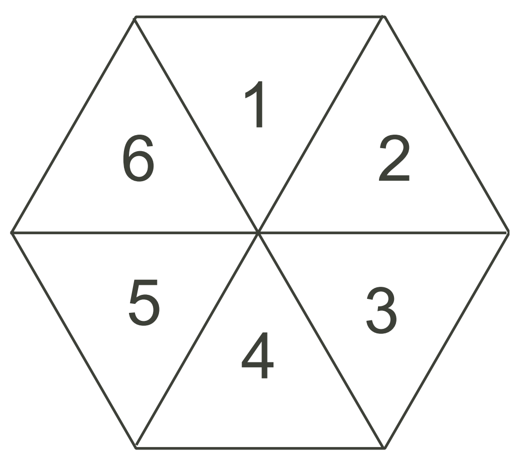
Punch a hole in the centre of each spinner.
Use a paper fastener to fasten the two pieces together.



The connection should be tight enough so the arrow doesn't wobble, but loose enough so that it spins freely.







BLM 18: Tree Diagrams

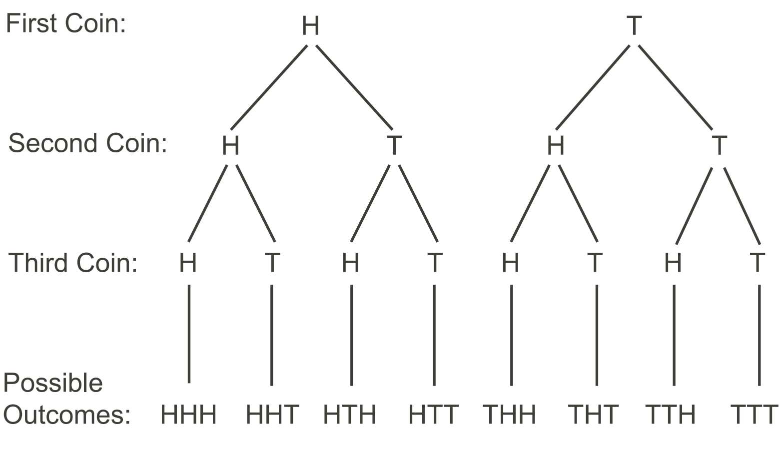
A tree diagram is a way of counting all possible outcomes for a simple experiment. For example, suppose we want to identify all possible outcomes for flipping three coins.

Step 1: List the possible outcomes for the first of the three coins.

Step 2: Draw 'branches' from each of these outcomes. The number of branches will be the number of possible outcomes for the second coin — that is, two.

Step 3: Draw branches for each of the possible outcomes of flipping the third coin.

Step 4: Read down the chart from the top to identify 8 different combinations — that is, the eight possible outcomes when three coins are flipped.



The number of possibilities for the first coin is 2.

For each of these, the number of possibilities for the second coin is 2.

For each of these, the number of possibilities for the third coin is 2.

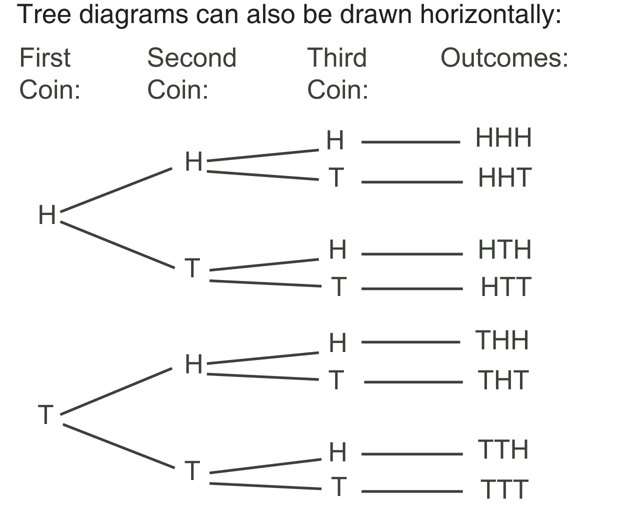
Thus, there are 8 possible outcomes:

Tree diagrams are not useful if there are too many outcomes — for example, rolling three dice (216 outcomes) or even rolling two dice (36 outcomes).

Tree diagrams are useful, however, for determing the outcomes of experiments like the following:

(a) flipping a coin and rolling a die;
(b) the number of outfits possible with rust, green, black, and cream t-shirts and brown, green, and orange shorts;
(c) spinning two or three spinners like the ones on BLM 1.

Tree diagrams can also be drawn horizontally:



Solutions & Notes

Activity 1: Fair Spinners

BLM 1

Students should find that Spinners A and B give each colour about the same number of times. Even the number of spins recorded from the whole class may not give exactly the same number of spins for each colour. That is, "experimental probability" (the number of spins for each colour) may not match the "theoretical probability" (equal numbers of spins for all colours).

