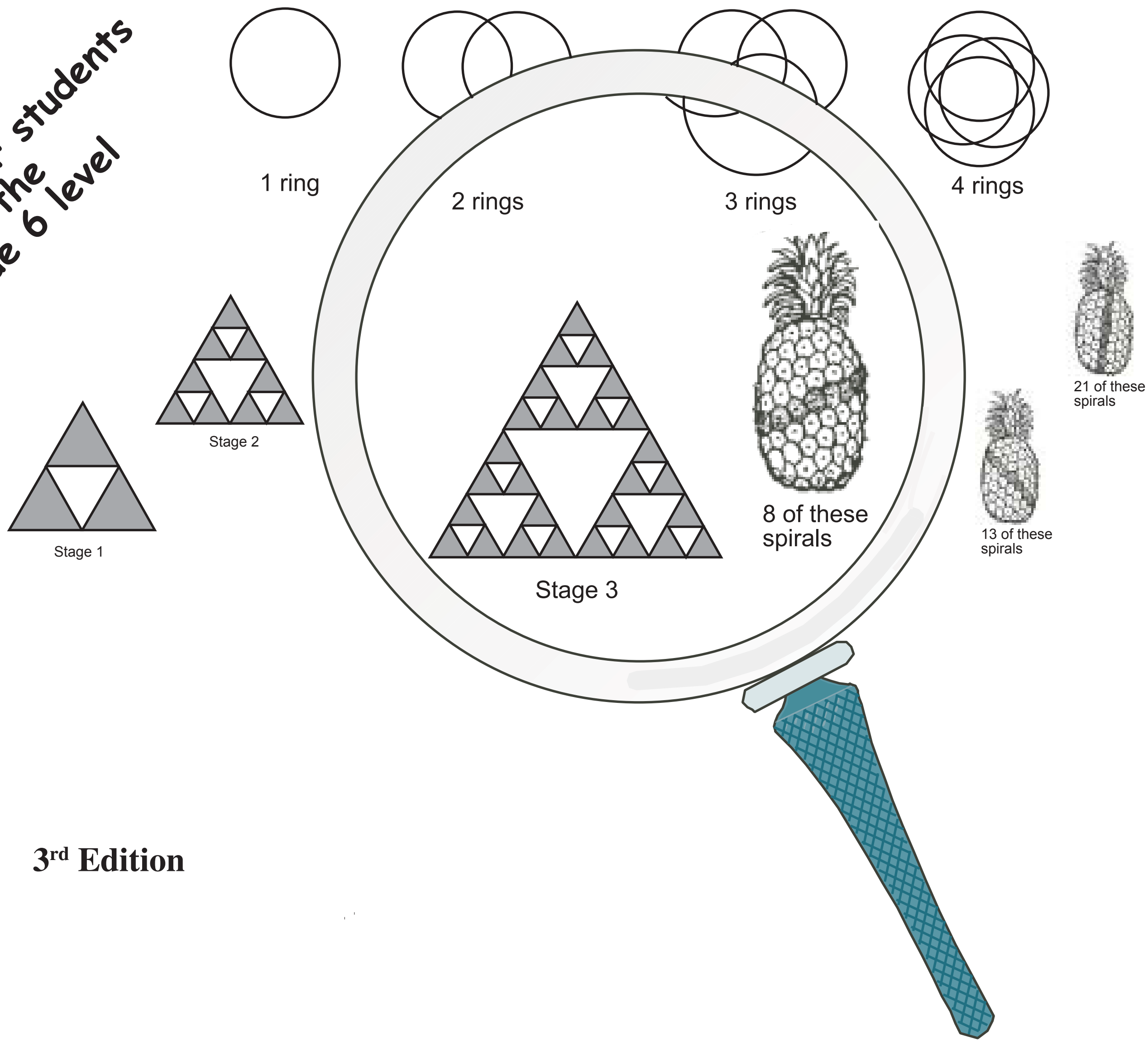
Invitations to Mathematics

Investigations in Patterns and Algebra

"Patterns All Over"

Suggested for students at the Grade 6 level



3rd Edition

An activity of

The CENTRE for EDUCATION in MATHEMATICS and COMPUTING

Faculty of Mathematics, University of Waterloo

Waterloo, Ontario, Canada N2L 3G1

Preface

The Centre for Education in Mathematics and Computing at the University of Waterloo is dedicated to the development of materials and workshops that promote effective learning and teaching of mathematics. This unit is part of a project designed to assist teachers of Grades 4, 5, and 6 in stimulating interest, competence, and pleasure in mathematics among their students. While the activities are appropriate for either individual or group work, the latter is a particular focus of this effort. Students will be engaged in collaborative activities which will allow them to construct their own meanings and understanding. This emphasis, plus the extensions and related activities included with individual activities/projects, provide ample scope for all students' interests and ability levels. Related "Family Activities" can be used to involve the students' parents/care givers.

Each unit consists of a sequence of activities intended to occupy about one week of daily classes; however, teachers may choose to take extra time to explore the activities and extensions in more depth. The units have been designed for specific grades, but need not be so restricted. Activities are related to the Ontario Curriculum but are easily adaptable to other locales.

"Investigations in Patterns and Algebra" is comprised of activities which explore numeric and geometric patterns in both mathematical and everyday settings, including, for example, architectural structures, biology, and experimental data. The recognition, description, and extension of patterns is arguably one of the most fundamental skills needed in mathematics, or for any problem solving situation; the activities in this unit are aimed specifically at developing this skill.

Acknowledgements

Contributing Teachers

Nancy Dykstra (Waterloo County Board of Education)
Kelly Lantink (Waterloo County Board of Education)
Ron Sauer (Waterloo County Board of Education - retired)
Barry Scully (York Region Board of Education/York University)
Jan Scully (York Region Board of Education/OISE/University of Toronto)
Mary Thompson (University of Waterloo)

Authors/Co-editors

Bev Marshman (University of Waterloo)
Lorna Morrow (Mathematics Consultant)

We wish to acknowledge the support of the **Centre for Education in Mathematics and Computing,** and in particular the efforts of Ron Scoins, Gord Nichols, and Carolyn Jackson. A special thank you goes to Bonnie Findlay for prompt, accurate type-setting and creative diagrams.

Table of Contents

Overview

Common Beliefs

These activities have been developed within the context of certain beliefs and values about mathematics generally, and pattern and algebra specifically. Some of these beliefs are described below.

Mathematics is the science of patterns. Recognizing, describing and generalizing patterns are thus key to developing students' understanding of mathematics, and also provide powerful tools for problem solving.

Patterns can be investigated through concrete materials, number tables, and experiments, and in a wide variety of contexts, such as music, art, natural science, and architecture. Patterns inherent in cultural artifacts have a relevance for students that helps them realize the importance of patterns, and motivates them to explore patterns further and to make connections between and among patterns. Verbalizing such patterns through descriptions, analysis, and predictions helps lead to the understanding necessary for generalizations expressed in the language of algebra.

As students attempt to justify their choices and patterns they develop a facility with, and understanding of, both the nature of proof and the language of mathematics.

Essential Content

The activities in this unit allow students to discover how patterns arise in a variety of mathematical and everyday contexts, and to establish the rules which govern them. In addition, there are Extensions in Mathematics, Cross-Curricular Activities and Family Activities. These may be used prior to or during the activity as well as following the activity. They are intended to suggest topics for extending the activity, assisting integration with other subjects, and involving the family in the learning process.

During this unit, the student will:

- explore, identify, and describe patterns in Fibonacci Numbers;
- use calculators to explore patterns in repeating decimals, the results of dividing by, for example, 3, 7, and 9;
- describe and analyze multiplication patterns that can be used in mental math;
- create and analyze 'growing patterns' from simple motifs (i.e. fractals);
- identify and generalize number patterns based on spatial patterns generated by, for example, constructing a patio, cutting a pizza, designing a logo;
- collect data, describe observed patterns, and make predictions from a variety of physical experiments;
- justify choice of pattern.

Notes

Curriculum Expectations

"Curriculum Expectations" are based on current Ontario curricula.

ACTIVITY	DESCRIPTION OF THE ACTIVITY	CURRICULUM EXPECTATIONS
Activity 1 **Patterns in Fibonacci Numbers**	• describing patterns in a special sequence of numbers called the "Fibonacci sequence"	• analyze and discuss patterning rules • recognize relationships and use them to summarize and generalize patterns • describe patterns encountered in any context • identify and extend patterns to solve problems in meaningful contexts • analyse number patterns and state the rule for any relationships • discuss and defend the choice of a pattern rule
Activity 2 **Decimal Patterns**	• exploring patterns in repeating and non-repeating decimals	• recognize relationships and use them to summarize and generalize patterns • use a calculator and computer applications to explore patterns • pose and solve problems by recognizing a pattern • analyse number patterns and state the rule for any relationships • discuss and defend the choice of a pattern rule
Activity 3 **Fractals**	• creating and analyzing growing patterns from simple motifs • exploring such patterns in two and three dimensions	• display pattern relationships graphically and numerically • apply patterning strategies to problem-solving situations • recognize relationships and use them to summarize and generalize patterns • given a rule expressed in mathematical language, extend a pattern
Activity 4 **Stairs and Pizzas**	• modelling, identifying, describing, and generalizing number patterns based on spatial patterns	• recognize relationships and use them to summarize and generalize patterns • identify and extend patterns to solve problems in meaningful contexts • analyse number patterns and state the rule for any relationships • discuss and defend the choice of a pattern rule
Activity 5 **Swinging Patterns**	• collecting data from experiments • recording data in tables of values • identifying and describing patterns in the table	• identify, extend, and create patterns in a variety of contexts • analyse and discuss patterning rules • recognize relationships and use them to summarize and generalize patterns • discuss and defend the choice of a pattern rule • state a rule for the relationship between terms in a table of values

Overview

Prerequisites

For Activity 1, it would be helpful for students to understand that a fraction such as can be interpreted as 2 ÷ 3, and that this is how the decimal equivalent of a fraction is determined. Otherwise, only basic computation skills and general knowledge are needed for these activities.

Logos

The following logos, which are located in the margins, identify segments related to, respectively:

Problem Solving Communication Assessment Use of Technology

Materials

ACTIVITY	MATERIALS
Activity 1 **Patterns in Fibonacci Numbers**	• Copies of BLMs 1 and 2 for each student • Copies of BLM 3 (optional)
Activity 2 **Decimal Patterns**	• Copies of BLMs 4 and 5 for each student • Calculators • Copies of BLM 6 (optional) • Piano, or other musical instrument (optional)
Activity 3 **Fractals (Growing Patterns)**	• Copies of BLMs 7 and 8 • Acetate copies of BLMs 7 and 8 for overhead projector • Copies of BLMs 9, 10 and 19 (optional) • Post-its, chart paper, counters, coins or linking cubes (optional) • Coloured paper or bristol board (optional)
Activity 4 **Stairs and Pizzas**	• Copies of BLMs 11, 12, 13, and 14 for each student or group • Copies of BLM 8 or graph paper (optional)
Activity 5 **Swinging Patterns**	• Paperclips, string, washers, elastics • Small flashlight, metre tape/stick • Copies of BLMs 15 and 16 for each group of students • Copies of BLMs 17 and 18 (optional)

Notes

Overview

Letter to Parents

SCHOOL LETTERHEAD

DATE

Dear Parent(s)/Guardian(s):

For the next week or so, students in our classroom will be participating in a unit titled “Patterns All Over”. The classroom activities will focus on the wide variety of contexts in which number patterns occur, including the Fibonacci sequence (as seen in plant growth and genetics), patterns in repeating decimals, growing patterns (i.e., fractals), patterns in architecture and logos, and in observations from physical experiments.

You can assist your child in understanding the relevant concepts by working together to perform simple experiments, and play games, and helping to locate everyday ways patterns are used.

Various family activities have been planned for use throughout this unit. Helping your child with the completion of these will enhance his/her understanding of the concepts involved.

If you work with patterns in your daily work or hobbies, please encourage your child to learn about this so that he/she can describe these activities to his/her classmates. If you would be willing to visit our classroom and share your experience with the class, please contact me.

Sincerely,

Teacher’s Signature

A Note to the Teacher:

If you make use of the suggested Family Activities, it is important to schedule class time for sharing and discussion of results.

Activity 1: Patterns in Fibonacci Numbers

Focus of Activity:

- Detecting patterns in a special sequence of numbers (Fibonacci sequence)

What to Assess:

- Identification of patterns
- Descriptions of patterns
- Accuracy of calculations

Preparation:

- Make copies of BLM 1 and BLM 2.
- Make copies of BLM 3 (optional).

Activity:

Tell students that there is a special sequence of numbers called the Fibonacci numbers (pronounced Fi-bow-natch-ee, with accent on the third syllable). These numbers are found in art, music, nature, science, etc. Give them the first few numbers in the sequence:

1, 1, 2, 3, 5, 8, 13 , . . .

Ask students to look for a pattern that would help them list the next ten numbers. If students are working in pairs or small groups, every student in a group should agree on the pattern that will be shared with the rest of the class. Have students explain how they devised that particular pattern.

A commonly used description of the pattern is

> Except for the first two terms, which must be given, each term in the Fibonacci sequence is the sum of the two immediately preceding terms.

Students usually identify this pattern though they will probably word it differently.

Distribute copies of BLM 1, and direct students to the column of numbers down the side of the page. This gives several numbers of the Fibonacci sequence. Students may wish to keep this as a reference when working on the problems on BLM 2 or 3.

Numbers from the Fibonacci sequence occur frequently in plants, in architecture, in art, in music, in the genealogy of male bees (drones), and elsewhere. Four examples are given on BLM 1. Others could be found by researching the topic in the library or on the web. For example, in the centre of a sunflower, the seeds spiral in two directions. One spiral has 55 parallel rows of seeds; the other has 89; both of these numbers are part of the Fibonacci sequence.

Notes

For information about Fibonacci, see "Solutions and Notes."

Use of Technology

Activity 1: Patterns in Fibonacci Numbers

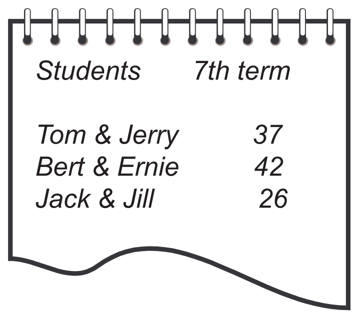
The traditional Fibonacci sequence begins with “1, 1” but a Fibonacci-type sequence could begin with any two numbers. To be sure all students recognize the same rule, give them the first two terms and ask them to write the first 10 terms of one of the following Fibonacci-type sequences. Several terms of each sequence are given here for your convenience in checking their work.

2 , 2, 4, 6, 10, 16, 26, 42, 68, 110, . . .

or 3, 7, 10, 17, 27, 42, 69, 111, 180, 291, . . .

or 5, 0, 5, 5, 10, 15, 25, 40, 65, 105, . . .

Have students select any two starting numbers for a Fibonacci-type sequence of their own, and write the first ten terms in a column. While you are walking around the room watching the students at work, make a note of the 7th term for each of several students/pairs/groups.



For an algebraic proof of the “trick” see “Soluions and Notes.”

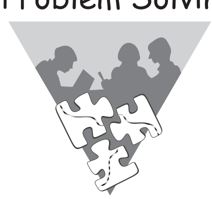
Then, when they have finished writing the 10 terms, tell them that you have already calculated the totals of the 10 numbers for many of them. Mentally multiply the 7th term of one pair by 11 (See BLM 3 for a “short cut”) . This will be the total of the 10 terms. Repeat for several of the students you observed, then have them calculate their totals. Assuming all the arithmetic is correct, their answers should match yours.

Have the students test this number “trick” or number “magic” by choosing another pair of starting numbers, and repeating the exercise.

Comments in italics are explanatory, and need not be conveyed to the students.

BLM 3,”Eleven Patterns” explores patterns in multiplying by 11 and will give students a technique for multiplying by 11 mentally. You may wish to introduce this now, though it could equally well be used later.

Problem Solving



Distribute copies of BLM 2 so that students can explore more patterns using Fibonacci numbers. You may wish to review the idea that $\frac{2}{3}$ can be interpreted as 2÷3, and that this is the way to determine a decimal equivalent for a fraction. If this is not a review, you will need to explain this to students for them to complete #3 on BLM 2. Alternatively, you may wish to postpone this problem until students have explored fraction-decimal equivalents further in Activity 2 (BLMs 4, 5, 6).

Activity 1: Patterns in Fibonacci Numbers

OPTIONAL

Ask students if they think the adding "trick" (multiplying the 7th term by 11) will work if the first two terms are fractions. Have students select two fractions as the first two terms and then write the first ten terms and determine their total, to see if the total is eleven times the 7th term.

Some samples are given below:

Example 1: $\frac{1}{2}$, $\frac{1}{2}$, 1, $1\frac{1}{2}$, $2\frac{1}{2}$, 4, . . .

Example 2: $\frac{1}{3}$, $\frac{1}{4}$, $\frac{7}{12}$, $\frac{10}{12}$, . . .

Ask: "Are the calculations you made to test the trick easier or harder than you thought they would be? Would it be harder if the first two terms were fractions like $\frac{5}{7}$ and $\frac{9}{13}$? Why or why not?"

For example 1, ask students why every third number is a whole number.

Although students may be prepared for some "ugly" calculations, this problem will not occur. Once the first two terms are added, the denominator does not change, so all the addition can be done using numerators only.

Example 3: $\frac{1}{5}\left(\text{or } \frac{3}{15}\right)$, $\frac{1}{3}\left(\text{or } \frac{5}{15}\right)$, $\frac{8}{15}$, $\frac{21}{15}$, $\frac{34}{15}$, . . .

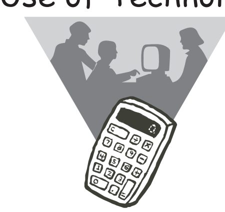
Example 4: $\frac{2}{7}\left(\text{or } \frac{6}{21}\right)$, $\frac{5}{3}\left(\text{or } \frac{35}{21}\right)$, $\frac{41}{21}$, $\frac{76}{21}$, $\frac{117}{21}$, . . .