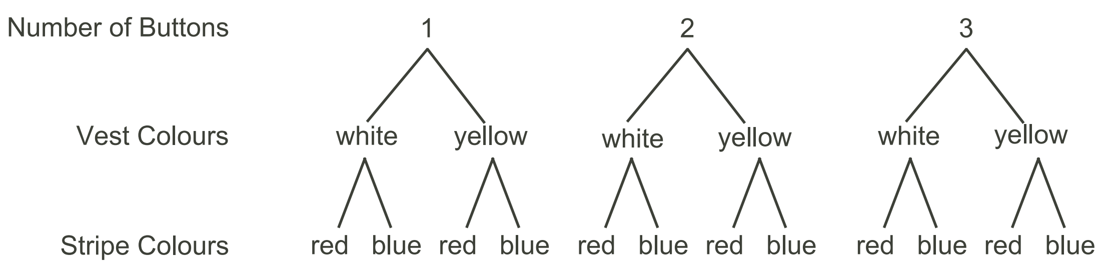
Spinner C should give red about twice as often as yellow or blue. Of course, the number of 'green spins' will be zero. You may wish to introduce the idea that something that cannot happen is considered to have a probability of zero.

Spinner D gives about twice as many 'blue spins' as red or yellow. Again, the probability of a 'green spin' is zero.

Activity 2: Tree Diagrams

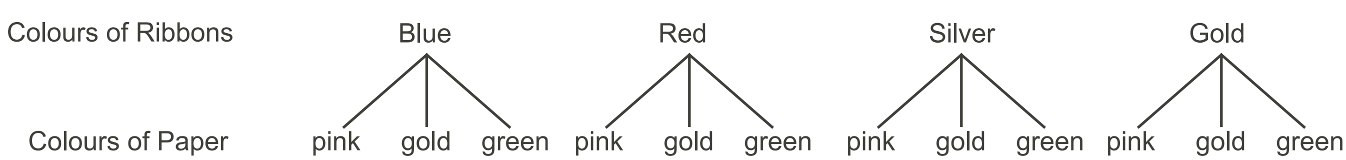
BLM 5

A completed tree diagram is given below. It clearly shows the 12 possible teddy bear outfits.



If the use of 1, 2, and 3 as numbers of buttons gets confused in the students' minds with the 3 choices for numbers of buttons, alter this to 3 types of buttons: round, square, and triangular.

Sandy can make 12 party favours. The following tree diagram shows how.



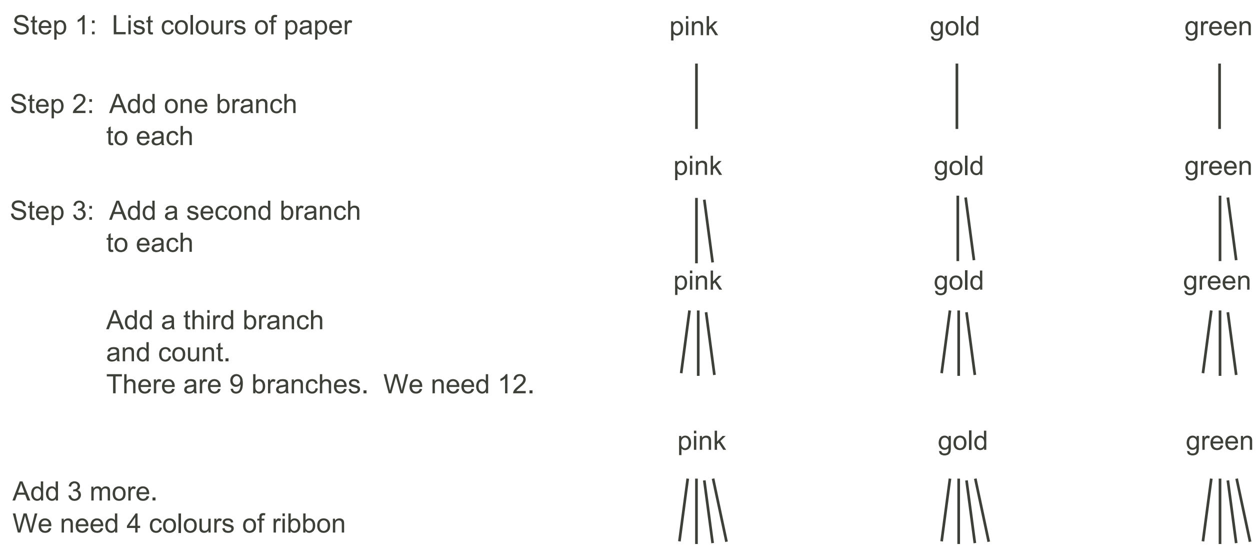
Notice that the tree diagram lists ribbon colour first. Listing the items for which there is the greater choice first sometimes allows students to find sufficient space for the rest of the tree diagram.

Solutions & Notes

The variation on this problem could be approached as follows:

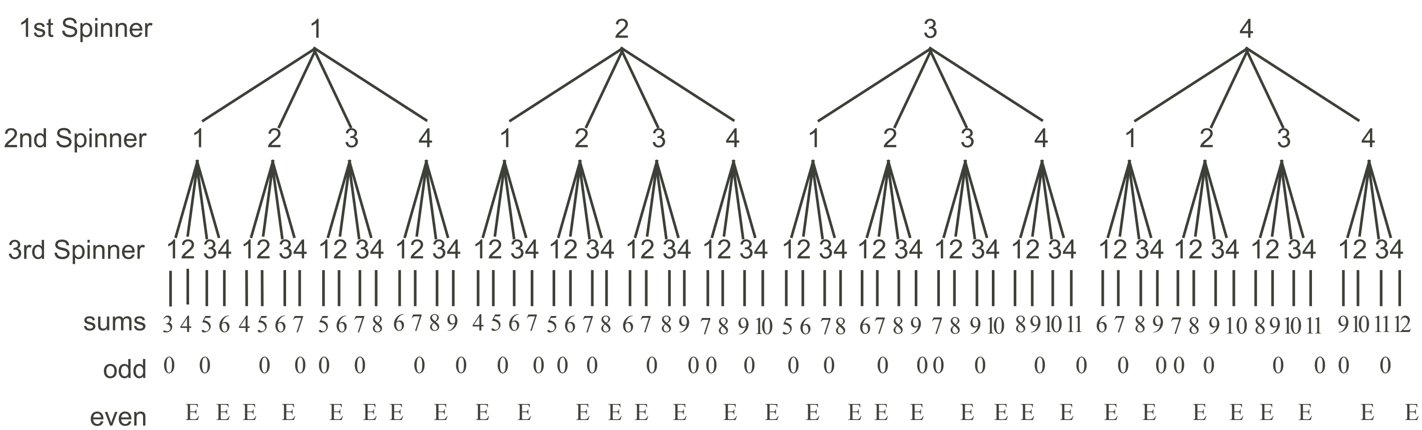
Sandy has 3 colours of paper. She needs 12 party favours.
She needs to make 4 party favours with each colour of paper.
She needs 4 colours of ribbon.

An alternate technique would be to start a tree diagram with the colours of paper and add branches, making sure each colour paper has the same number of branches, until there are 12 altogether.



Extension 1

Using 3 spinners, the tree diagram might look like this



Since half the outcomes are even and half are odd, this is a fair game.

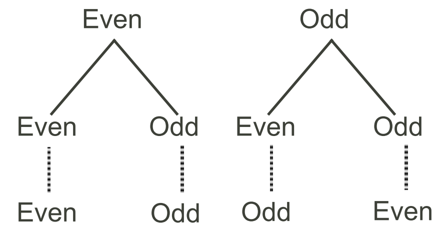
Solutions & Notes

Another tree diagram, less crowded, could start with the outcomes listed earlier for the totals of two spinners.

Outcomes of the sums of 2 spinners:

Outcomes of the third spinner:

Sums of 3 spinners:
Half the results are even and half are odd.



Note: 'Even' and 'Odd' can be listed as the only outcomes of the 1st two spins because they are equally likely. Similarly, since the numbers on the 3rd spinner are 1, 2, 3, and 4, half even and half odd, and equally likely, the outcomes for the third spinner can be given simply as 'Even' and 'Odd'.

Terry's T-shirts (Extension 2)

If students have difficulty getting started, suggest that they make lists of possible colours, sizes, and neck styles. The number of each should be small enough to use a tree diagram. For example, up to 6 colours, 3 sizes (small, medium, large) and 2 neck styles (round and vee).

Pete's Pizza Parlour (Family Activities 1)

1. (i) Since there are 5 possible toppings, there are 5 possible pizzas with one extra topping each.
   (ii) To determine the number of pizzas with 2 extra toppings, draw a tree diagram, or reason as follows: For every pizza with 1 extra topping, we have 4 possible second toppings (unless we want the same topping again). This gives 20 possible choices.
   (iii) Similarly, there will be 3 choices left for the third extra topping, so there will be 20 x 3 or 60 possible pizzas.

2. If pepperoni is not permitted, then Chris has a choice of
   1 pizza with no extra toppings, or
   4 pizzas with one extra topping each, or
   16 pizzas with two extra toppings each.
   He has a choice of 21 different pizzas.

License Plates (Family Activities 2)

1. The first digit can be any one of 10 digits.
   For each of these, the second digit can be any one of 10 digits, making 100 combinations.
   For each of these, the third digit can be any one of 10 digits, making 1000 combinations.