You might wish to have students try this with decimal fractions as well.

Example 5: 0.5, 0.9, 1.4 , 2.3, 3.7, . . .

Cross-Curricular Activities:

Use of Technology



1. Students could do some library research or search the web to discover more about the mathematician Fibonacci, whose real name was Leonardo of Pisa. He was born about 1175 A.D. and used a pen-name of Fibonacci, possibly a contraction of "filius Bonacci" or "son of Bonacci" (his father).

2. Collect pine cones or sunflowers and count the numbers of spirals. Do all examples show Fibonacci numbers? What is the average of all the pine cones or sunflowers?

3. Fibonacci's first book, "Liber Abaci", was written before the invention of mechanical printing techniques. If the book was, say, 100 pages, estimate how long it might have taken for a scribe to make a copy.

Activity 1: Patterns in Fibonacci Numbers

Family Activities:

1. Encourage students to have family members write the first ten terms of a Fibonacci-type sequence, so that they (the students) can show how rapidly they can "add" these ten terms.

2. Each student could try to draw a family tree for himself/herself, and add names to it. Ask them how many great, great, great, great grandparents each of them had (like the bee in #4, BLM 1.) In what years might these particular ancestors have lived? In what country?
   The family tree will be simplest if, like the bee's, brothers and sisters, aunts and uncles are left out.

Other Resources:

For additional ideas, see annotated "Other Resources" list on page 61, numbered as below.

1. Patterns, Addenda Series, Grades K-6.
5. Fascinating Fibonaccis: Mystery and Magic in Numbers.

Activity 2: Decimal Patterns

Focus of Activity:

- Exploring patterns in repeating decimals

What to Assess:

- Correct use of calculators and interpretation of calculator displays
- Validity of patterns identified
- Clarity of student descriptions of patterns

Preparation:

- Make copies of BLMs 4 and 5.
- Make copies of BLM 6 (optional).

Activity:

Distribute copies of BLM 4 and calculators. You may wish to do #1 with the whole class to make sure students understand the task. Students should see a pattern in the calculator displays, namely,

$1 \div 9 = 0.1111111$, $2 \div 9 = 0.2222222$, and $3 \div 9 = 0.3333333$

The number of digits in the display will vary with the calculators used. Answers given in this booklet show an 8-digit display.

You may wish to remind students that $1 \div 9$ can also be written as $\frac{1}{9}$, and that they are calculating decimal equivalents for fractions during this activity.

Most students will be able to predict correct decimal equivalents for the divisions from $4 \div 9$ to $8 \div 9$. However, some will focus on the pattern so much that they will record $9 \div 9$ as 0.9999999 and will be surprised when the calculator shows '1'.

In answering #5 (and #9) on BLM 4, students will need to use a paper-and-pencil algorithm to check their predictions. This will reinforce the idea that the patterns repeat indefinitely.

Division by 11 produces similar patterns:

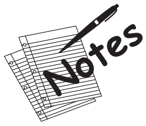
$1 \div 11 = 0.0909090$
$2 \div 11 = 0.1818181$
$3 \div 11 = 0.2727272$

Students may identify the pattern as related to the product of 9 times the dividend. For example,

$1 \div 11 = 0.$**09**$09090 \rightarrow 1 \times 9 =$ **9** or **09**
$2 \div 11 = 0.$**18**$18181 \rightarrow 2 \times 9 =$ **18**
$3 \div 11 = 0.$**27**$27272 \rightarrow 3 \times 9 =$ **27**

Use of Technology

You may wish to have students look for patterns in the 9 times table at this point. See "Solutions and Notes" for further suggestions.



Activity 2: Decimal Patterns

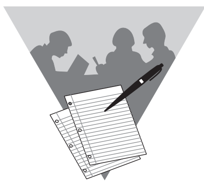
At this point, introduce the term "repeating decimal" and illustrate different ways to write a repeating decimal to indicate its repeating nature. For example,

$$1 \div 9 = 0.\overline{1} \text{ or } 0.\dot{1} \text{ or } 0.1111...$$

$$2 \div 11 = 0.\overline{18} \text{ or } 0.\dot{1}\dot{8} \text{ or } 0.1818...$$

$$3 \div 11 = 0.\overline{27} \text{ or } 0.\dot{2}\dot{7} \text{ or } 0.2727...$$

Assessment



Students should be able to distinguish between, say, 0.5 and $0.\overline{5}$, and should be able to describe the difference.

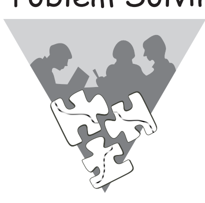
Distribute BLM 5. Be sure students know that they should use division as they did for BLM 4 to determine which of these divisors will produce repeating decimals. You may wish to have students predict which they think will be repeating and why.

Communication

Ask students how many actual divisions they will be testing for each divisor. They will need 1 division for each whole number less than the divisor. For example, when they tested division by 9, they tested 9 different division questions. However, 9 ÷ 9 (or 11 ÷ 11 or 3 ÷ 3, etc.) is not an essential division. Thus, to test division by 3, students should perform two divisions — 1 ÷ 3 and 2 ÷ 3; for division by 7, they should test 6 divisions, and so on. Students should realize that they do not need to calculate the decimal equivalent for an improper fraction such as $\frac{4}{3}$ or $\frac{9}{6}$. They should know that $\frac{4}{3}$ is really $1+\frac{1}{3}$ (or one plus one-third more) and therefore $\frac{4}{3}$ as a decimal will be 1.33333...

To cut down the time involved, you may wish to assign division by, for example, 3, 5 and 7 to half the class and division by 4, 6, and 8 to the other half. Once the students have completed problems 1 and 2 on BLM 5, they should report to the class. Record results on a large chart.

Problem Solving



Give students time to complete problems 3 to 6 on BLM 5. They will need to use a paper-and-pencil algorithm for #1(e) and #5 on BLM 5 to prove that each decimal repeats six digits. To indicate this, use a bar over the whole of the period (the digits that repeat) or a dot over the first and 6th as shown below.

$$\frac{1}{7} = 0.\overline{142857} \quad \text{or} \quad \frac{1}{7} = 0.\dot{1}4285\dot{7}$$

See "Solutions and Notes" following the BLMs for full explanations for #6 . You might wish to add another True/False statement: "All fractions with prime number denominators give repeating decimals." These questions force students to think about the nature of numbers in a way that is different from everyday computation.

Activity 2: Decimal Patterns

Extensions in Mathematics:

1. BLM 6 explores the use of the pattern in repeating decimals to predict specific digits of the decimal. Students will find it trivial to predict the 100th digit of the decimal equivalent for $\frac{1}{3}$, but determining the 10th or 50th or 100th digit of the decimal equivalent of $\frac{1}{7}$ takes some thought. See "Solutions and Notes" for techniques students might use.

Cross-Curricular Activities:

1. Have students experiment with creating musical patterns by assigning number values to the notes of the C major scale, starting at the B below middle C, as follows:
   B - 0, C - 1, D - 2, E - 3, F - 4, G - 5, A - 6, B - 7, C - 8, D - 9.
   Then have them 'play' the repeating decimal patterns they found on BLMs 4, 5 and 6, on a piano or other musical instrument. Ask which patterns they liked best and why. Students could carry this idea further by incorporating the black keys into their numbering system to see how that changes the sound of the different musical patterns. Alternatively, they could invent longer repetitive chains of digits and experiment with how they 'sound'.

Family Activities:

1. Discuss the following problem with students and have them try the problem with family members.

   Remind students of the "Make a smaller problem" strategy, and suggest that they
   (i) break down the following problem into smaller steps,
   (ii) record the steps and the totals,
   (iii) find a pattern, and
   (iv) use the pattern to determine the sum of the fractions.

   Problem: Determine the sum of $\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\frac{1}{4\times 5}+\ldots+\frac{1}{19\times 20}$.

Other Resources:

For additional ideas, see annotated "Other Resources" list on page 61, numbered as below.

2. Patterns and Functions, Addenda Series, Grades 5-8.

Notes

Problem Solving

Activity 3: Fractals (Growing Patterns)

Focus of Activity:
- Creating and analyzing growing patterns from simple motifs

What to Assess:
- Identification of relevant number patterns
- Accurate construction of the designs

Preparation:
- Make copies of BLMs 7 and 8 for students, and an acetate copy for use on the overhead projector.
- Make copies of BLMs 9, 10 and 19 (optional).

Activity:

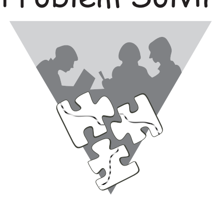
An area of mathematics that is currently receiving a great deal of attention is the study of fractals. These are self-replicating designs — that is, they repeat themselves indefinitely, thus producing a new kind of pattern. The study of naturally occurring fractals (e.g., coastlines, growth patterns of plankton) is an important branch of this topic. Much simpler versions are presented below for students to construct and explore. The term “fractal” was coined by Benoit Mandelbrot, a professor of mathematics at Yale University.

Distribute copies of BLMs 7 and 8. The growth of the fractal in #1 can be illustrated using post-its on chart paper or the blackboard. Students could use counters or coins or linking cubes to construct the stages of the fractal. They can then record these stages on graph paper (BLM 8).

Communication

Although examples of the fractals on BLM 7 can be found in “Solutions and Notes”, this activity is well suited for students checking each other, and explaining and justifying their answers to each other. If students begin the problems in pairs, then two pairs could compare results and try to come to a common solution, before reporting to the class.

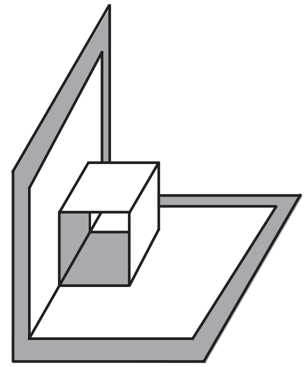
Problem Solving



OPTIONAL:
BLM 9, “The Sierpinski Triangle”, illustrates one of the best known fractals. It differs from the fractals on BLM 7 in that the size of the figure remains constant, but its complexity increases. Students are not asked to draw the stages of this fractal, but are asked to find a pattern to indicate the number of black triangles in the 100th stage. The ‘Challenge’ problem presents a variation.

Activity 3: Fractals (Growing Patterns)

Extensions in Mathematics:



1. BLM 10, "3D Fractals" involves some paper folding and cutting to make "pop-up" fractals in much the same way pop-up greeting cards or books are made. If the fractal paper is pasted inside some coloured paper or bristol board, a very dramatic greeting card can be made. The diagram at right shows the fractal at Stage 1 pasted into a coloured card. BLM 19 provides templates for the students to cut, if desired. Note that these fractals work best if made from heavy paper. For clean cuts, Xacto™ knives work better than scissors. Score the paper on fold lines for precise folds.

Cross-curricular Activities:

1. Students could explore Origami, another type of paper folding.

Other Resources:

For additional ideas, see annotated "Other Resources" list on page 61, numbered as below.

6. Fractal Cuts

7. Pop-Up Paper Engineering: Cross-Curricular Activities in Design Technology, English and Art.

Notes

Notes

Activity 4: Stairs and Pizzas

Focus of Activity:

- Generalizing number patterns based on spatial patterns

What to Assess:

- Identification of patterns
- Accurate descriptions of patterns
- Accuracy of predictions based on patterns

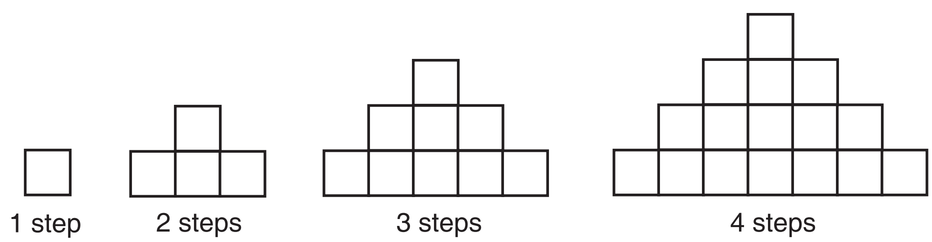
Preparation:

- Make copies of BLMs 11, 12, 13, and 14.
- Make copies of BLM 8 or graph paper (optional).

Activity:

This activity consists of problems in which students manipulate simple materials, record data, and then identify a pattern that will help them predict further values.

See Figure 4.1. Sketch the diagram on the blackboard or chart paper and present the problem to the students.



Jamie was using grey and coloured concrete blocks to make a patio. Jamie needed to figure out how many coloured blocks would be needed to form a stair pattern like the ones shown, but with 8 steps. How many coloured blocks do you think will be needed?

Figure 4.1

When offering their predictions, the students should explain why they think their answers are correct. You might wish to discuss problem-solving strategies that could be useful, such as ‘make a smaller problem’ or ‘find a pattern’.

Start a table to encourage students to look for a pattern.

No. of steps	1	2	3	4	5	6	7	8
No. of blocks	1	4	9	__	__	___	___	___

Problem Solving

Students could work in pairs to determine the number of blocks needed for 5 steps and 6 steps. Then have them look for a pattern. They should see that the number of steps multiplied by itself will give the number of blocks needed. Using graph paper helps students to make accurate drawings.

Activity 4: Stairs and Pizzas

Algebraically, this means that for n steps, the number of blocks needed is n x n or n^2. Students may express this as "the number of steps times itself."
Stress that, in the problems they will be interpreting, an algebraic expression may be asked for, or introduced. The letter 'n' can be thought of as 'any number'.

Ask students to give the meaning of each of the following

$n \times n$

$n + n$

$2 \times n$

n^2

$n^2 + 1$

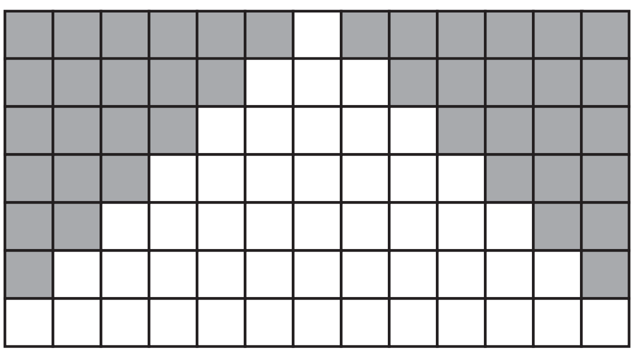
Responses should be similar to the following

"a number times itself" $(n \times n \text{ or } n^2)$

"two times a number" $(2 \times n)$

"a number added to itself" $(n + n)$

Extend the problem by asking students to determine the number of grey blocks needed if the patio is to be rectangular, measuring 8 blocks by 15 blocks.



Distribute BLM 11, A Mayan Staircase. This presents a problem similar to the one just finished. However, the algebraic expression for n steps in this array is more complicated. Students are not asked to determine the expression; they are asked to select one of 3 given expressions similar to the ones they have already interpreted. (You may wish to have students write the meanings opposite the expressions.) They should realize that a true generalization must hold for all numbers, and so they must test each algebraic expression more than once.

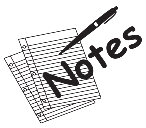
The sequence of numbers that appears in the second row of the table (i.e., 1, 3, 6, 10, 15, 21, 28, 36, ...) is called the set of "Triangular Numbers". For more on this see Reference 1 below.

BLMs 12, 13, and 14 give more problems, all of which involve finding a pattern, and developing a generalization. These can be used in a variety of ways such as assigning different problems to different groups, assigning one problem per day, sending a problem home as a Family Activity, or placing the problems and materials in a math centre.

Notes

Communication

Triangular Numbers can also be found in the Grade 5 book of this series: "Picture the Pattern".



Use of Technology

Activity 4: Stairs and Pizzas

Extensions in Mathematics:

1. Have students determine the average height of their classmates, and pretend the photo on BLM 11 shows two grade 6 students on the pyramid. Ask how this would change their estimates of the pyramid's height.

Cross-Curricular Activity:

1. Investigate the mathematics and the calendar of the Mayan civilization, either in the library, or on the web.

Family Activities:

1. See the suggestion above for BLMs 12, 13, and 14.

Other Resources:

For additional ideas, see annotated "Other Resources" list on page 61, numbered as below.

1. Patterns, Addenda Series, Grades K-6.
8. A Teacher's "Try" Angles.

Activity 5: Swinging Patterns

Focus of Activity:

- Collecting data, describing observed patterns, and making predictions

What to Assess:

- Care with which experiments are conducted
- Identification of trends in observed data
- Justification of conclusions
- Communication of results

Preparation:

- Make copies of BLMs 15 and 16.
- Make copies of BLMS 17 and 18 (optional).

Activity:

This activity consists of experiments for students to carry out. They should record their data and look for a pattern in the table they have completed. You may not want to take the time for all students to do all experiments. The experiments are arranged on the BLMs so that they can be easily cut apart and one or two given to each pair/group of students. One or more could also be sent home as a Family Activity.

Three of the experiments deal with pendulums, and there is some vocabulary that students will need:

- a *cycle* is one swing back and forth;
- the number of cycles per unit of time is the *frequency*
- the *amplitude* is the width of the swing.

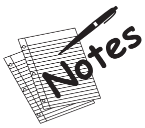
These meanings should be discussed with students before they begin the experiments, and could be recorded on blackboard or chart paper so they are always accessible to students as they are working.

One way to help make the meanings clear is to use experiment 1 as a demonstration with the whole class. That way students will also learn how important it is to measure carefully. Students should also realize, however, that their measurements will be made under less-than-accurate circumstances, given the unsophisticated materials and measuring devices. Even scientists in their well equipped laboratories are limited by the accuracy of their instruments, and must be willing to accept a certain margin of error. So, too, should the students. They should examine their results to see if a trend is showing up. They should not be looking for a hard-and-fast rule.

The three experiments on BLM 15 explore what happens when one variable is changed: for #1, the length of the pendulum; for #2, the number of washers (i.e.,the weight of the pendulum bob); and for #3, the amplitude of the swing. Students must be careful to keep everything constant except the variable they are changing.

Notes

Problem Solving



Assessment

Use of Technology

Activity 5: Swinging Patterns

See “Solutions and Notes” for expected results for the experiments.