> **Re: License Plates**
> Since publication, Ontario license plates have gone to 4 letters. You may wish to discuss how this will change the answers, and why the change was necessary.

2. To calculate the total number of combinations, compute 26 x 26 x 26, which gives 17 576.
3. For each letter combination, there are 1000 digit combinations, giving 17 576 000 possible license plates.
4. Students will need to know the population of Ontario to answer this.
   They should be encouraged to check various sources.

Solutions & Notes

Activity 3: Frequencies

BLM 7

2. There should be enough telephone numbers on one page to give the expected result - that each digit should appear about as often as any other digit.

3. Students should be able to list all possible totals from 0 to 18 but they may not realize that some occur more frequently than others. Completion of the chart as suggested in the notes for Activity 3 lets them see that some totals are far more likely than others.
   The frequencies in the chart on BLM 8 should show that the low and high sums (e.g., 0, 1, 2, 3 and 16, 17, 18) occur rarely compared with the middle sums (e.g. 7, 8, 9, 10, 11).

Activity 4: Probability Experiments

BLM 9: Experiment 1: Coin Flipping

Students should come to two conclusions

(i) The number of heads and the number of tails should be close to equal.
(ii) The more data they collect, the closer the experimental probability will come to the theoretical probability.

BLM 10: Experiment 2: It's In The Bag

1. Students should see that there is a better chance of drawing the red since there are more red in the bag than blue.

2. The final results should be distributed in a way that reflects the beads in the bag. Theoretical probability suggests that: 3 out of 12 or 15 out of 60 will be red
   2 out of 12 or 10 out of 60 will be blue
   2 out of 12 or 10 out of 60 will be green.
   Ideally, the students' results will be close to this. If students results should differ to any great degree, you might wish to collect data from all groups and examine the frequency for each colour.

BLM 11: Experiment 3: Cup Tossing

The results cannot be predicted at all accurately, but the cup is expected to land on its side most of the time. Results of 60 tosses by the authors gave 50 on the side, 6 upside down, and 4 right side up.

Solutions & Notes

BLM 12: Experiment 4: Number Cards

Under ideal conditions, each number has an equal chance of being drawn. As with other experiments, students should find that the totals of several trials will be closer to the expected probability than the results of only one trial.

Extension 1

The 12 outcomes are black, black, black, black, black, red, red, red, yellow, yellow, yellow, and orange.
The probability of drawing a red is 3 out of 12 (or 1 out of 4).
The probability of drawing an orange is 1 out of 12.
The simplest spinner would be one divided into 12 sections, of which 5 are coloured black, 3 are red, 3 are yellow and 1 is orange.

Activity 5: Probability in Games

BLM 15: Experiment 4: Number Cards

Answers will vary.

BLM 16:

1. (a) (i) 'less than 70' gives 24 choices
   (ii) 'more than 50' gives 20 choices
   (iii) 'more than 30' gives 28 choices
   (iv) 'less than 20' gives 4 choices
   (v) 'more than 80' gives 8 choices

2. and 3.
   If students interpret these to mean what rolls from #1 are most or least favourable they should indicate that 'more than 30' is one of the most favourable, and 'less than 20' is one of the least favourable. However, the best roll of all is 'greater than 20' which gives 32 choices.

4. Since this choice allows any number to be covered, it reduces the game to tic-tac-toe.

Suggested Assessment Strategies

Investigations

Investigations involve explorations of mathematical questions that may be related to other subject areas. Investigations deal with problem posing as well as problem solving. Investigations give information about a student's ability to:

- identify and define a problem;
- make a plan;
- create and interpret strategies;
- collect and record needed information;
- organize information and look for patterns;
- persist, looking for more information if needed;
- discuss, review, revise, and explain results.

Journals

A journal is a personal, written expression of thoughts. Students express ideas and feelings, ask questions, draw diagrams and graphs, explain processes used in solving problems, report on investigations, and respond to open-ended questions. When students record their ideas in math journals, they often:

- formulate, organize, internalize, and evaluate concepts about mathematics;
- clarify their thinking about mathematical concepts, processes, or questions;
- identify their own strengths, weaknesses, and interests in mathematics;
- reflect on new learning about mathematics;
- use the language of mathematics to describe their learning.

Observations

Research has consistently shown that the most reliable method of evaluation is the ongoing, in-class observation of students by teachers. Students should be observed as they work individually and in groups. Systematic, ongoing observation gives information about students':

- attitudes towards mathematics;
- feelings about themselves as learners of mathematics;
- specific areas of strength and weakness;
- preferred learning styles;
- areas of interest;
- work habits — individual and collaborative;
- social development;
- development of mathematics language and concepts.

In order to ensure that the observations are focused and systematic, a teacher may use checklists, a set of questions, and/or a journal as a guide. Teachers should develop a realistic plan for observing students. Such a plan might include opportunities to:

- observe a small number of students each day;
- focus on one or two aspects of development at a time.