Students should be prepared to report their results to the class and to justify their conclusions. Whether this is a formal or informal presentation, it is an ideal opportunity for assessment.

Cross-Curricular Activities:

1. Distribute BLM 18. Have students read through the BLM and discuss it briefly with them. This is intended as a long-term activity. Students might be allowed a few minutes a day over a week or more, and also be expected to do some work on their own. Students could also work in groups. It is not essential that a student give examples of every type of pattern suggested. The purpose of the activity is to make students aware of the many different types of pattern. Students could also search the web for information about patterns.

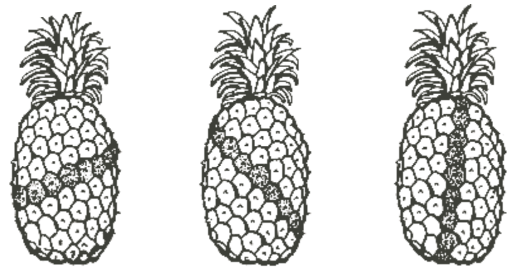
Family Activities:

1. BLM 17 presents some string-cutting experiments that students could take home to work on with other family members. Pattern “A” is a simple one in which each cut doubles the number of pieces of string. Patterns “B” and “C” are slightly more complex but build on Pattern A.

BLM 1: Fibonacci All Over

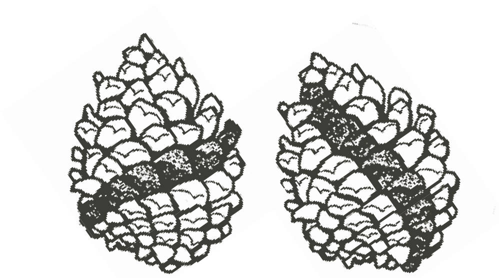
Fibonacci numbers can be found in many places in nature. Four are given here.

1. The scales of a pineapple are arranged in spirals. There are three sets of spirals, each going in a different direction. If you count carefully you will find that the number of each type of spiral is a Fibonacci number.



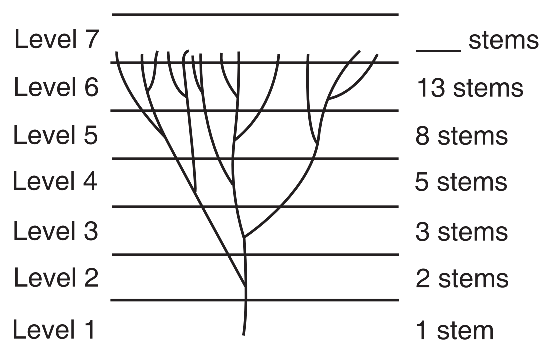
8 of these spirals — 13 of these spirals — 21 of these spirals

2. The bracts of a pine cone are also arranged in spirals, but only two different ones. The number of each type of spiral is a Fibonacci number.



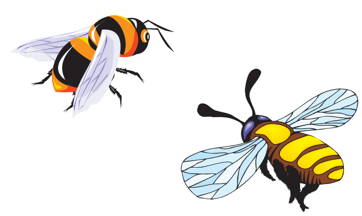
8 of these spirals — 13 of these spirals

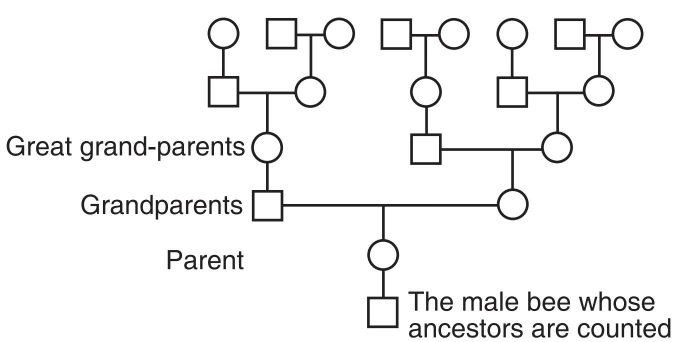
3. A Sneezewort bush sprouts new stems in such a way that the total number of stems at each level of development is a Fibonacci number. How many stems should there be at Level 7?



A Sneezewort Plant

4. Male bees have only a mother; female bees have a mother and a father. In the genealogy chart below, the ancestors for a male bee are given back to his great, great, great grandparents. The numbers in the table are from the Fibonacci sequence. How many great, great, great, great grandparents would the male bee have had? How many of these would be male?





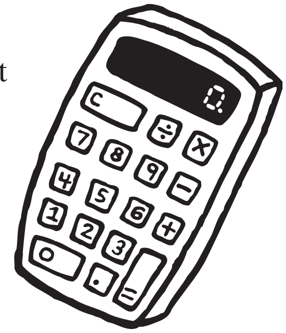
Male (□)	Female (○)	Total ___
3	5	8
2	3	5
1	2	3
1	1	2
0	1	1

The Fibonacci Sequence

1
1
2
3
5
8
13
21
34
55
89
144
233
377
610
987
1 597
2 584
4 181
6765
10 946
17 711
28 657
46 368
75 025
.
.
.

BLM 2: Fibonacci Patterns

Use the Fibonacci sequence beginning 1, 1, 2, 3, 5, 8, 13, 21, 34, . . . for these problems.

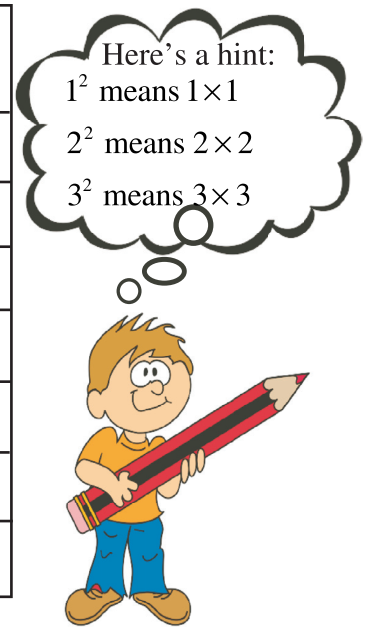


1. An interesting pattern in Fibonacci numbers has to do with their divisors. It is said that
   every 3rd Fibonacci number is divisible by 2: (2, 8, 34, 144, 610, ...)
   every 4th Fibonacci number is divisible by 3: (3, 21, ...)
   every 5th Fibonacci number is divisible by 5
   every 6th Fibonacci number is divisible by 6
   every 7th Fibonacci number is divisible by 7

   Is this accurate? Does the pattern continue beyond the divisors 2 and 3?

2. Continue the table to identify a pattern in the sums of the squares of Fibonacci numbers. How are the numbers in the third column related to the Fibonacci numbers? Give the third column a title.

The Fibonacci numbers squared	The sum	
1^2	1	1×1
1^2+1^2	2	1×2
$1^2+1^2+2^2$	6	2×3
$1^2+1^2+2^2+3^2$	15	3×5



3. Fibonacci fractions are produced in the following way. Choose one Fibonacci number as the numerator; then the next Fibonacci number will be the denominator.

   For example, $\frac{1}{1}$, $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{5}$, $\frac{5}{8}$, . . . ,

   (a) Will any of the Fibonacci fractions have a value greater than 1? Why or why not?
   (b) Which Fibonacci fraction has the greatest value?
   (c) Using a calculator, determine the value of each of the first eight Fibonacci fractions as a decimal to the nearest thousandth.

fraction	$\frac{1}{1}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{5}$	$\frac{5}{8}$
decimal	1.0	0.5			

   (d) What pattern can you see in the decimal numbers?

BLM 3: Eleven Patterns

1. (a) Complete the following:

   2 x 11 = _______   5 x 11 = _______   4 x 11 = _______

   9 x 11 = _______   3 x 11 = _______   7 x 11 = _______

   (b) Describe the result when you multiply a single digit by 11.

   (c) What do you think will be the result if you multiply a two-digit number by 11? Why?

2. (a) Complete the following:

   23 x 11 = _______   34 x 11 = _______   63 x 11 = _______

   71 x 11 = _______   44 x 11 = _______   90 x 11 = _______

   (b) Describe the result when you multiplied one of these two-digit numbers by 11. Compare this with your prediction from 1(c).

   (c) Do you think this will be true for all two-digit numbers when multiplied by 11? Why?

3. (a) Complete the following:

   27 x 11 = _______   47 x 11 = _______   65 x 11 = _______

   39 x 11 = _______   73 x 11 = _______   94 x 11 = _______

   (b) Describe the results. Compare this with your prediction from 2(c).

   (c) How are the results from 2(a) and 3(a) alike? How are they different?

4. (a) Use what you have learned about multiplication by 11 to write answers for the following:

   123 x 11 = _______   241 x 11 = _______   304 x 11 = _______

   352 x 11 = _______   579 x 11 = _______   827 x 11 = _______

   (b) Check your answers using a calculator.

5. Repeat 4(a) and (b) for the following:

   1234 x 11 = _______   2431 x 11 = _______   3041 x 11 = _______

   3252 x 11 = _______   5779 x 11 = _______   2872 x 11 = _______

BLM 4: Decimal Patterns

1. Using a calculator, complete the following, recording the complete calculator display for each.

   1 ÷ 9 = ___________ 2 ÷ 9 = ____________ 3 ÷ 9 = _________

2. Describe any patterns you see.

3. Predict answers for the following, and then check with your calculator.

   4 ÷ 9 = ___________ 5 ÷ 9 = ___________ 6 ÷ 9 = ___________

   7 ÷ 9 = ___________ 8 ÷ 9 = ___________ 9 ÷ 9 = ___________

4. Were you surprised by any results? Why or why not?

5. If your calculator display showed 10 places, what would it display for 1 ÷ 9? Why? How could you check this?

6. Using a calculator, complete the following, recording the complete calculator display

   1 ÷ 11 = ___________ 2 ÷ 11 = ___________ 3 ÷ 11 = ___________

7. Describe any pattern you see.

8. Using the pattern, predict the results for the following, and then check with your calculator.

   4 ÷ 11 = ___________ 5 ÷ 11 = ___________ 6 ÷ 11 = ___________

   7 ÷ 11 = ___________ 8 ÷ 11 = ___________ 9 ÷ 11 = ___________

   10 ÷ 11 = ___________ 11 ÷ 11 = ___________

9. If your calculator display showed 12 digits, what would it display for 1 ÷ 11? Why? How could you check this?

BLM 5: Repeating or Not?

1. Test each of the following divisors to determine which produce repeating decimals. Record the division and the repeating or non-repeating decimals. Look for patterns in the decimal answers to help you predict results.

(a) ÷ 3	(b) ÷ 4	(c) ÷ 5
(d) ÷ 6	(e) ÷ 7	(f) ÷ 8

2. Write how you determined which divisors in #1 were repeating decimals. How could you check your work?

3. If you knew that $\frac{1}{13}$ was equivalent to a repeating decimal, what would you say about $\frac{3}{13}$? $\frac{4}{13}$? $\frac{10}{13}$? Explain.

4. If $\frac{1}{6} = 0.1\overline{6}$ or 0.16666... then $\frac{2}{6}$ should be $2 \times 0.1666\ldots$. The multiplication on the right shows the result when your calculator has an 8-digit display.

$$\begin{array}{r} 0.1666666 \\ \times 2 \\ \hline 0.3333332 \end{array}$$

What would happen with a 10-digit display? a 12-digit display? an endless display? How would you write this as a repeating decimal?

5. An unusual repeating decimal is 1/7 = 0.142857.

(a) Complete the following chart without using a calculator, and without looking at your answers for #1.

$\frac{1}{7}$	$1 \times 0.\overline{142857} = 0.\overline{142857}$
$\frac{2}{7}$	$2 \times 0.\overline{142857} = 0.\overline{285714}$
$\frac{3}{7}$	$3 \times 0.\overline{142857} =$
$\frac{4}{7}$	$4 \times \qquad =$

(b) How are all the repeating decimals alike? How are they different? Use these similarities and differences to help you write decimal values for $\frac{5}{7}$ and $\frac{6}{7}$ without any calculation. Explain your method.

(c) How does the number of digits in your calculator display make it difficult to decide whether or not $\frac{1}{7}$ gives a repeating decimal?

6. Tell which of the following statements you think is/are true. Tell why. Then test your predictions and explain any differences between your results and your predictions.

(a) All fractions with odd numbers in the denominator give repeating decimals.

(b) No fractions with even numbers in the denominator give repeating decimals.

BLM 6: Repeating and Repeating and Repeating

As you have seen, a repeating decimal has a pattern.

For example, $\frac{4}{9} = 4 \div 9 = 0.363636\ldots$ has a repeating pattern of two digits;

and, $\frac{1}{7} = 1 \div 7 = 0.142857142857\ldots$ has a repeating pattern of six digits.

1. How could you use the pattern for $\frac{4}{9}$ or $4 \div 9$ to predict the 20th digit in this repeating decimal? the 100th digit? the 199th digit?

2. (a) What will be the 18th digit of $\frac{1}{7}$ or $1 \div 7$ as a repeating decimal?

   (b) What will be the 20th digit?

   (c) Explain how you got your answers.

3. Each of the following gives a repeating decimal. Use a calculator to help you determine what the repeating decimal is. Then predict the 100th digit of the repeating decimal.

   (a) $\frac{1}{3} = 1 \div 33 =$

   (b) $\frac{1}{37} = 1 \div 37 =$

   (c) $\frac{5}{41} = 5 \div 41 =$

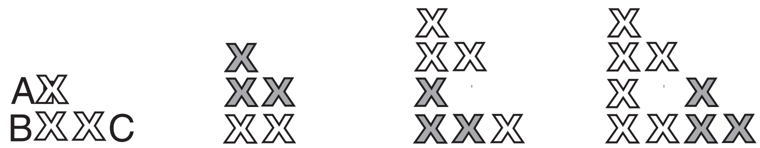
   (d) $\frac{8}{27} =$

   (e) $\frac{1}{101} =$

   (f) $\frac{10}{111} =$

BLM 7: Growing Patterns

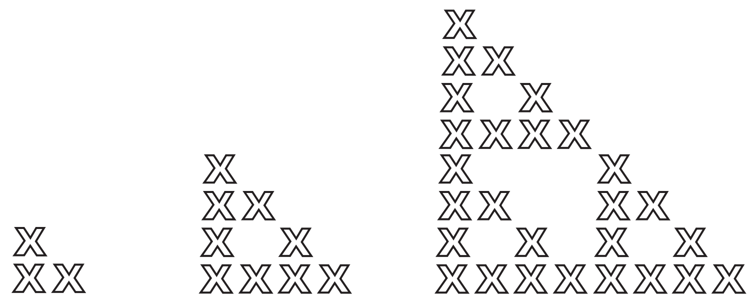
1. A fractal is a design made by repeating the design itself over and over. For example, suppose we start with a very simple design of three crosses, labelled A, B, and C. (See 1 below.)



1 2 3 4

Diagram 2 shows cross A replaced by the whole design. Diagram 3 shows each of crosses A and B replaced by the whole design, and diagram 4 shows each of crosses A, B, and C replaced by the whole design. Diagram 4 is Stage 2 of the fractal.

(a) The first three stages are shown below. To construct Stage 4, replace every cross in Stage 3 by the design in Stage 1.



Stage 1 Stage 2 Stage 3

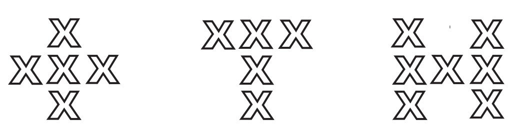
(b) Complete the table to show the number of crosses needed for Stage 3 and Stage 4

Stage	1	2	3	4	5
Number of crosses	3	9			243

(c) Can you see a pattern in the table to help you figure out the number of crosses needed for the next stages? Describe your pattern.

(d) How will your pattern help you decide how many crosses would be needed for Stage 10? Tell how many crosses are needed for Stage 10.

2. (a) To construct another fractal, choose one of the following designs for Stage 1.



(i) (ii) (iii)

(b) To construct Stage 2, replace every cross in Stage 1 with the whole design that you chose for Stage 1. Complete a chart for your fractal:

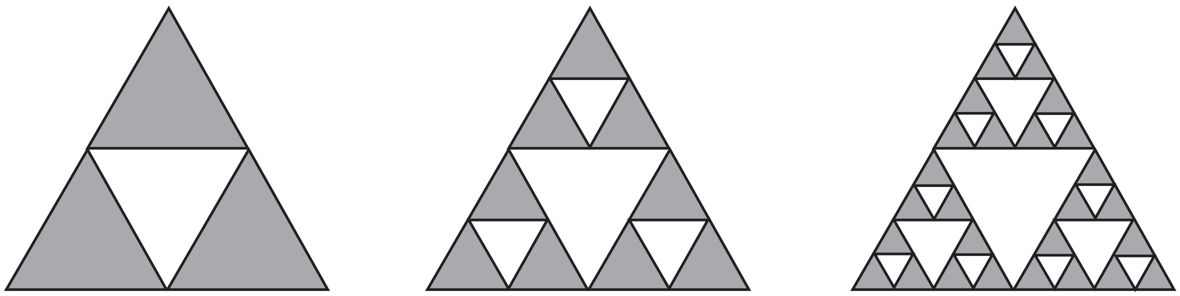
Stage	1	2	3	4	5
Number of crosses					

(c) Look for a pattern in your table. How is it like the pattern for #1? How is it different?

BLM 8: Grid

BLM 9: The Sierpinski Triangle

1. A famous fractal is called the Sierpinski Triangle. The first three stages are shown below.



Stage 1 Stage 2 Stage 3

(a) Complete the table below, first by counting, then by finding a pattern that will let you predict the number of black triangles in various stages.

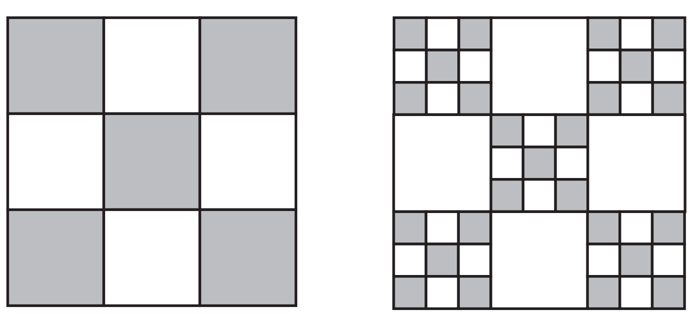
Stage	1	2	3	4	5	10	100
Number of black triangles							

(b) How is the pattern for the Sierpinski Triangle like the patterns you found for the fractals on BLM 7? How is it different?

(c) If someone asked you what kind of patterns you found in fractals, what would you say?

A Challenge:

2. A fractal similar to the Sierpinski Triangle is composed of squares rather than triangles. The first 2 stages are shown below. Every dark square is replaced by a square divided into 9 small squares, alternately coloured and plain.



Stage 1 Stage 2

(a) Use graph paper to draw Stage 3 of this fractal. Start by drawing the large square 27 squares by 27 squares.

(b) Compare the pattern of the number of dark squares to the pattern in the number of triangles above. How are the patterns alike? How are they different?

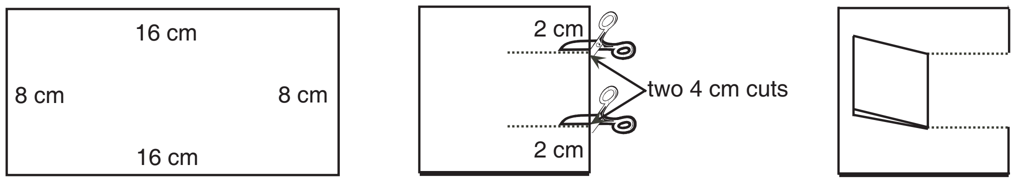
BLM 10: A Three-dimensional Fractal

To construct this fractal, you will need a pencil, heavy paper, a ruler, and scissors.
Follow the instructions, being sure that your drawing, folding, and cutting are accurate.

1. Draw a rectangle 8 cm by 16 cm, and cut this out.

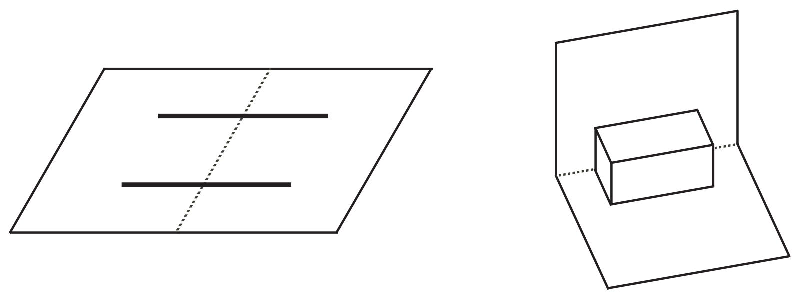
2. Fold the rectangle in half to form a square 8 cm by 8 cm.

3. Measure along the fold 2 cm in from each edge. Draw two lines perpendicular to the fold. Make these lines 4 cm long. Cut along both these lines. You will be cutting through two layers of paper.

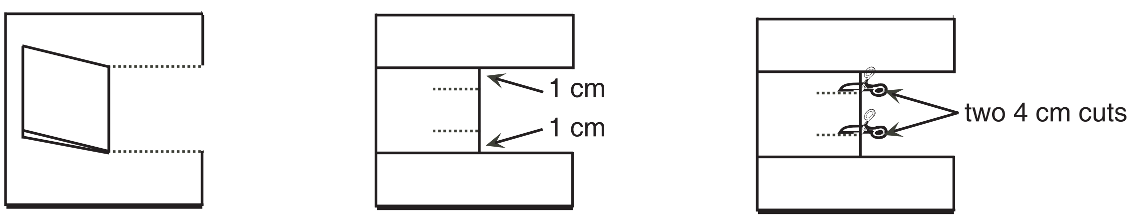


Fold the cut pieces back

4. Fold the cut pieces back and forth. Now, open the first fold and lay the paper flat. Then start folding on the first fold, pushing the inner piece of the card so it folds in the opposite direction.



5. Follow the instructions below for the next step. When you cut as shown you will be cutting through four layers of paper.

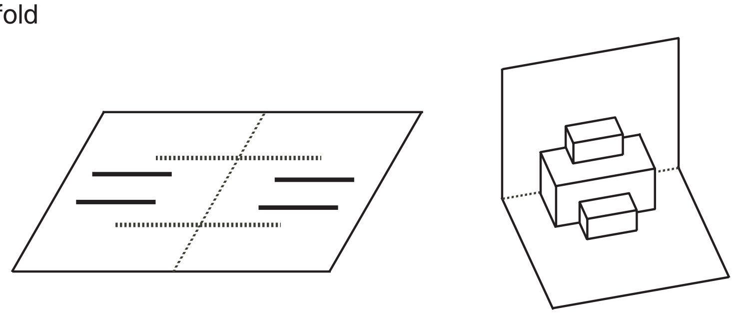


Fold the cut pieces back

Measure 1 cm in from each end of the new fold

Cut two 2 cm lines as shown

6. Fold the new cut pieces back and forth. Unfold and lay the paper flat. Now fold up the left half, pushing the cut pieces out, as shown.

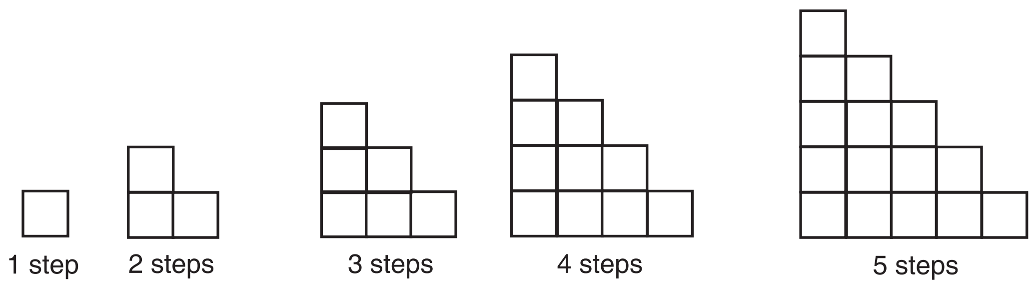


7. You are now at Stage 2 of the fractal. To complete Stage 3, repeat 5 and 6 starting with the last fold you made. The cuts you make will be through 8 layers of paper. In Stage 3, when you open your paper you should be able to see 7 'steps': 1 large one, two middle-sized ones, and 4 small ones.

8. Do you think you could expand this fractal to Stage 4? Why or why not?

BLM 11: A Mayan Staircase

Mayan temples, built in what is now Mexico, often had extended staircases made of stone. The diagrams below show how many large blocks of stone were needed for small staircases. Use this information to determine the number of blocks needed to make a staircase 100 steps high.



1. Record the given data in the table below. Try to predict the number of blocks needed for 6, 7, and 8 steps. Then draw the staircases and check your predictions.

Number of steps:	1	2	3	4	5	6	7	8
Number of blocks:	1	3	6	__	__	__	__	__

2. Predict the number of blocks needed for a 20-step staircase. Explain how you did this.

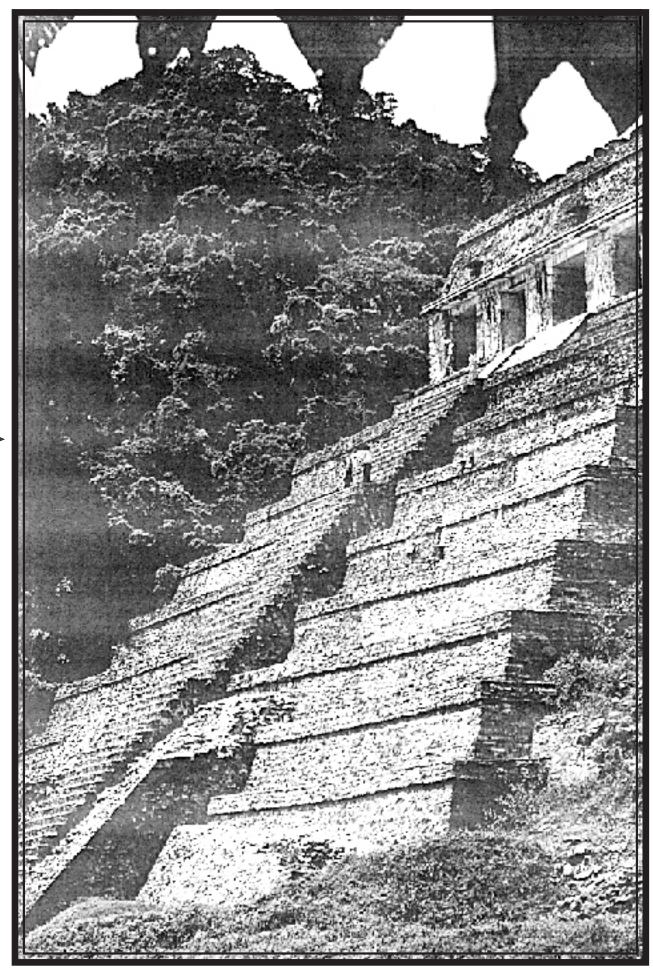
3. One of the following algebraic expressions tells how many blocks will be needed for n steps. Which expression is the correct one? How do you know?

(a) n^2

(b) $n^2 + n$

(c) $\left(n^2 + n\right) \div 2$

→



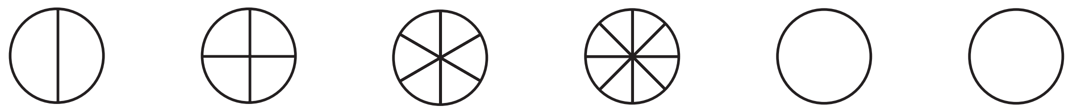
4. How many blocks are needed for a 100-step staircase?

A Challenge:

The picture shows a pyramid in Palenque with a long staircase and a temple at the top. Palenque is now part of Chiapas state in Mexico. The arrow on the left of the picture points to two people on the staircase. Assuming they are of average size, estimate the height of the pyramid.

BLM 12: Pete's Pizza Parlour 1

1. Pete of Pete's Pizza Parlour knows how to calculate the number of cuts needed to produce a given number of pieces. The diagrams below show how his pizzas are cut.



1 cut   2 cuts   3 cuts   4 cuts   5 cuts   6 cuts

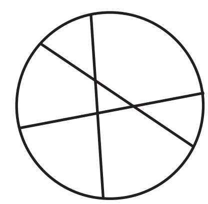
(a) Complete the table, predicting the number of pieces made by 5, 6, 7, or 8 cuts.

Number of cuts	1	2	3	4	5	6	7	8
Number of pieces	2	4						

(b) Write an algebraic expression to show how many pieces Pete will get with 'n' cuts.

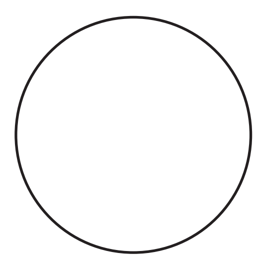
(c) If Pete needs 50 pieces from one pizza, how many cuts will he have to make?

(d) Sometimes Pete and one of his favourite customers play jokes on each other. The other day, Chris asked for 7 pieces and Pete cut the pizza as shown. Are there 7 pieces? How many cuts did Pete make?



(e) Complete this table, showing the maximum number of pieces Pete will get for the number of cuts given. The cuts do not need to go through the centre.

Number of cuts	1	2	3	4	5	6	7	8
Maximum number of pieces	2	4	7					



A Challenge:

(f) Pete thought that the maximum number of pieces he could get for n cuts would be $n^2 - 2$. Chris thought the expression should be $(n^2 + n + 2) \div 2$. Who was correct? How do you know?

BLM 13: Pete's Pizza Parlour, 2 and 3

2. One day Pat noticed that one of the tables in Pete's Pizza Parlour was wobbly. Pete said to put a piece of card underneath one leg. The table was still wobbly, so Pete said to fold the card in half and put the two layers underneath the table leg. It was still wobbly. After the card was folded again and put under the table leg, the table no longer wobbled.

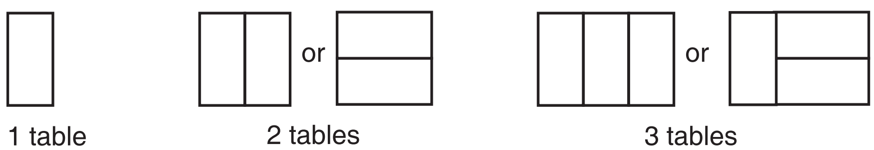
   (a) How many layers of card were needed to fix the table leg? How do you know?

   (b) Suppose Pat had to fold the card 8 times. How many layers of the card would there be? How do you know? If this were written as 2^n, what would be the value of n?

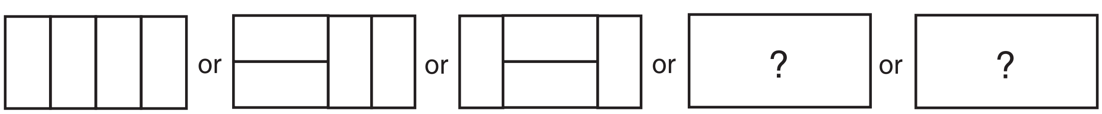
   (c) How many folds would be needed to make 2^{10} layers? 2^{50} layers? Do you think it would be possible to fold any piece of card or paper that many times? Explain your answer.

   (d) How many folds would you have if you folded the card n times?

3. Pete has several tables available for banquets or parties. He pushes the tables together, as shown below. Each table is twice as long as it is wide.



   (a) How many different arrangements of 4 tables could Pete make?



   (b) Complete the following chart:

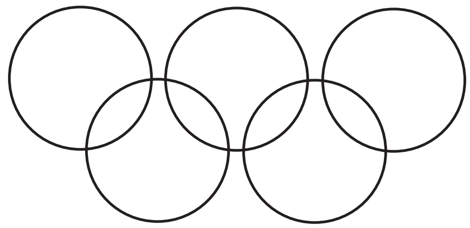
Number of tables	1	2	3	4	5	6
Number of arrangements	1	2	3			

   (c) Look at the numbers in the second row of the chart. Where have you seen these numbers before? Use what you know about these numbers to predict the number of arrangements Pete could make with 10 tables.

BLM 14: Pete's Pizza Parlour, 4

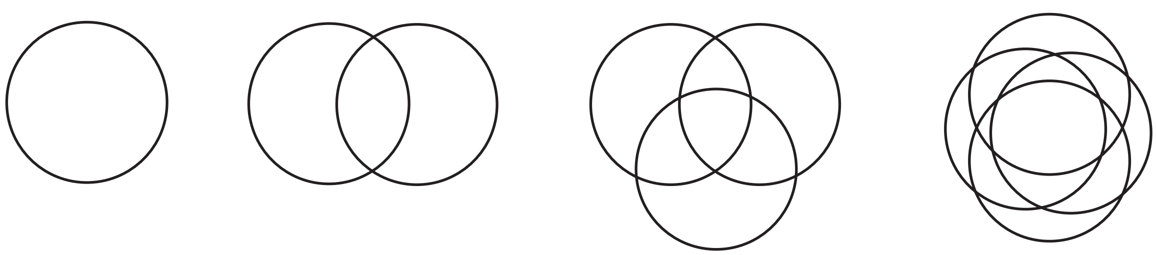
4. Pete hired Ari to help him design a new logo for his Pizza Parlour. Pete liked the Olympic rings and wanted to show one of his pizza topics in each different region.
   Ari drew the rings and counted the regions.

   (a) How many different toppings could Pete advertise if each topping was drawn in a different region?



   (b) Pete had 20 different toppings. How many rings would he need if
   (i) he wanted the pattern of the rings to be continued, and
   (ii) each region had one topping drawn in it?
   Explain how you got your answer.

   (c) Pete thought this would make the logo too big for the space he had, and he asked Ari to arrange the rings in a different way to get at least 20 regions with fewer circles. Ari drew some diagrams and started the following chart. What is the least number of rings that would would be needed for each of Pete's 20 toppings to be in a separate region?



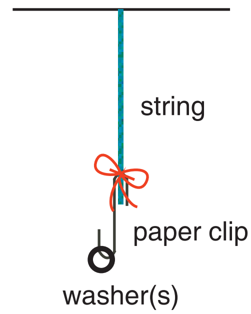
1 ring 2 rings 3 rings 4 rings

Number of rings	1	2	3	4	5	6
Number of regions	1	3	7	____	____	____

A Challenge:
Ari said that Pete could figure out how many rings would be needed for n toppings, but that the pattern was different from the pattern with the rings all in a row. Ari would not tell Pete the expression needed, but did give a hint. For n toppings, Pete would need $n^2 - \boxed{} + 1$ rings. Can you help Pete find the missing parts of the expression? How would you do this?

BLM 15: Paper Clips and Washers

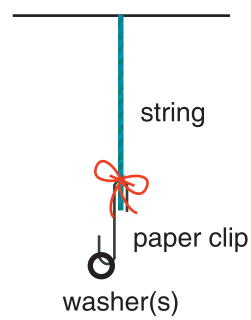
1. For this experiment you will need string, a paper clip, some washers all of one weight (or other weights that are all the same), a metre stick/tape, and a watch or clock with a second hand.
   (a) Construct the pendulum as shown on the right. The length of the pendulum should be 30 cm to start.
   (b) Gradually lengthen the pendulum and complete the table.



Length of pendulum (cm)	30	40	50	60	70	80	90
Number of cycles in 10 seconds							

   (c) Describe any patterns you see.
   (d) What happens if you make the pendulum shorter than 30 cm? How short can it be?

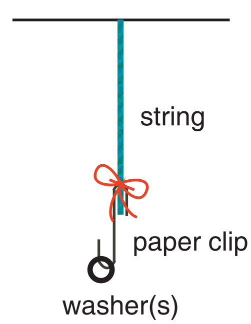
2. For this experiment you will need string, a paper clip, some washers all of one weight (or other weights that are all the same), a metre stick/tape, and a watch or clock with a second hand.
   (a) Construct the pendulum as shown on the right. The length of the pendulum should be 30 cm.
   (b) Add washers one at a time and complete the table.



Number of washers	1	2	3	4	5	6
Number of cycles in 10 seconds						

   (c) Describe any patterns you see.
   (d) What do you think would happen if you removed all the washers? Why? Check your prediction.