Suggested Assessment Strategies

Student Self-Assessment

Student self-assessment promotes the development of metacognitive ability (the ability to reflect critically on one's own reasoning). It also assists students to take ownership of their learning, and become independent thinkers. Self-assessment can be done following a co-operative activity or project using a questionnaire which asks how well the group worked together. Students can evaluate comments about their work samples or daily journal writing. Teachers can use student self-assessments to determine whether:

- there is change and growth in the student's attitudes, mathematics understanding, and achievement;
- a student's beliefs about his or her performance correspond to his/her actual performance;
- the student and the teacher have similar expectations and criteria for evaluation.

Resources for Assessment

"For additional ideas, see annotated Other Resources list on page 54, numbered as below."

1. The Ontario Curriculum, Grades 1-8: Mathematics.

2. *Assessment Standards for School Mathematics*, NCTM, 1995.

3. *Linking Assessment and Instruction in Mathematics: Junior Years*, Ontario Association of Mathematics Educators/OMCA/OAJE, Moore et al., 1996.

4. *Mathematics Assessment: Myths, Models, Good Questions, and Practical Suggestions*, by Jean Kerr Stenmark (Ed.), NCTM, 1991.

5. "Assessment", *Arithmetic Teacher* Focus Issue, February 1992, NCTM.

6. *How to Evaluate Progress in Problem Solving*, by Randall Charles et al., NCTM, 1987.

7. *Assessment in the Mathematics Classroom*, Yearbook, NCTM, 1993.

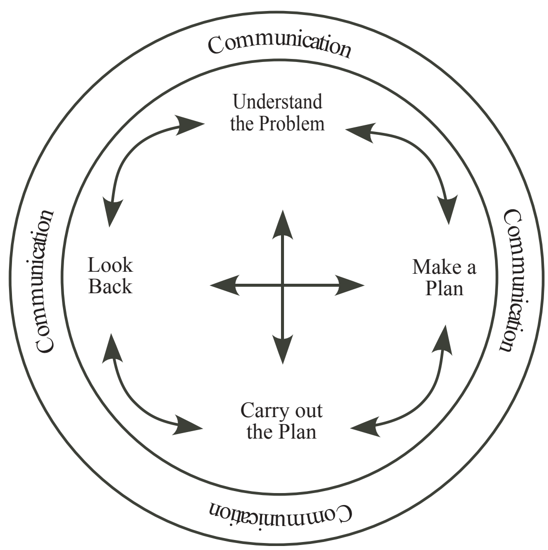
A GENERAL PROBLEM SOLVING RUBRIC

This problem solving rubric uses ideas taken from several sources. The relevant documents are listed at the end of this section.

"US and the 3 R's"

There are five criteria by which each response is judged:
Understanding of the problem,
Strategies chosen and used,
Reasoning during the process of solving the problem,
Reflection or looking back at both the solution and the solving, and
Relevance whereby the student shows how the problem may be applied to other problems, whether in mathematics, other subjects, or outside school.

Although these criteria can be described as if they were isolated from each other, in fact there are many overlaps. Just as communication skills of one sort or another occur during every step of problem solving, so also reflection does not occur only after the problem is solved, but at several points during the solution. Similarly, reasoning occurs from the selection and application of strategies through to the analysis of the final solution. We have tried to construct the chart to indicate some overlap of the various criteria (shaded areas), but, in fact, a great deal more overlap occurs than can be shown. The circular diagram that follows (from OAJE/OAME/OMCA "Linking Assessment and Instruction in Mathematics", page 4) should be kept in mind at all times.



There are four levels of response considered:

Level 1: Limited identifies students who are in need of much assistance;

Level 2: Acceptable identifies students who are beginning to understand what is meant by 'problem solving', and who are learning to think about their own thinking but frequently need reminders or hints during the process.

Level 3: Capable students may occasionally need assistance, but show more confidence and can work well alone or in a group.

Level 4: Proficient students exhibit or exceed all the positive attributes of the **Capable** student; these are the students who work independently and may pose other problems similar to the one given, and solve or attempt to solve these others.