3. For this experiment you will need string, a paper clip, some washers all of one weight (or other weights that are all the same), a metre stick/tape, and a watch or clock with a second hand.
   (a) Construct the pendulum as shown on the right. The length of the pendulum should be about 30 cm to start.
   (b) Gradually increase the amplitude, and complete the table



Amplitude of swing (cm)	10	20	30	40	50
Number of cycles in 10 seconds					

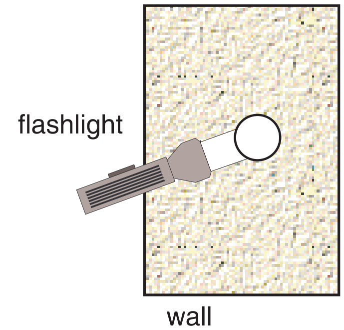
   (c) Describe any patterns you see.
   (d) What is the greatest amplitude possible? Explain.

BLM 16: Flashlights and Elastics

4. For this experiment you will need a small flashlight, a metre tape/stick, flat surface to shine the light on.

   (a) Shine the light against a flat surface.

   (b) Measure the distance of the flashlight from the wall and the diameter of the light on the wall. Record this in the table below.



   (c) Gradually move the light farther from the wall and complete the table.

Distance from wall (cm)						
Diameter of light (cm)						

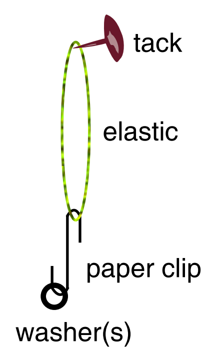
   (d) Describe any patterns you see.

   (e) Use your pattern to predict the diameter of the light when the flashlight is 10 m from the wall. Test your prediction. Explain your results.

5. For this experiment you will need an elastic, a paper clip, some washers all of one weight (or other weights that are all the same), a metre stick/tape.

   (a) Construct the pendulum as shown.

   (b) Measure the length of the elastic when there is one washer on the pendulum. Record this in the table.

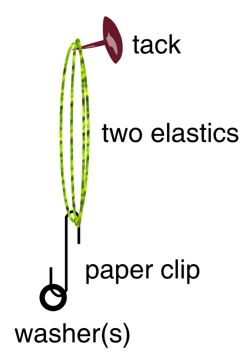


   (c) Add washers one at a time and complete the table.

Number of washers	1	2	3	4	5
Length of elastic (cm)					

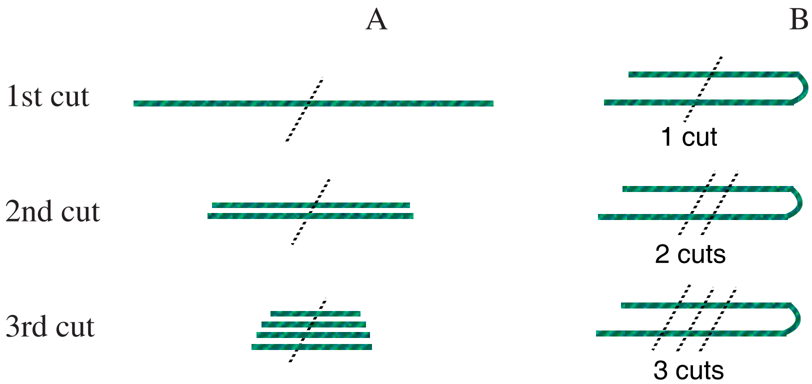
   (d) Describe any pattern you see.

   (e) What do you think would happen if you used 2 identical elastics? Check your prediction.



BLM 17: Pieces of String

Two ways of cutting pieces of string are shown below.



1. Record the number of pieces of string after each cut.

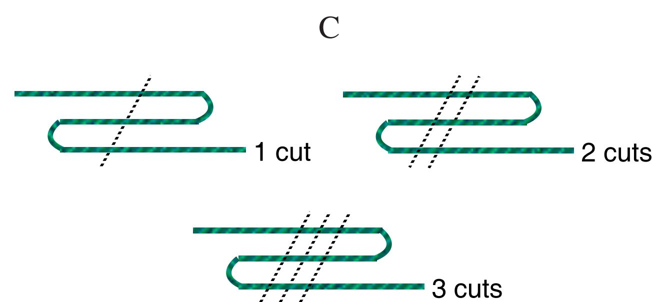
Number of Cuts	0	1	2	3	4	5	6	7	8	9	10
A											
B											

2. As soon as you can see a pattern in the table, predict what the next values will be. Check your prediction by cutting pieces of string.

3. Predict the number of pieces of string after 100 cuts, if that were possible. Explain why you think your predictions are correct.

4. Try to write an expression to show the number of pieces of string after *n* cuts.

5. **A Challenge:**
   Suppose the string were to be cut as shown on the right.
   What changes, if any, would this make in the table?
   Complete the table below.



Number of Cuts	0	1	2	3	4	5	6	7	8	9	10
C											

6. Compare the patterns for A, B, and C.

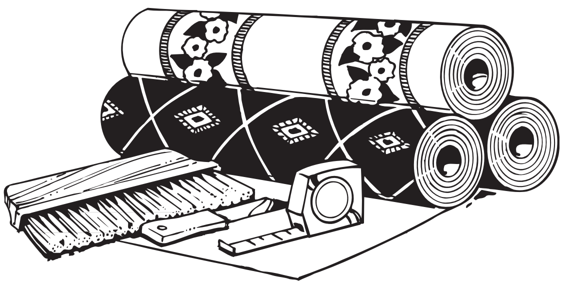
BLM 18: Patterns Everywhere

1. What is a pattern? Describe as many different kinds of patterns as you can.

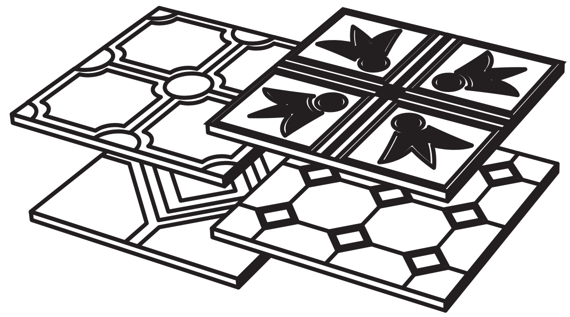
2. Check a dictionary or thesaurus for other meanings of "pattern". Record any types of patterns that are different from the ones you described in #1.

3. Which of the following could be called patterns? Explain.

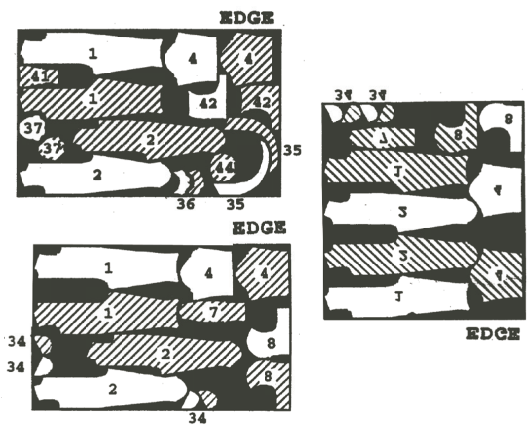
(a) a wallpaper design



(b) a tiled floor

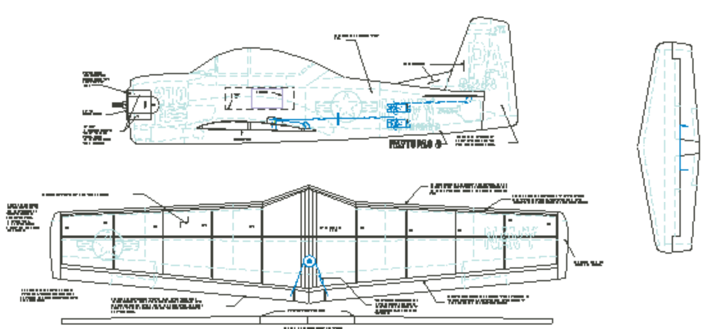


(c) a costume design



(d) written music

(e) instructions for building a model airplane



(f) a recipe

Rice Krispies[*] Squares

50 ml (1/4 cup) butter or margarine
1000 ml (4 cups) miniature marshmellows
2 ml (1/2 tsp.) vanilla extract
1250 ml (5 cups) KELLOGG'S[*] RICE KRISPIES[*]

1. Melt butter or margarine in a 4 L (3-quart) saucepan. Add marshmellows, and cook over low heat, stirring constantly, until marshmellows are melted and mixture is well blended. Remove from heat: stir in vanilla.
2. Add RICE KRISPIES[*] and stir until well coated with marshmellow mixture.
3. Press warm marshmellow treats mixture lightly into buttered 3 L (13 x 9 inch) pan. Cut into squares when cool.

YIELD: 24 - 5 cm (two-inch) squares.

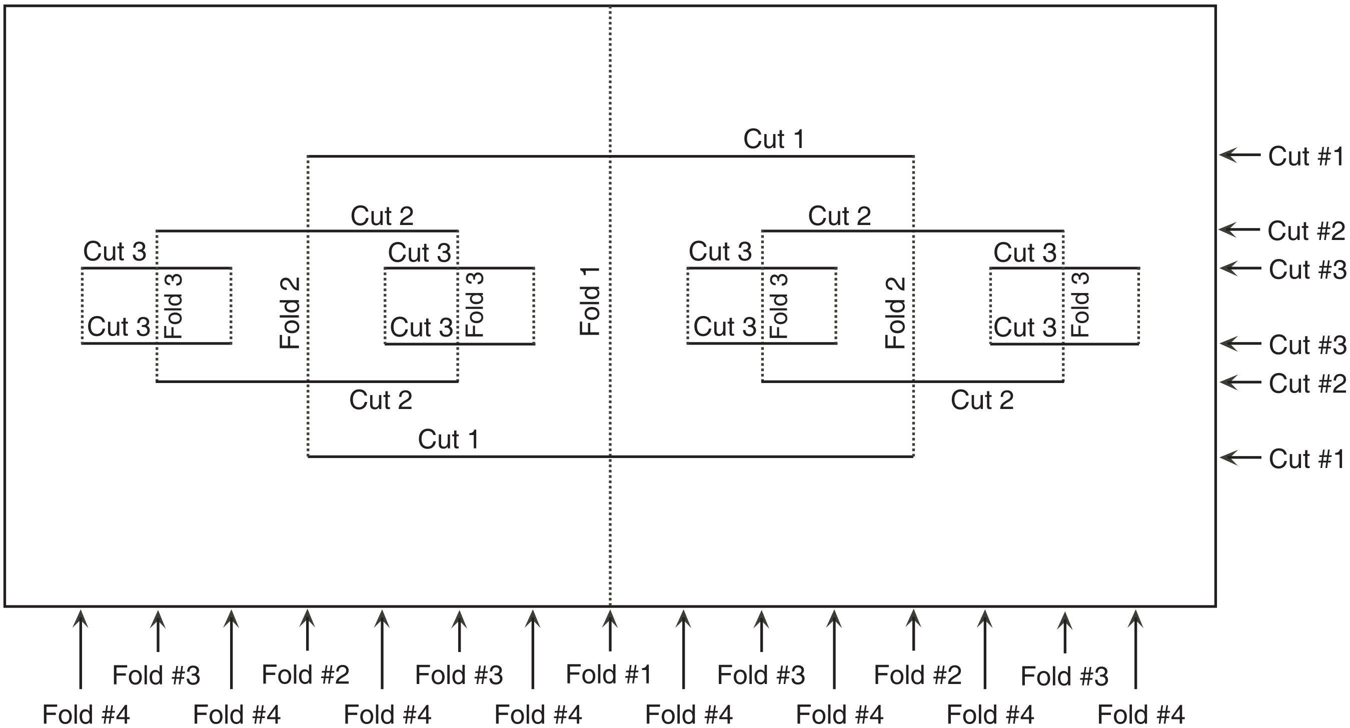
4. Give an example of a pattern in

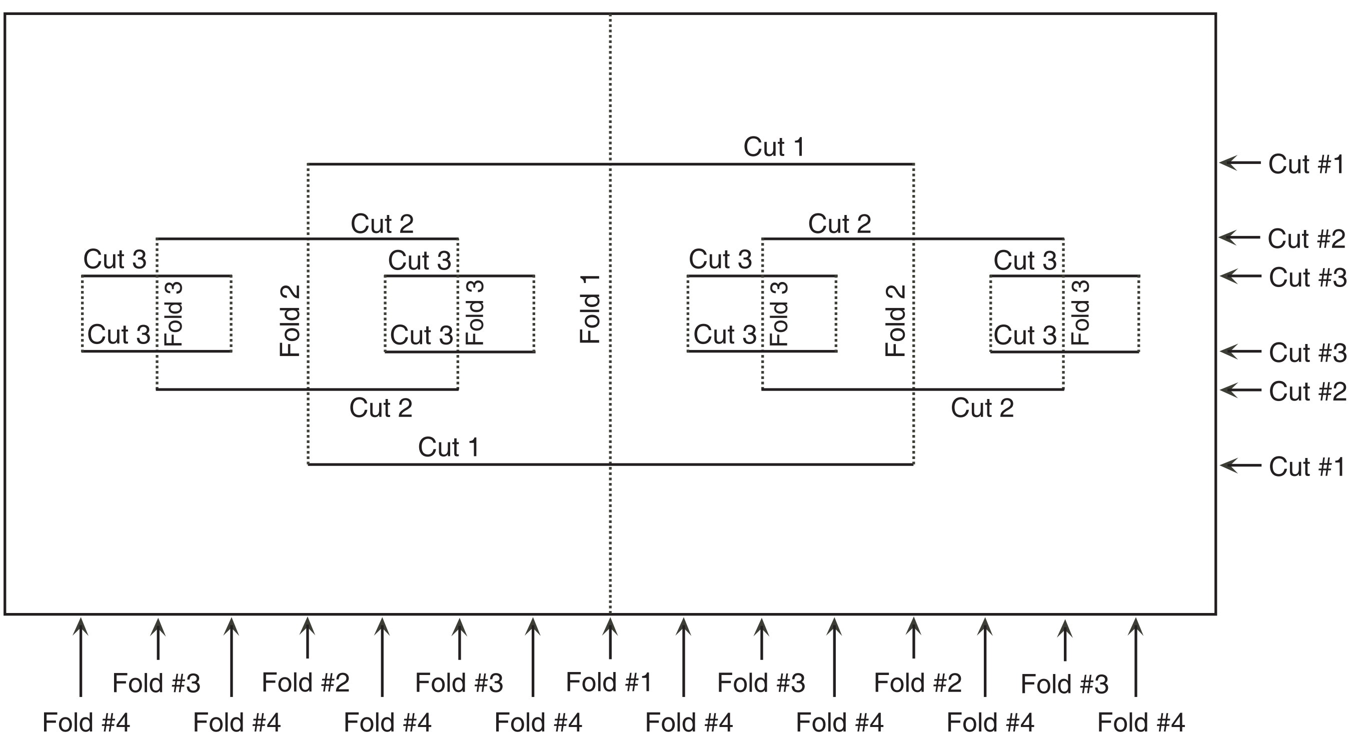
(a) History	(c) Science	(e) Music
(b) English	(d) Sport	(f) Art

5. Check the meanings of the following words in a dictionary, and tell whether each could be the name of a type of pattern. For those that could, give an example of that type of pattern.

(a) prototype	(c) exemplar	(e) standard
(b) original	(d) example	(f) repetend

BLM 19: 3D Fractal Template





Solutions & Notes

Activity 1: Patterns in Fibonacci Numbers

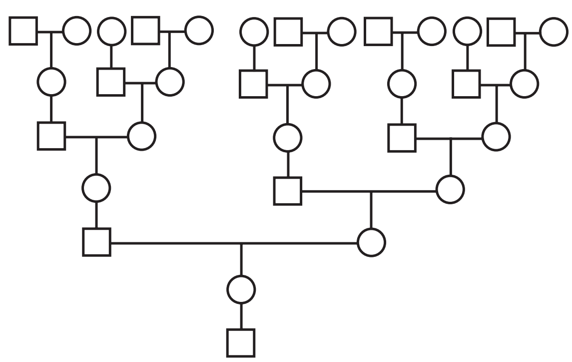
'Fibonacci' was the pen name of Leonardo, son of Bonacci, born in Pisa about 1175 A.D. During his youth he travelled with his father, a customs official, to many parts of the Mediterranean such as Greece, Turkey, Syria, Egypt, and Algeria. Thus he learned the Hindu-Arabic numerals on which our own numbers are based, and he realized how much easier they were to work with than the Roman Numerals which were then used throughout much of Europe. In 1202 he published a book, "Liber Abaci" ("Free from the Abacus") so called because when Roman Numerals were used, it was almost essential to have an abacus. The main message of his book was that the Hindu-Arabic system, which included a zero and place value, was much easier to use. He wrote many other books, but it is for "Liber Abaci" that he is principally remembered since this book was instrumental in convincing Europeans to use Hindu-Arabic numbers. It wasn't until the mid-1800s that the "Fibonacci Sequence" became known by that name, though Fibonacci himself had written about it many years earlier. One of his other contributions to mathematics was the use of a fraction line to separate numerator from denominator. It is believed he died between 1240 and 1250 A.D.

BLM 1: Fibonacci All Over

3. There should be 21 stems at Level 7.

4. Great, great, great, great grandparents →

Great, great, great grandparents →



Male	Female	Total
5	8	13

Number "Magic"

Suppose the first two numbers of the sequence are 'a' and 'b'. Then the column on the right shows consecutive numbers in the sequence up to the tenth term.

The seventh term is $5a+8b$.
Eleven times this is $55a+88b$, which is the sum of the first ten terms.