Suggested Assessment Strategies

LEVEL OF RESPONSE →

CRITERIA FOR ASSESSMENT ↓

	Level 1: Limited	Level 2: Acceptable	Level 3: Capable	Level 4: Proficient
UNDERSTANDING	• requires teacher assistance to interpret the problem • fails to recognize all essential elements of the task	• shows partial understanding of the problem but may need assistance in clarifying	• shows a complete understanding of the problem	• shows a complete understanding of the problem
STRATEGIES	• needs assistance to choose an appropriate strategy	• identifies an appropriate strategy	• identifies an appropriate strategy	• identifies more than one appropriate strategy
STRATEGIES	• applies strategies randomly or incorrectly • does not show clear understanding of a strategy[1] • shows no evidence of attempting other strategies	• attempts an appropriate strategy, but may not complete it correctly[2] • tries alternate strateges with prompting	• uses strategies effectively • may attempt an inappropriate strategy, but eventually discards it and tries another without prompting	• chooses and uses strategies effectively[3] • recognizes an inappropriate strategy quickly and attempts others without prompting
REASONING	• makes major mathematical errors • uses faulty reasoning and draws incorrect conclusions • may not complete a solution	• may present a solution that is partially incorrect	• produces a correct and complete solution, possibly with minor errors	• produces a correct and complete solution, and may offer alternative methods of solution
REASONING	• describes[4] reasoning in a disorganized fashion, even with assistance • has difficulty justifying[5] reasoning even with assisstance	• partially describes[4] a solution and/or reasoning or explains fully with assistance • justification[5] of solution may be inaccurate, incomplete or incorrect	• is able to describe[4] clearly the steps in reasoning; may need assistance with mathematical language • can justify[5] reasoning if asked; may need assistance with language	• explains reasoning in clear and coherent mathematical language • justifies[5] reasoning using appropriate mathematical language
REFLECTION	• shows no evidence of reflection or checking of work • can judge the reasonableness of a solution only with assistance	• shows little evidence of reflection or checking of work • is able to decide whether or not a result is reasonable when prompted to do so	• shows some evidence of reflection and checking of work • indicates whether the result is reasonable, but not necessarily why	• shows ample evidence of reflection and thorough checking of work • tells whether or not a result is reasonable, and why
RELEVANCE	• unable to identify similar[6] problems	• unable to identify similar[6] problems	• identifies similar[6] problems with prompting	• identifies similar[6] problems, and may even do so before solving the problem
RELEVANCE	• unlikely to identify extensions[7] or applications of the mathematical ideas in the given problem, even with assistance	• recognizes extensions[7] or applications with prompting	• can suggest at least one extension[7], variation, or application of the given problem if asked	• suggests extensions[7], variation, or applications of the given problem independently

Suggested Assessment Strategies

Notes on the Rubric

1. For example, diagrams, if used, tend to be inaccurate and/or incorrectly used.

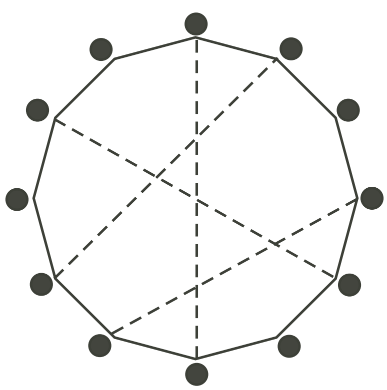
2. For example, diagrams or tables may be produced but not used in the solution.

3. For example, diagrams, if used, will be accurate models of the problem.

4. To *describe* a solution is to tell *what* was done.

5. To *justify* a solution is to tell *why* certain things were done.

6. *Similar* problems are those that have similar structures, mathematically, and hence could be solved using the same techniques.
   For example, of the three problems shown below right, the better problem solver will recognize the similarity in structure between Problems 1 and 3. One way to illustrate this is to show how both of these could be modelled with the same diagram:



Problem 1: There were 8 people at a party. If each person shook hands once with each other person, how many handshakes would there be? How many handshakes would there be with 12 people? With 50?

Problem 2: Luis invited 8 people to his party. He wanted to have 3 cookies for each person present. How many cookies did he need?

Problem 3: How many diagonals does a 12-sided polygon have?

Each dot represents one of 12 people and each dotted line represents either a handshake between two people (Problem 1, second question) or a diagonal (Problem 3).

The weaker problem solver is likely to suggest that Problems 1 and 2 are similar since both discuss parties and mention 8 people. In fact, these problems are alike only in the most superficial sense.

7. One type of extension or variation is a “what if...?” problem, such as “What if the question were reversed?”, “What if we had other data?”, “What if we were to show the data on a different type of graph?”.

Suggested Assessment Strategies

Suggested Adapted Rubric for Activity 2

The rubric below has been adapted for the problem on BLM 5 of Activity 2: How Many Teddies? The rubric examines understanding of the problem, the application of chosen strategies, and generalization of the strategies and their solutions.