This will be true no matter what the values of 'a' and 'b' are – that is, no matter what the first two terms are.

a } add these two to get
add these two to get { b
$a+b$
$a+2b$
$2a+3b$
$3a+5b$
$5a+8b$ ← 7th term
$8a+13b$
$13a+21b$
$21a+34b$ ← 10th term
$55a+88b$ ← total of first 10 terms

Solutions & Notes

BLM 2: Fibonacci Patterns

1. Every 5th Fibonacci number (5, 55, 610, 6765, 75 025, …) is divisible by 5.
   Every 6th Fibonacci number (8, 44, 2584, 46 368, …) is not divisible by 6.
   Every 7th Fibonacci number (13, 377, 10 946, 317 811, …) is not divisible by 7.
   The pattern does not continue past division by 5.

2.

The Fibonacci numbers squared	The Sum	
1^2	1	1×1
1^2+1^2	2	1×2
$1^2+1^2+2^2$	6	2×3
$1^2+1^2+2^2+3^2$	15	3×5
$1^2+1^2+2^2+3^2+5^2$	40	5×8
$1^2+1^2+2^2+3^2+5^2+8^2$	104	8×13
$1^2+1^2+2^2+3^2+5^2+8^2+13^2$	273	13×21

The numbers in the third column are products of consecutive Fibonacci numbers.
Students may record products in the third column that do not use Fibonacci numbers.
If they can show that there is a pattern to their answers, this should be accepted.

3. (a) Since the numbers keep increasing in value, the first number of a pair (the numerator) will always be less than the second number (the denominator) and hence no fraction will be greater than 1.

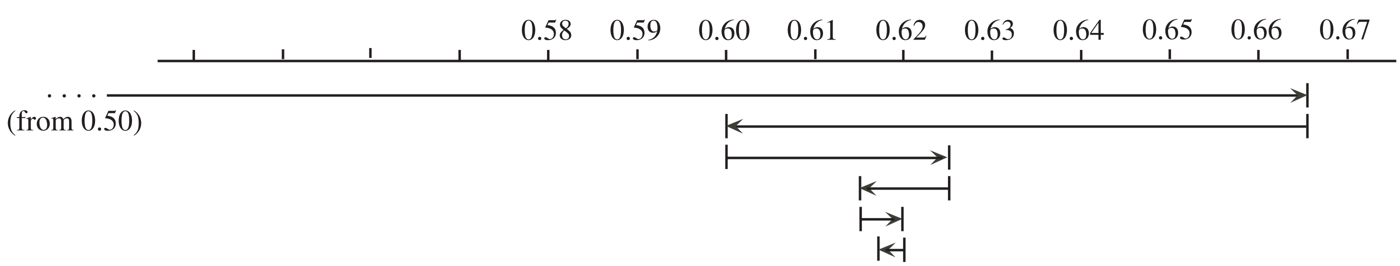
   (b) The fraction made up of the first pair of Fibonacci numbers will be $\frac{1}{1}$ or 1.

   (c)

fraction	$\frac{1}{1}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{5}$	$\frac{5}{8}$	$\frac{8}{13}$	$\frac{13}{21}$	$\frac{21}{34}$	$\frac{34}{55}$
decimal	1	0.5	0.667	0.6	0.625	0.615	0.619	0.618	0.618

   (d) The decimal equivalents alternately increase and decrease while getting closer and closer to 0.618.

   A way to picture this uses the number line. The first two decimals (1 and 0.5) are not shown.



This graph also shows that the differences between consecutive Fibonacci fractions decrease.

Solutions & Notes

BLM 3: Eleven Patterns

1. (a) $2\times11=22$ $5\times11=55$ $4\times11=44$
$9\times11=99$ $3\times11=33$ $7\times11=77$

(b) Students should see that the answer is composed of two identical digits.

(c) Answers will vary. Students should be encouraged to justify their predictions.

2. (a) $23\times11=253$ $34\times11=374$ $63\times11=693$
$71\times11=781$ $44\times11=484$ $90\times11=990$

(b) The tens digit of the number being multiplied by 11 becomes the hundreds digit of the answer.
The ones digit of the given number remains the ones digit of the answer.
The sum of the tens and ones digits becomes the tens digit of the answer.
Writing out the multiplication showing partial products illustrates why this is so.
For example, $\begin{array}{r} 34 \\ \underline{\times11} \\ 34 \\ \underline{34} \\ 374 \end{array}$ or $\begin{array}{r} 63 \\ \underline{\times11} \\ 63 \\ \underline{63} \\ 693 \end{array}$

(c) When the sum of the tens and ones digits of the given number is greater than ten, it appears that the pattern does not hold. However, if we explore this on a place-value chart, we can see that the pattern does hold, but we must do some regrouping as a final step.
For example, $\begin{array}{r} 84 \\ \underline{\times11} \\ 84 \\ \underline{84} \\ 924 \end{array}$ ←

100	10	1
8	12	4
9	2	4

regroup

Thus, the technique can still be used as a way of multiplying by 11 mentally.

3. (a) $27\times11=297$ $47\times11=517$ $65\times11=715$
$39\times11=429$ $73\times11=803$ $94\times11=1034$

(b)(c) See notes on regrouping for question 2(c). Notice that 94×11 involves regouping in two places.

Solutions & Notes

4. (a) This is a simple extension of the technique. Students should be encouraged to picture the partial products (or write them if necessary). For example,

for 123×11, a student should picture $\begin{array}{r} 123 \\ 123 \end{array}$ (as would occur in ordinary multiplication)

for 827×11, the mental picture should be $\begin{array}{r} 827 \\ 827 \end{array}$

$123 \times 11 = 1353$ $241 \times 11 = 2651$ $304 \times 11 = 3344$
$352 \times 11 = 3872$ $579 \times 11 = 6369$ $827 \times 11 = 9097$

5. $1234 \times 11 = 13\,574$ $2431 \times 11 = 26\,741$ $3041 \times 11 = 33\,451$
$3252 \times 11 = 35\,772$ $5779 \times 11 = 63\,569$ $2872 \times 11 = 31\,592$

A short-cut for checking with the calculator involves using an "automatic constant" that is available for most four-function calculators, and allows repeated multiplication by the same number. To test yours enter $\boxed{5}\boxed{\times}\boxed{2}\boxed{=}\boxed{=}\boxed{=}$. If the display shows, consecutively, 10, 50, 250 then your calculator has this automatic constant and each time you press $\boxed{=}$ your calculator multiplies the number in the display by 5.

To use this to check answers for #5 for example, enter one of the problems, entering 11 first so that it becomes the automatic constant.

Enter $\boxed{1}\boxed{1}\boxed{\times}\boxed{1}\boxed{2}\boxed{3}\boxed{4}\boxed{=}$ and read the answer from the display.

Then, ***without clearing the calculator or entering 11 again***, just enter $\boxed{2}\boxed{4}\boxed{3}\boxed{1}\boxed{=}$ and the answer to 2431×11 will be shown.

Enter $\boxed{3}\boxed{0}\boxed{4}\boxed{1}\boxed{=}$ and the answer to 3041×11 will be shown.
Continue till you have completed all the questions.

Activity 2: Decimal Patterns

BLM 4: Decimal Patterns

Answers are as seen for an 8-digit calculator display.

1. $1 \div 9 = 0.1111111$ $2 \div 9 = 0.2222222$ $3 \div 9 = 0.3333333$

3. $4 \div 9 = 0.4444444$ $5 \div 9 = 0.5555555$ $6 \div 9 = 0.6666666$

$7 \div 9 = 0.7777777$ $8 \div 9 = 0.8888888$ $9 \div 9 = 1.$

Solutions & Notes

5. A 10-digit display would simply extend the repeated digits.

6. $1 \div 11 = 0.090909$ $2 \div 11 = 0.1818181$ $3 \div 11 = 0.2727272$

   Ask them why the display for $1 \div 11$ shows only 7 digits. (The eighth digit would be zero and calculators do not generally show a 'final' zero in a decimal fraction)

8. $4 \div 11 = 0.3636363$ $5 \div 11 = 0.4545454$ $6 \div 11 = 0.5454545$

   $7 \div 11 = 0.6363636$ $8 \div 11 = 0.7272727$ $9 \div 11 = 0.8181818$

   $10 \div 11 = 0.9090909$ $11 \div 11 = 1$

   If students do not spot the 'pairing' of certain values, ask them how the decimal values for $2 \div 11$ and $9 \div 11$ are alike and how they are different. Ask them to identify similarly related pairs (e.g., $4 \div 11$ and $7 \div 11$, $5 \div 11$ and $6 \div 11$; $3 \div 11$ and $8 \div 11$).

   If your calculator has an automatic constant for division, this will show in the following sequence:
   $\boxed{2}\boxed{4}\boxed{\div}\boxed{2}\boxed{=}\boxed{=}\boxed{=}$
   If your calculator shows, successiely 12, 6, and 3, then it is continuing to divide by 2.
   To use this capability to check answers for # 8, enter $\boxed{4}\boxed{\div}\boxed{1}\boxed{1}\boxed{=}^{*}\boxed{5}\boxed{=}^{*}\boxed{6}\boxed{=}^{*}$ and so on.
   Stop at '*' to read the display. Compare the automatic constant for multiplication, with the one for division.
   If $a \times b$ is entered, then 'a' becomes the automatic constant for multiplication.
   If $a \div b$ is entered, then 'b' becomes the automatic constant for division.

   If you are curious about automatic constants, read on.
   Most four-function calculators have automatic constants for addition and subtraction as well.
   If $a + b$ is entered, then 'b' becomes the automatic constant for addition.
   If $a - b$ is entered, then 'b' becomes the automatic constant for subtraction.
   Notice that, for addition, subtraction, and division, it is the number entered after the operation sign that becomes the constant. For multiplication, the first number entered becomes the constant. This makes sense if we think about the meaning of, for example, 6×2 as "<u>6 groups of</u> 2", whereas $6 \div 2$ is "6 <u>divided by 2</u>", $6 + 2$ is "6 <u>add 2</u>", and $6 - 2$ is "6 <u>take away 2</u>" or "6 <u>subtract 2</u>". The number that becomes the automatic constant is the one that is interpreted as 'belonging with' the operation.

BLM 5: Repeating or Not?

1. Division by 3, 6, and 7 will produce repeating decimals (except for $3 \div 6$).
   Division by 4, 5, and 8 will not.
   All possible divisions are given here for your convenience.

   (a) $1 \div 3 = 0.3333333$
   $2 \div 3 = 0.6666666$

   (b) $1 \div 4 = 0.25$
   $2 \div 4 = 0.50$
   $3 \div 4 = 0.75$

   (c) $1 \div 5 = 0.2$
   $2 \div 5 = 0.4$
   $3 \div 5 = 0.6$
   $4 \div 5 = 0.8$

Solutions & Notes

(d) $1 \div 6 = 0.6666666$
$2 \div 6 = 0.3333333$
$3 \div 6 = 0.5$
$4 \div 6 = 0.6666666$
$5 \div 6 = 0.8333333$

(e) $1 \div 7 = 0.1428571$
$2 \div 7 = 0.2857142$
$3 \div 7 = 0.4285714$
$4 \div 7 = 0.5714285$
$5 \div 7 = 0.7142857$
$6 \div 7 = 0.8571428$

(f) $1 \div 8 = 0.125$
$2 \div 8 = 0.25$
$3 \div 8 = 0.375$
$4 \div 8 = 0.5$
$5 \div 8 = 0.625$
$6 \div 8 = 0.75$
$7 \div 8 = 0.875$

Ask students why $3 \div 6$ is not a repeating decimal. They should see that $3 \div 6$ or $\frac{3}{6}$ is equivalent to $1 \div 2$ or $\frac{1}{2}$.

3. Since $\frac{3}{13}$ is $3 \times \frac{1}{13}$ and $\frac{1}{13}$ is a repeating decimal, it makes sense that $\frac{3}{13}$ would also be a repeating decimal. Similarly for $\frac{4}{13}$ and $\frac{10}{13}$. Note that if any of these could be reduced, that ***might*** mean they would not produce repeating decimals.
For example, $\frac{1}{6} = 0.16666\ldots$ a repeating decimal,
but $\frac{3}{6} = \frac{1}{2} = 0.5$ a terminating decimal.

4. Students should realize that, since the decimal fraction repeats indefinitely, the final number for $2 \times 0.16666\ldots$ will never actually be reached and therefore will not be '2'.

5. (b) The numbers in the period (the numbers that repeat) are the same for $1 \div 7$, $2 \div 7$, $3 \div 7$, …, $6 \div 7$ but in different orders. (See 1(e) above)

(c) Since the period is 6 digits long, an 8-digit display will show only one of the repeating digits a second time.

6. (a) Since division by 5 (see #1(c)) does not give repeating decimals, this statement is false.

(b) Since division by 6 (see #1(c)) sometimes gives repeating decimals, this statement is false.

From Activity notes: "All fractions with prime number denominators give repeating decimals".

Both 2 and 5 are prime numbers but $\frac{1}{2}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}$ and $\frac{4}{5}$ give non-repeating decimals. Therefore the statement is false.

Solutions & Notes

Extensions in Mathematics

BLM 6: Repeating and Repeating and Repeating

1. Every second number of $\frac{4}{9}$ or 0.363636 … is 6. Therefore the 20th digit will be 6 and so will the 100th, but the 199th (an odd number) will be 3.

2. (a) $1 \div 7 = 0.\overline{142857}$
Every 6th digit will be 7. Therefore the 18th digit will be 7.

(b) Since we know the 18th digit is 7, we go on two more digits to determine the 20th. The 20th digit is 4.

3. (a) $\frac{1}{33} = 1 \div 33 = 0.030303\ldots$ Every 2nd digit is 3, so the 100th digit is 3.

(b) $\frac{1}{37} = 1 \div 37 = 0.027027\ldots$ Every 3rd digit is 7, so the 99th digit is 7; then the 100th digit must be zero.

(c) $\frac{5}{41} = 5 \div 41 = 0.1219512\ldots$
$= 0.\overline{12195}$
Every 5th digit is 5, so the 100th digit is 5.

(d) $\frac{8}{27} = 8 \div 27 = 0.2962962\ldots$ Every 3rd digit is 6, so the 99th digit is 6; then the 100th digit must be 2.

(e) $\frac{1}{101} = 1 \div 101 = 0.0099009\ldots$
$= 0.\overline{0099}$ Every 4th digit is 9, so the 100th digit is 9.

(f) $\frac{10}{111} = 0.09009009\ldots$
$= 0.09\overline{009}$
This is an unusual repeating decimal. Since the repeating part is just '009', the 5th, 8th, 11th, 14th … digits will be 9.

One way to determine the 100th digit is to begin by 'deleting' the non-repeating zero and nine (e.g. 0.**09**009009 …) and trying to determine the 98th term of the repeating part. Then, every 3rd term will be 9 so the 99th term will be 9, and the 98th term will be zero.

Solutions & Notes

Family Activities

1. The solution below follows the strategies suggested.

$$\frac{1}{1\times2}=\frac{1}{2}$$

$$\frac{1}{1\times2}+\frac{1}{2\times3}=\frac{1}{2}+\frac{1}{6}=\frac{2}{3}$$

$$\frac{1}{1\times2}+\frac{1}{2\times3}+\frac{1}{3\times4}=\frac{1}{2}+\frac{1}{6}+\frac{1}{12}=\frac{3}{4}$$

$$\frac{1}{1\times2}+\frac{1}{2\times3}+\frac{1}{3\times4}+\frac{1}{4\times5}=\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}=\frac{4}{5}$$

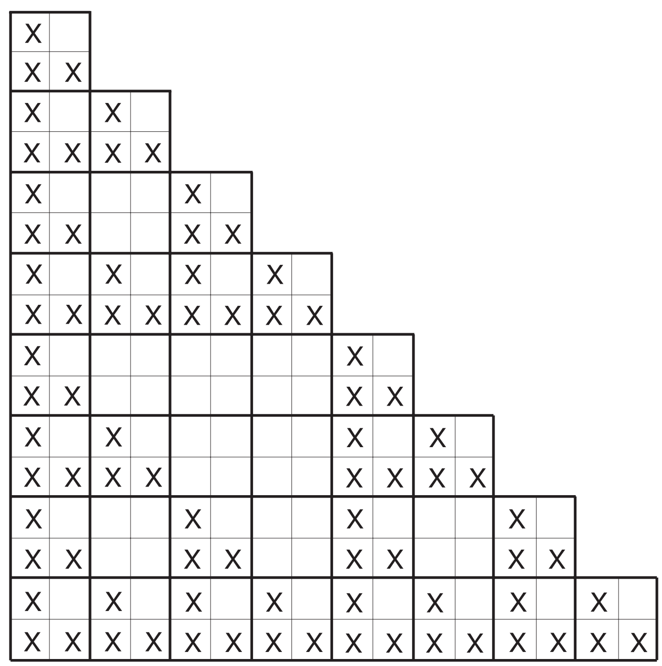
Compare the looped denominations with the answers.

This suggests that $\frac{1}{1\times2}+\frac{1}{2\times3}+\frac{1}{3\times4}+\ldots+\frac{1}{19\times20}=\frac{19}{20}$ which is, in fact, true.

Students may wish to check this, but the work could become onerous without a fraction calculator or a computer.

Activity 3: Fractals (Growing Patterns)

BLM 7: Growing Patterns



1. (a) Stage 4 is shown on the right.
Each X on Stage 3 has been replaced by

X	
X	X

(b)

Stage	1	2	3	4	5
Number of Crosses	3	9	27	81	243

(c) Each successive stage uses three times as many crosses as the preceding stage.
Thus the second row of the table could be written as

	3	3×3	$3\times3\times3$	$3\times3\times3\times3$	$3\times3\times3\times3\times3$
or	3^1	3^2	3^3	3^4	3^5

Solutions & Notes

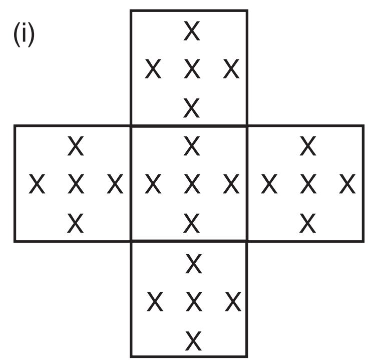
(d) The number of crosses needed for Stage 10 will be
$3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3$ or 3^{10}.
If students wish to multiply this out, they should be allowed to use calculators, or note that $3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3$ can also be written as $3^5 \times 3^5$ or 243×243 which can be multiplied with pencil and paper.

$3^{10} = 59\ 049$

Using the automatic constant for multiplication this can be calculated by entering
$\boxed{3}\boxed{\times}\boxed{=}\boxed{=}\boxed{=}\boxed{=}\boxed{=}\boxed{=}\boxed{=}\boxed{=}\boxed{=}$, using the $\boxed{=}$ key 9 times, since the first $\boxed{=}$ will give 9 or 3^2.
Students should be encouraged to look for visual patterns as well.

2. (a) Stage 2 of each figure is shown on the right. It is not necessary for students to draw any of the figures except to check, since the numerical patterns are very similar to the pattern in #1. For example, for (i) and (ii) each cross in Stage 1 will be replaced by 5 crosses in Stage 2, and each of these will be replaced by 5 crosses in Stage 3.

(i)

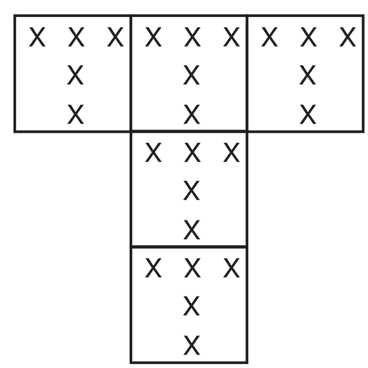


Stage 2
Each square
represents one cross
from Stage 1

(b)

Stage		1	2	3	4	5
Number Of Crosses	(i)	$5^1 = 5$	$5^2 = 25$	$5^3 = 125$	$5^4 = 625$	$5^5 = 3125$
	(ii)	$5^1 = 5$	$5^2 = 25$	$5^3 = 125$	$5^4 = 625$	$5^5 = 3125$
	(iii)	$7^1 = 7$	$7^2 = 49$	$7^3 = 343$	$7^4 = 2401$	$7^5 = 16\ 807$

(ii)

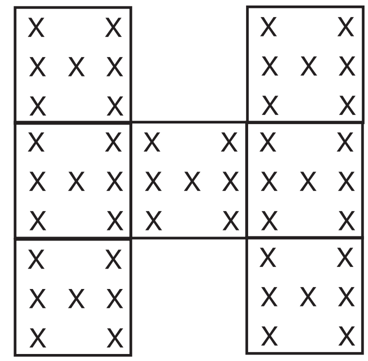


Stage 2
Each square
represents one cross
from Stage 1

Solutions & Notes

(c) Students should see the tables for #1 and #2 can be extended simply by multiplying the 'base number' by itself. ('Base number' is used here to denote the number of crosses in a figure at Stage 1)

(iii)



Stage 2
Each square represents one cross from Stage 1

BLM 9: The Sierpinski Triangle

1. (a)

Stage	1	2	3	4	5	10	100
Number of black triangles	3	9	27	81	243	59 049	3^{100}

These numbers should be identical with those in the table for #1, BLM 7.

Note: 3^{100} is a valid answer. Students who try to work this out with a calculator will find that a calculator with an 8-digit display will overload after 3^{16}.

However, students could be asked to estimate an answer:

3^{10} is about 60 000

3^{100} is $3^{10} \times 3^{10} \times 3^{10} \times 3^{10} \times 3^{10} \times 3^{10} \times 3^{10} \times 3^{10} \times 3^{10} \times 3^{10}$

or about $60\,000 \times 60\,000 \times 60\,000 \ldots\ldots$ (to ten such factors)

Ask students how many zeros would be on the end of the product [40].

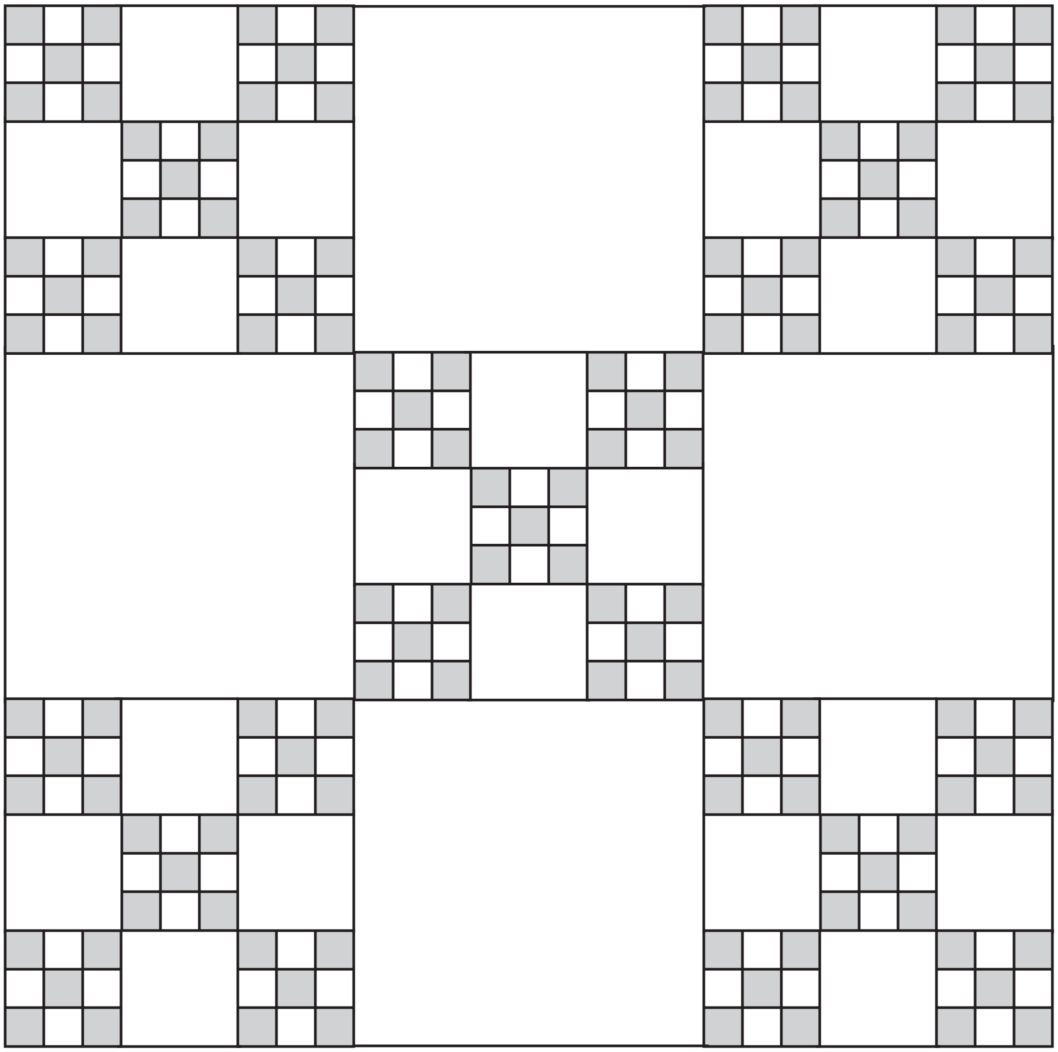
Have them calculate 6^{10} with their calculators [6 0 4 6 6 1 7 6].

Thus, an estimate for 3^{100} is 60,466,176 with 40 zeros or 60,000,000 with 40 more zeros or 6 followed by 47 zeros.

This should help students see why mathematicians use number patterns rather than trying to draw some fractals (even with a computer).

Solutions & Notes

2. (a) Stage 3 of given fractal



(b) The number of dark squares is multiplied by 5 each time.

Stage	1	2	3	4	5
Number of black squares	5	25	125	625	3125

These numbers should be identical with those in the table for #2(b) (i) and (ii), BLM 7.

BLM 10: A Three-dimensional Fractal

Typing paper is a good weight for this construction. It is easy to cut and fold but strong enough to hold its shape.

Solutions & Notes

Activity 4: Stairs and Pizzas

BLM 11: A Mayan Staircase

1.

Number of steps	1	2	3	4	5	6	7	8
Number of blocks	1	3	6	10	15	21	28	36

2. Extending the chart is one way to determine this:

Number of steps	9	10	11	12	13	14	15	16	17	18	19	20
Number of blocks	45	55	66	78	91	105	120	136	153	171	190	210

Another but similar way is to note that the number of blocks increases by one more each time:

Number of blocks $1 \xrightarrow{+2} 3 \xrightarrow{+3} 6 \xrightarrow{+4} 10 \xrightarrow{+5} 15 \xrightarrow{+6} 21$ and so on.

It is important for students to explain and justify their techniques.

3. The correct expression is (c) : $(n^2 + n) \div 2$.
Some students may have developed a similar ‘formula’ for #2 but may have expressed it in words rather than algebraic symbols.

For example,

“Multiply the number of steps by itself, add the number of steps, and divide by two.”
“Multiply the number of steps by itself and divide by two; then add half the number of steps.”

4. Using the expression from #3(c), calculation of the number of blocks needed for 100 steps is simple:

$$\begin{aligned}&(100^2 + 100) \div 2\\&= (10\,000 + 100) \div 2\\&= 10\,100 \div 2\\&= 5050\end{aligned}$$

A Challenge:

Answers will vary since it is not clear where the base of the pyramid is or whether the building on top is to be included. One strategy is to measure the height of the people (about 4 mm), the height of the pyramid (about 130 mm with the building) and estimate that the pyramid is about 32 times as tall as the people. This would be an acceptable answer. However, we could continue. If we assume the people are between 160 and 180 cm (about 5 feet 3 inches to 6 feet), then the pyramid, with building, is between 5120 cm and 5760 cm or 51.2 m and 57.6 m.

Solutions & Notes

BLM 12: Pete's Pizza Parlour 1

1. (a)

Number of cuts	1	2	3	4	5	6	7	8
Number of pieces	2	4	6	8	10	12	14	16

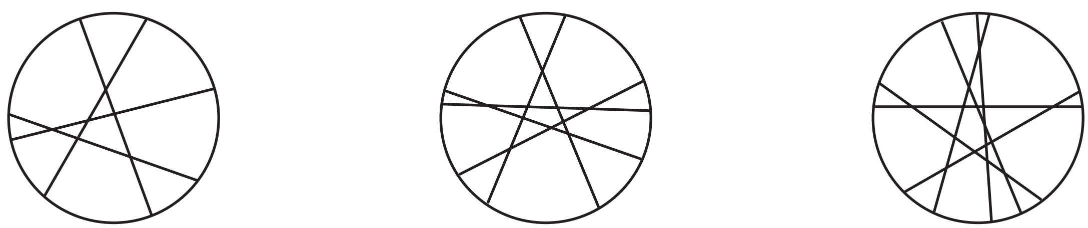
(b) n cuts will give $2 \times n$ or $2n$ pieces

(c) If Pete needs 50 pieces from a pizza, he will have to make 25 cuts.
Ask students how big this pizza would have to be in order that the pieces are a reasonable size. This will provide an opportunity to use measurement and estimation skills.

(d) Pete made 3 cuts

(e)

Number of cuts	1	2	3	4	5	6	7	8
Maximum number of pieces	2	4	7	11	16	22	29	37



4 cuts 5 cuts 6 cuts

The strategy in drawing these is to be sure each added cut crosses each earlier cut separately.

(f)

n	1	2	3	4	5	6
$n^2 - 2$	–1	2	7	14	23	34
$(n^2 + n + 2) \div 2$	2	4	7	11	16	22

The values for $n^2 - 2$ are correct for only 3 cuts, whereas the values for $(n^2 + n + 2) \div 2$ are correct for all values. Chris had the correct expression.

Solutions & Notes

BLM 13: Pete's Pizza Parlour, 2 and 3

2. (a) 4 layers of card were needed:

(b) A chart is helpful here:

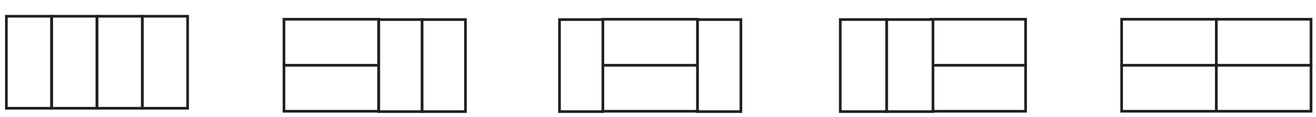
Number of folds	1	2	3	4	5	6	7	8
Number of layers	2	4	8	16	32	64	128	256
Number of layers could also be given as	2^1	2^2	2^3	2^4	2^5	2^6	2^7	2^8

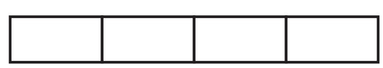
Ask students if they think it would be possible to fold a piece of card 8 times. Have them try this with a sheet of paper.

(c) An expression to give the number of layers is 2^n, where n is the number of folds.
That is, 2 folds give 2^2 or 2×2 or 4 layers
6 folds give 2^6 or $2\times2\times2\times2\times2\times2$ or 64 layers.
Thus, 10 folds would give 2^{10} layers;
50 folds would give 2^{50} layers.

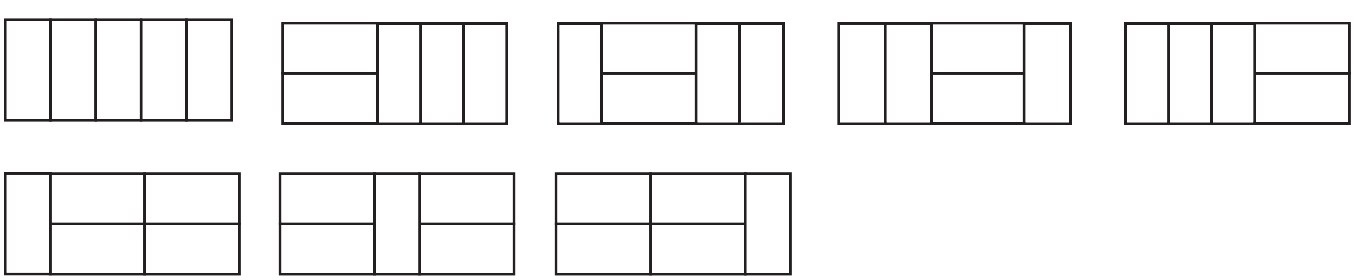
(d) If you folded the card n times, you would have 2^n layers.

3. (a) There are 5 arrangements with 4 tables.



This assumes that the arrangement is a long table whose width is double the width of a single table. If students begin adding other arrangements such as  they will not find it easy to detect a pattern in the table.

There are 8 possible arrangements with 5 tables:



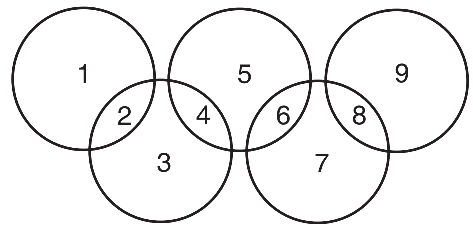
Solutions & Notes

(b)

Number of tables	1	2	3	4	5	6
Number of arrangements	1	2	3	5	8	13

(c) Students should recognize the numbers in the second row of the chart as Fibonacci numbers. Continuing the sequence, we find that Pete can make 89 different arrangements with 10 tables.

BLM 14: Pete's Pizza Parlour, 4



4. (a) Pete could advertise 9 toppings.

(b) One way to solve the problem is with a chart:

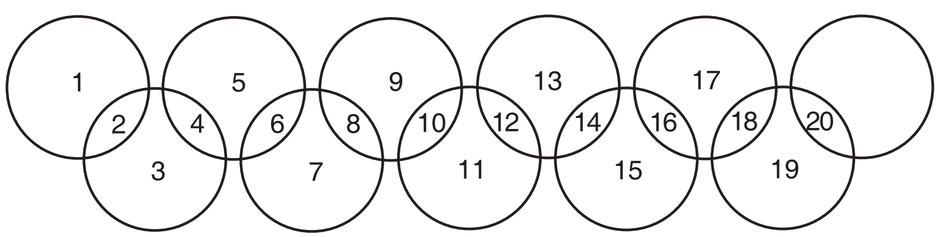
Number of rings	1	2	3	4	5
Number of toppings	1	3	5	7	9

Extend the table:

Number of rings	6	7	8	9	10	11
Number of toppings	11	13	15	17	19	21

Count the number of entries in the second row to determine the number of rings. Pete will need 11 rings for 20 toppings, and he will have one region left empty.

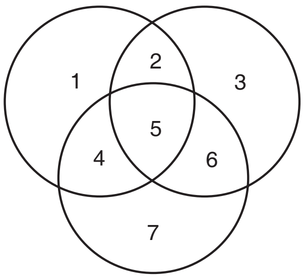
A second solution is to keep adding rings and counting regions:



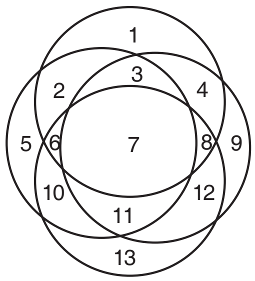
A third solution is to look for a pattern rule that will show how many regions there will be for any number of rings. One such pattern rule states that for n rings you will get $2n-1$ regions.

Solutions & Notes

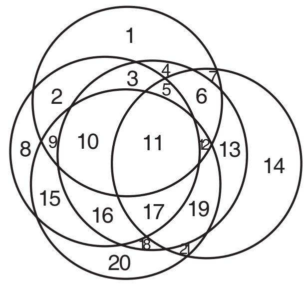
(c) To determine the fewest rings needed, each added ring should intersect with each of the preceding rings separately. However, students should be able to identify a pattern in the chart and extend it without any further drawings.



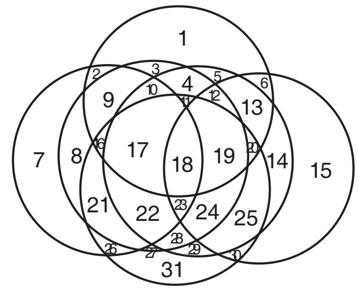
3 rings



4 rings



5 rings



6 rings

Number of rings	1	2	3	4	5	6
Number of regions	1	3	7	13	21	31
One pattern:	+2	+4	+6	+8	+10	

Add the next greatest even number each time.