Level 1: Limited	Level 2: Acceptable	Level 3: Capable	Level 4: Proficient
• does not understand that teddies are to have both buttons and vest colours to start, and may begin by colouring some of the vests and drawing buttons on others	• may not understand the problem initially, but can respond to direct questions by referring to the appropriate part of the problem	• understands the problem and can answer questions about the nature of the problem	• fully understands the problem and can express it in his/her own words
• shows no organization in the solution; buttons and colours are added haphazardly • addition of stripes is done haphazardly • may have duplicates and may or may not recognize these	• shows evidence of thinking about vest combinations of buttons and colours before marking up the bears • shows evidence of trying to relate the addition of stripes to the combinations already found • may have duplicates but can identify these without help	• shows some organization in the colouring and drawing buttons • initially, is not sure of the effects of adding stripes, but eventually concludes that this doubles the number of combinations	• may not need to draw buttons or colour the vests to be certain all possibilities are given • understands that adding stripes simply doubles the number of combinations
• misses 2 or 3 combinations	• misses at most one combination	• finds all combinations; list may not be organized	• can list all combinations in an organized manner
• is not sure of finding all solutions or cannot give a sensible reason why he/she is sure that all combinations are given • is unable to connect a list of possibilities with a tree diagram or cannot draw the tree diagram without help	• reason given for confidence in having all solutions is "I really tried, but I can't think of any more." • can draw a tree diagram if one is suggested; may need a hint to get started	• shows some organization in listing the combinations in order to 'prove' that he/she has all solutions • can construct a tree diagram if one is suggested	• uses an organized list or a tree diagram to 'prove' that he/she has all solutions

Other Resources

1. The Ontario Curriculum, Grades 1-8: Mathematics.

2. *Assessment Standards for School Mathematics*, NCTM, 1995.
This comprehensive document examines the purposes of assessment and the nature of various types of assessment. Actual experiences in the classroom are used to clarify and augment suggestions made.

3. *Linking Assessment and Instruction in Mathematics: Junior Years*, Ontario Association of Mathematics Educators/OMCA/OAJE, Moore et al., 1996.
The document provides a selection of open-ended problems (including Fair Games activities) tested at the Junior Years (grades 4 to 6) level. Performance Rubrics are used to assess student responses (which are included) at four different levels.

4. *Mathematics Assessment: Myths, Models, Good Questions, and Practical Suggestions*, by Jean Kerr Stenmark (Ed.), NCTM, 1991.
Included are notes on observation techniques, interviews, conferences, portfolios. Sample student work is included with assessment to show how to apply the suggestions in the book. Includes an annotated bibliography.

5. "Assessment", *Arithmetic Teacher* Focus Issue, February 1992, NCTM.
This issue contains articles on linking assessment with instruction, alternative forms of assessment, self-evaluation, using manipulatives on tests, and suggestions for assessing cooperative problem solving.

6. *How to Evaluate Progress in Problem Solving*, by Randall Charles et al., NCTM, 1987.
Chapter headings include "What are you trying to evaluate?", "What are some evaluation techniques?" and "How do you organize and manage an evaluation program?" Sample 'test' items are included that show how to assess students' understanding; holistic scoring scales and student opinion surveys suggest ways of assessing progress and affective domain.

7. *Assessment in the Mathematics Classroom*, Yearbook, NCTM, 1993.
The chapters in this book deal with a wide range of forms of assessment, techniques for implementing different forms, and examples from the classroom showing how observation, interview, journal writing, and tests, among others, can be used to make a meaningful assessment of a student and his/her work.

8. *Making Sense of Data: Addenda Series, Grades K-6*, Mary Lindquist et al., NCTM, 1992
Students explore different spinners in Grade 3 and determine which outcomes are more likely than others. Older students could design spinners to give specified outcomes. A grade 5 activity uses a "Random walk" to explore the nature of random numbers:: Logo is used to explore this idea further with a computer.

Other Resources

9. "Roll the Dice — an introduction to Probability", Andrew Freda, in *Mathematics Teaching in the Middle School*, NCTM, October, 1998, pp. 85-89.
   This describes how the introduction of an unfair game that appears, at first, to be fair, is used to intersect students in exploring how probabilities depend on relative frequencies of outcomes. Computer simulation leads to the conclusion that, the greater the sample, the closer the experimental probability is to theoretical probability. Suitable for Grades 5 or 6.

10. *Dealing with Data and Chance: Addenda Series, Grades 5-8*, Judith S. Zawojewski et al, NCTM, 1991.
    One activity deals with determining the fairness or "unfairness" of games, another with determining how many purchases you must make to collect all the enclosed cards necessary to "win" a prize. The game of "Montana Red Dog" (p. 41) involves students in predicting probabilities.

11. *Measuring Up: Prototypes for Mathematics Assessment*, Mathematical Sciences Education Board and National Research Council, Washington, DC.
    Performance assessment activities are given that deal with a number of mathematics topics. The "Hog" Game uses dice to encourage students to determine winning strategies while dealing with simple probability.

12. *What Are My Chances?, Book A,* and *What Are My Chances?, Book B*, Creative Publications, 1977.
    These books provide a number of activities (black line masters) suitable for a wide range of activities. Students flip coins or two-colour chips, roll number cubes, draw marbles from closed containers (with and without replacement) and explore simple probability activities dealing with the weather and ice-cream flavour combinations. Book B also explores dependence and independence of events.