Five rings would give Pete the 20 spaces he wants

A CHALLENGE:

The expression is $n^2 - n + 1$.

Activity 5:

BLM 15: Paper Clips and Washers

1. Students should discover that increased length leads to higher frequency (number of cycles per unit time – or, in this case, number of cycles in 10 seconds.)

2. Students should discover that the weight of the pendulum bob (the number of washers) does not affect the frequency.

3. Students should discover that amplitude does not affect the frequency of a pendulum.

BLM 16: Flashlights and Elastics

4. With most flashlights, the diameter of the light on the flat surface will increase as the flashlight is moved farther away. However, at some point the light will be so dispersed it will be difficult, it not impossible, to measure the projected light on the surface.

5. As washers are added, the elastic will stretch. In fact, the length of the elastic should increase the same amount for each added washer.

   (e) With two elastics, the stretch per washer will be less than for one elastic, but it should be the same for each washer.

Solutions & Notes

BLM 17: Pieces of String

1.

Number of cuts	0	1	2	3	4	5	6	7	8	9	10
A	1	2	4	8	16	32	64	128	256	512	1024
B	1	3	5	7	9	11	13	15	17	19	21

3. For A, after 1 cut there are 2^1 pieces; after 2 cuts there are 2^2 pieces; after 10 cuts there are 2^{10} pieces; after 100 cuts there would be 2^{100} pieces. The answer can be left in this form (i.e. 2^{100}) unless you wish students to estimate the value.

   For B, after 1 cut there are $2\times1+1$ pieces, after 2 cuts there are $2\times2+1$ pieces; after 10 cuts there are $2\times10+1$ pieces; after 100 cuts there would be $2\times100+1$ or 201 pieces.

4. For A, after n cuts there will be 2^n pieces.
   For B, after n cuts there will be $2\times n+1$ pieces.

5.

Number of cuts	0	1	2	3	4	5	6	7	8	9	10
C	1	4	7	10	13	16	19	22	25	28	31

   After n cuts, there will be $3\times n+1$.

6. Answers will vary.

Suggested Assessment Strategies

Investigations

Investigations involve explorations of mathematical questions that may be related to other subject areas. Investigations deal with problem posing as well as problem solving. Investigations give information about a student's ability to:

- identify and define a problem;
- make a plan;
- create and interpret strategies;
- collect and record needed information;
- organize information and look for patterns;
- persist, looking for more information if needed;
- discuss, review, revise, and explain results.

Journals

A journal is a personal, written expression of thoughts. Students express ideas and feelings, ask questions, draw diagrams and graphs, explain processes used in solving problems, report on investigations, and respond to open-ended questions. When students record their ideas in math journals, they often:

- formulate, organize, internalize, and evaluate concepts about mathematics;
- clarify their thinking about mathematical concepts, processes, or questions;
- identify their own strengths, weaknesses, and interests in mathematics;
- reflect on new learning about mathematics;
- use the language of mathematics to describe their learning.

Observations

Research has consistently shown that the most reliable method of evaluation is the ongoing, in-class observation of students by teachers. Students should be observed as they work individually and in groups. Systematic, ongoing observation gives information about students':

- attitudes towards mathematics;
- feelings about themselves as learners of mathematics;
- specific areas of strength and weakness;
- preferred learning styles;
- areas of interest;
- work habits — individual and collaborative;
- social development;
- development of mathematics language and concepts.

In order to ensure that the observations are focused and systematic, a teacher may use checklists, a set of questions, and/or a journal as a guide. Teachers should develop a realistic plan for observing students. Such a plan might include opportunities to:

- observe a small number of students each day;
- focus on one or two aspects of development at a time.

Suggested Assessment Strategies

Student Self-Assessment

Student self-assessment promotes the development of metacognitive ability (the ability to reflect critically on one's own reasoning). It also assists students to take ownership of their learning, and become independent thinkers. Self-assessment can be done following a co-operative activity or project using a questionnaire which asks how well the group worked together. Students can evaluate comments about their work samples or daily journal writing. Teachers can use student self-assessments to determine whether:

- there is change and growth in the student's attitudes, mathematics understanding, and achievement;
- a student's beliefs about his or her performance correspond to his/her actual performance;
- the student and the teacher have similar expectations and criteria for evaluation.

Resources for Assessment

"For additional ideas, see annotated Other Resources list on page 61, numbered as below."

1. The Ontario Curriculum Grades 1-8: Mathematics, Ministry of Education and Training, 1997.

2. "Linking Assessment and Instruction in Mathematics: Junior Grades" by OAME/OMCA, Crompton et al, 1996.
   The document provides a selection of open-ended problems tested in grades 4, 5, and 6. Performance Rubrics are used to assess student responses (which are included) at four different levels. Problems could be adapted for use at the Junior Level. Order from OAME/AOEM, P.O. Box 96, Rosseau, Ont., P0C 1J0. Phone/Fax 705-732-1990.

3. Mathematics Assessment: Myths, Models, Good Questions, and Practical Suggestions, by Jean Karr Stenmark (Ed.), NCTM, 1991.
   This book contains a variety of assessment techniques and gives samples of student work at different levels. Order from Frances Schatz, 56 Oxford Street, Kitchener, Ont., N2H 4R7. Phone 519-578-5948; Fax 519-578-5144. email: frances.schatz@sympatico.ca

4. "Assessment", Arithmetic Teacher Focus Issue, February 1992, NCTM.
   This copy of NCTM's journal for elementary school addresses several issues dealing with assessment. It also includes suggested techniques and student activities.

5. How to Evaluate Progress in Problem Solving, by Randall Charles et al., NCTM, 1987.
   Suggestions for holistic scoring of problem solutions include examples of student work. Also given are ways to vary the wording of problems to increase/decrease the challenge. A section on the use of multiple choice test items shows how these, when carefully worded, can be used to assess student work.

Suggested Assessment Strategies

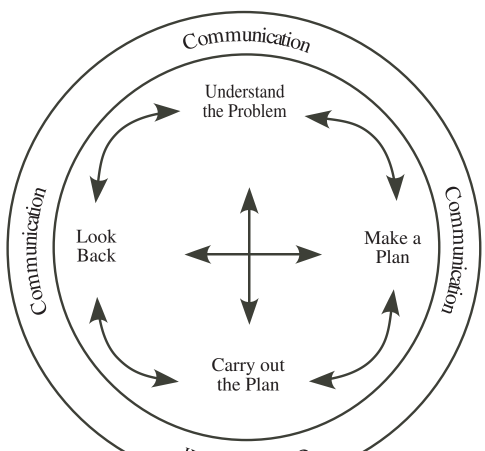
A GENERAL PROBLEM SOLVING RUBRIC

This problem solving rubric uses ideas taken from several sources. The relevant documents are listed at the end of this section.

"US and the 3 R's"

There are five criteria by which each response is judged:
Understanding of the problem,
Strategies chosen and used,
Reasoning during the process of solving the problem,
Reflection or looking back at both the solution and the solving, and
Relevance whereby the student shows how the problem may be applied to other problems, whether in mathematics, other subjects, or outside school.

Although these criteria can be described as if they were isolated from each other, in fact there are many overlaps. Just as communication skills of one sort or another occur during every step of problem solving, so also reflection does not occur only after the problem is solved, but at several points during the solution. Similarly, reasoning occurs from the selection and application of strategies through to the analysis of the final solution. We have tried to construct the chart to indicate some overlap of the various criteria (shaded areas), but, in fact, a great deal more overlap occurs than can be shown. The circular diagram that follows (from OAJE/OAME/OMCA "Linking Assessment and Instruction in Mathematics", page 4) should be kept in mind at all times.



There are four levels of response considered:

Level 1: Limited identifies students who are in need of much assistance;

Level 2: Acceptable identifies students who are beginning to understand what is meant by 'problem solving', and who are learning to think about their own thinking but frequently need reminders or hints during the process.

Level 3: Capable students may occasionally need assistance, but show more confidence and can work well alone or in a group.

Level 4: Proficient students exhibit or exceed all the positive attributes of the **Capable** student; these are the students who work independently and may pose other problems similar to the one given, and solve or attempt to solve these others.

Suggested Assessment Strategies

LEVEL OF RESPONSE →

CRITERIA FOR ASSESSMENT ↓

	Level 1: Limited	Level 2: Acceptable	Level 3: Capable	Level 4: Proficient
UNDERSTANDING	• requires teacher assistance to interpret the problem • fails to recognize all essential elements of the task	• shows partial understanding of the problem but may need assistance in clarifying	• shows a complete understanding of the problem	• shows a complete understanding of the problem
STRATEGIES	• needs assistance to choose an appropriate strategy	• identifies an appropriate strategy	• identifies an appropriate strategy	• identifies more than one appropriate strategy
	• applies strategies randomly or incorrectly • does not show clear understanding of a strategy[1] • shows no evidence of attempting other strategies	• attempts an appropriate strategy, but may not complete it correctly[2] • tries alternate strateges with prompting	• uses strategies effectively • may attempt an inappropriate strategy, but eventually discards it and tries another without prompting	• chooses and uses strategies effectively[3] • recognizes an inappropriate strategy quickly and attempts others without prompting
REASONING	• makes major mathematical errors • uses faulty reasoning and draws incorrect conclusions • may not complete a solution	• may present a solution that is partially incorrect	• produces a correct and complete solution, possibly with minor errors	• produces a correct and complete solution, and may offer alternative methods of solution
	• describes[4] reasoning in a disorganized fashion, even with assistance • has difficulty justifying[5] reasoning even with assisstance	• partially describes[4] a solution and/or reasoning or explains fully with assistance • justification[5] of solution may be inaccurate, incomplete or incorrect	• is able to describe[4] clearly the steps in reasoning; may need assistance with mathematical language • can justify[5] reasoning if asked; may need assistance with language	• explains reasoning in clear and coherent mathematical language • justifies[5] reasoning using appropriate mathematical language
REFLECTION	• shows no evidence of reflection or checking of work • can judge the reasonableness of a solution only with assistance	• shows little evidence of reflection or checking of work • is able to decide whether or not a result is reasonable when prompted to do so	• shows some evidence of reflection and checking of work • indicates whether the result is reasonable, but not necessarily why	• shows ample evidence of reflection and thorough checking of work • tells whether or not a result is reasonable, and why
RELEVANCE	• unable to identify similar[6] problems	• unable to identify similar[6] problems	• identifies similar[6] problems with prompting	• identifies similar[6] problems, and may even do so before solving the problem
	• unlikely to identify extensions[7] or applications of the mathematical ideas in the given problem, even with assistance	• recognizes extensions[7] or applications with prompting	• can suggest at least one extension[7], variation, or application of the given problem if asked	• suggests extensions[7], variation, or applications of the given problem independently

Suggested Assessment Strategies

Notes on the Rubric

1. For example, diagrams, if used, tend to be inaccurate and/or incorrectly used.

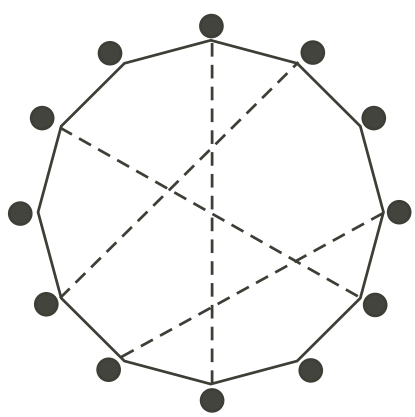
2. For example, diagrams or tables may be produced but not used in the solution.

3. For example, diagrams, if used, will be accurate models of the problem.

4. To *describe* a solution is to tell *what* was done.

5. To *justify* a solution is to tell *why* certain things were done.

6. *Similar* problems are those that have similar structures, mathematically, and hence could be solved using the same techniques.
   For example, of the three problems shown below right, the better problem solver will recognize the similarity in structure between Problems 1 and 3. One way to illustrate this is to show how both of these could be modelled with the same diagram:



Problem 1: There were 8 people at a party. If each person shook hands once with each other person, how many handshakes would there be? How many handshakes would there be with 12 people? With 50?

Problem 2: Luis invited 8 people to his party. He wanted to have 3 cookies for each person present. How many cookies did he need?

Problem 3: How many diagonals does a 12-sided polygon have?

   Each dot represents one of 12 people and each dotted line represents either a handshake between two people (Problem 1, second question) or a diagonal (Problem 3).

   The weaker problem solver is likely to suggest that Problems 1 and 2 are similar since both discuss parties and mention 8 people. In fact, these problems are alike only in the most superficial sense.

7. One type of extension or variation is a “what if...?” problem, such as “What if the question were reversed?”, “What if we had other data?”, “What if we were to show the data on a different type of graph?”.

Suggested Assessment Strategies

Suggested Adapted Rubric for Activity 4, BLMs 11, 12, 13, 14

The rubric below has been adapted for the problems on BLMs 11, 12, 13, 14 for Activity 4: Stairs and Pizzas. This rubric considers the understanding of the problem, the selection and application of strategies, reasoning, and reflection.

Level 1: Limited	Level 2: Acceptable	Level 3: Capable	Level 4: Proficient
• makes drawings that are not complete, or organized, or accurate	• makes drawings that are sometimes incomplete, or haphazard, or inaccurate	• makes neat organized drawings which may, occasionally, be incomplete	• makes neat, organized, accurate drawings
• fails to complete all tables or to complete them correctly; has difficulty drawing accurate diagrams	• completes most of the tables correctly, based on diagrams which may be incomplete	• completes the tables correctly, based on accurate drawings	• completes all tables correctly, based on accurate drawings or identification of a pattern
• tries to identify patterns from poor diagrams; may not see the inconsistencies, or recognize that the patterns will not help in solving the problems	• identifies patterns in tables that may have incorrect entries, and is not sure why such patterns are not useful in solving the problem	• if patterns derived from tables are inconsistent, can identify errors in tables with some assistance	• if patterns are inconsistent, can identify errors that may exist in tables or drawings, and adjust these to produce a pattern that can be used to solve the problem
• identifies patterns within the tables, but these may not be patterns that will help student predict the next consecutive value in a table	• identifies a pattern that allows student to predict the next consecutive value in the table	• identifies a pattern that allows student to predict, for example, the number of blocks needed for a 20-step staircase	• identifies a pattern that allows student to predict, for example, the number of blocks needed for a 20-step staircase
• has difficulty describing their results in any detail; can justify results only with assistance	• can describe their results, but may have difficulty justifying them	• can describe and justify their results; may need assistance with language	• can describe and justify their results using proper mathematical language

Other Resources

1. *Patterns, Addenda Series, K-6*, Miriam A Leiva (Ed.), 1993, NCTM
   Detailed notes for activities include patterns on a hundred chart, patterns built with manipulatives, patterns on the calculator, 'What's My Rule?' games, and tiling patterns. Three or more activities for each grade are given. Some Black Line Masters are included.

2. *Patterns and Functions, Addenda Series, Grades 5-8*, Frances R. Curcio (Ed.), 1991, NCTM
   This booklet contains problems for grades 5 to 8 dealing with various aspects of pattern and algebra. Topics include counting patterns, measurement and geometric patterns, patterns with fractions and repeating decimals, and the graphing of patterns.

3. *Multicultural Math Posters and Activities*, Wayne Peterson (Ed.), 1984, NCTM
   This package contains eighteen different 28 cm by 42 cm posters along with a book of suggestions for use. Some of the patterns explored are patterns in early number systems, Japanese and Arabic geometric patterns, Pascal's Triangle, and Russian Peasant multiplication (See Gr. 4 *Investigations in Pattern and Algebra*)

4. *Multicultural Mathematics Materials*, Marina C. Krause, 1983, NCTM
   This booklet contains Black Line Masters and Teacher Notes concerning patterns from several different areas, including Egypt, Japan, China, the Philippines, Norway, Peru, and ancient Rome, as well as several from the American Indians of south-west U.S.A.

5. *Fascinating Fibonaccis: Mystery and Magic in Numbers* by Trudi Hammel Garland, Dale Seymour Publications, 1987
   Many examples of the occurrence of Fibonacci Numbers are given here with delightful illustrations. These include the centres of sunflowers, the diatonic and other musical scales, the nautilus shell and starfish, architectural examples, and a BASIC program to generate Fibonacci fractions.

6. *Fractal Cuts*, Diego Uribe, 1995, Tarquin Publications
   Patterns for several three-dimensional fractals are given along with instructions.

7. *Pop-up Paper Engineering: Cross-Curricular Activities in Design Technology, English and Art* Paul Johnson, 1992, The Falmer Press (U.S.)
   Although the author seems to have neglected the tie to Mathematics and Pattern,these connections are certainly there. Simple patterns and sketches for making pop-up cards or books are given, along with examples designed and constructed by students ages 6 to 11. Suggestions for class pop-up story books are given.

8. "A Teacher's "Try" Angles", by Connie H. Yarema, Rhonda H. Adams, and Rachel Cagle, pp 299-302, *Teaching Children Mathematics*, January 2000, NCTM
   The article describes approaches to problems in an attempt to generalize them using algebraic notation. Patterns in the "handshake problem" (How many handshakes would there be among 8 people if each person shook each other's hand once?) and Pascal's Triangle are explored.

Other Resources

9. "Algebraic Thinking", Focus Issue of *Teaching Children Mathematics*, February 1977, NCTM
   Articles describe the use of spreadsheets to study patterns, explore ways students of different ages deal with a problem using square tiles to develop a pattern, illustrate the use of Logo and function machines, and suggest a way to introduce to students the use of variables (e.g., the letter 'x' or 'n' to represent a number).

10. "Algebraic Thinking: Opening the Gate", Focus Issue of *Mathematics Teaching in the Middle Grades*, February 1997, NCTM
    This volume contains articles on Exploring Patterns in Nonroutine Problems, Building Equations using M&Ms, and Exploring Algebraic Patterns through Literature (e.g., *Anno's Magic Seeds*)

11. "New Approaches to Algebra: Have We Missed the Point?" Stephen J. thornton, pp 388-392, *Mathematics Teaching in the Middle School*, March 2001
    This article illustrates how to use patterns developed from manupulatives to lead students to the use of a variable (e.g., *n*) to write generalizations. Different ways of writing the same generalization are shown. The importance of visualizing patterns is stressed.