13. "Truth or Coincidence?", Daniel J. Brohier, *Teaching Children Mathematics*, December 1996, pp. 180-183, NCTM.
    This activity for grades 3 to 6 involves children using heads and tails of coins to write "answer keys" for True-False tests. For example, if the test has 6 questions, how many possible combinations of True and False are possible? Applications to sports, social studies, and science are suggested.

14. "Exploring Random Numbers", William R. Speer, *Teaching Children Mathematics*, January 1997, pp. 242-245, NCTM.
    Students are introduced to the idea of randomness by trying to write random numbers. Other simple activities reinforce the idea that randomness is not easily achieved.

15. "Choice and Chance in Life: The Game of 'Skunk'", Dan Brutlag, *Mathematics Teaching in the Middle School*, April, 1994, pp. 28-33, NCTM.
    The game of "Skunk" uses "good" and "bad" rolls of the dice to create or eliminate a player's score. Analysis of the rules leads to a discussion of ways that "chance" enters our lives.

Other Resources

16. "Looking at Random Events with Logo Software", Thor Charischak and Robert Berkman, *Mathematics Teaching in the Middle School*, January-March 1995, pp. 318-322, NCTM.
    Students learn how to use Logo to simulate the rolling of up to 3 dice, and discuss the probabilities of different outcomes.

17. "Racing to Understand Probability", Laura R. Van Zoest and Rebecca K. Walker, *Mathematics Teaching in the Middle School*, October 1997, pp. 162-170, NCTM.
    Students play a game of racing sailboats that appears to be fair, but turns out to be skewed in favour of certain boats. Students make a chart of possible outcomes for 2 dice to try to explain why the game is unfair.

18. *Mathematics Teaching in the Middle School*, March 99, Focus Issue on "Data & Chance", NCTM.
    The entire journal is devoted to articles dealing with Data Management & Probability. One article explores probability through an even-odd game with dice of different shapes. Another uses animal crackers to simulate wild-life tagging and releasing to count animals in the wild.

19. "Mode Code", David Spangler, *Mathematics Teaching in the Middle School*, April, 1999, pp. 464-466, NCTM.
    Students explore the frequencies of letters of the alphabet in English. They apply this to decoding a message based on these frequencies, which are given as percents.

20. "Cat and Mouse", Brian Lannen. *Mathematics Teaching in the Middle School*, April 1999, pp. 456-459, NCTM.
    Students play a cooperative game to see whether or not the mouse finds the cheese. Aspects of probability include fairness or unfairness and tree diagrams to list outcomes. This activity is suitable for many different grades.

21. *Organizing Data & Dealing with Uncertainty*, NCTM, 1979
    The first section of the book deals with collecting and interpreting data. The second part explores probability through dice, license plates, paper cups, tacks, and beads. Black line masters are provided.

22. "Calendar Mathematics", Lorna J. Morrow, *Arithmetic Teacher*, March 1993, pp. 390-391.
    Problems with dice, or t-shirt and shorts combinations involve organized counting. Children must determine the number of outcomes and show how they are certain that they have all outcomes.

23. *DIME Probability Packs A and B*, available from Spectrum Educational Publishing, Aurora, ON 1988
    Each pack contains materials and activity cards for four different experiments. There are sufficient materials for 6 groups or pairs of students to work on the same experiment. The focus is on analyzing games as fair or unfair, and involves determining the probabilities of the outcomes in order to do so.

Curriculum Connections: Probability

ACTIVITY	DESCRIPTION OF THE ACTIVITY	CURRICULUM EXPECTATIONS
Activity 1 **Fair Spinners**	- identifying fair/unfair spinners by collecting experimental data - distinguishing between fair and unfair games using probabilities of various outcomes - identifying possible outcomes from a variety of spinners	• compare experimental results with predicted results • conduct simle probability experiments and use the results to make decisions
Activity 2 **Tree Diagrams**	- identifying all posible outcomes using tree diagrams - identifying all the variables needed for constructing a tree diagram	• use tree diagrams to organize data according to several criteria
Activity 3 **Frequencies**	- using tallies to record outcomes - counting relative frequencies of various outcomes	• demonstrate an understanding of probability and use language appropriate to situations involving probability experiments
Activity 4 **Probability Experiments**	- predicting probabilities for a variety of simple experiments - conducting probability experiments to determine experimental results - comparing experimental and theoretical probabilities	• compare experimental results with predicted results • conduct simle probability experiments and use the results to make decisions
Activity 5 **Probability in Games**	- listing all possible outcomes using tree diagrams or other method - identifying favourable outcomes	• use tree diagrams to organize data according to several criteria • use a knowledge of probability to ... solve simple problems