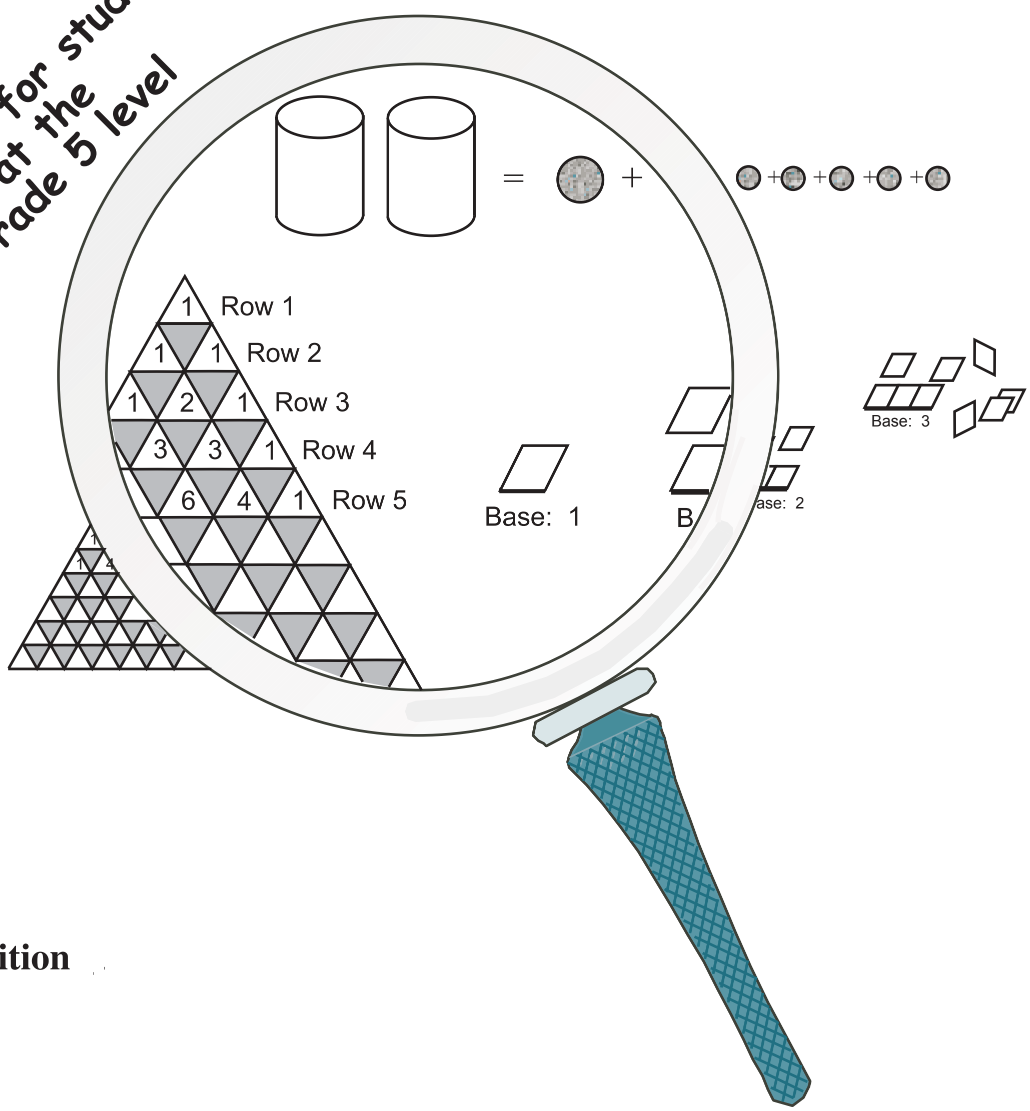
Invitations to Mathematics

Investigations in Patterns and Algebra

"Picture the Pattern"

Suggested for students at the Grade 5 level



3rd Edition

An activity of
The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
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Preface

The Centre for Education in Mathematics and Computing at the University of Waterloo is dedicated to the development of materials and workshops that promote effective learning and teaching of mathematics. This unit is part of a project designed to assist teachers of Grades 4, 5, and 6 in stimulating interest, competence, and pleasure in mathematics among their students. While the activities are appropriate for either individual or group work, the latter is a particular focus of this effort. Students will be engaged in collaborative activities which will allow them to construct their own meanings and understanding. This emphasis, plus the extensions and related activities included with individual activities/projects, provide ample scope for all students' interests and ability levels. Related "Family Activities" can be used to involve the students' parents/care givers.

Each unit consists of a sequence of activities intended to occupy about one week of daily classes; however, teachers may choose to take extra time to explore the activities and extensions in more depth. The units have been designed for specific grades, but need not be so restricted. Activities are related to the Ontario Curriculum but are easily adaptable to other locales.

"Investigations in Patterns and Algebra" is comprised of activities which explore numeric and geometric patterns in both mathematical and everyday settings, including, for example, architectural structures, biology, and experimental data. The recognition, description, and extension of patterns is arguably one of the most fundamental skills needed in mathematics, or for any problem solving situation; the activities in this unit are aimed specifically at developing this skill.

Acknowledgements

Contributing Teachers

Nancy Dykstra (Waterloo County Board of Education)
Kelly Lantink (Waterloo County Board of Education)
Ron Sauer (Waterloo County Board of Education - retired)
Barry Scully (York Region Board of Education/York University)
Jan Scully (York Region Board of Education/OISE/University of Toronto)
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We wish to acknowledge the support of the **Centre for Education in Mathematics and Computing,** and in particular the efforts of Ron Scoins, Gord Nichols, and Carolyn Jackson. A special thank you goes to Bonnie Findlay for prompt, accurate type-setting and creative diagrams.

Table of Contents

Overview

Notes

Common Beliefs

These activities have been developed within the context of certain beliefs and values about mathematics generally, and pattern and algebra specifically. Some of these beliefs are described below.

Mathematics is the science of patterns. Recognizing, describing and generalizing patterns are thus key to developing students' understanding of mathematics, and also provide powerful tools for problem solving.

Patterns can be investigated through concrete materials, number tables, and experiments, and in a wide variety of contexts, such as music, art, natural science, and architecture. Patterns inherent in cultural artifacts have a relevance for students that helps them realize the importance of patterns, and motivates them to explore patterns further and to make connections between and among patterns. Verbalizing such patterns through descriptions, analysis, and predictions helps lead to the understanding necessary for generalizations expressed in the language of algebra.

As students attempt to justify their choices and patterns they develop a facility with, and understanding of, both the nature of proof and the language of mathematics.

Essential Content

The activities in this unit allow students to discover how patterns arise in a variety of mathematical and everyday contexts, and to establish the rules which govern them. In addition, there are Extensions in Mathematics, Cross-Curricular Activities and Family Activities. These may be used prior to or during the activity as well as following the activity. They are intended to suggest topics for extending the activity, assisting integration with other subjects, and involving the family in the learning process.

During this unit, the student will:

- construct geometric models of square numbers, identify and extend their patterns;
- use triangle dot paper to explore the patterns of other figurate numbers;
- recognize and describe algorithmic patterns in various arithmetic short-cuts;
- explore the patterns in Pascal's Triangle, and relate them to many areas of mathematics;
- examine concrete ways to visualize simple equations, and explore their solutions;
- identify, and extend patterns in a variety of contexts;
- justify pattern rules selected.

Overview

"Curriculum Expectations" are based on current Ontario curricula.

Curriculum Expectations

ACTIVITY	DESCRIPTION OF THE ACTIVITY	CURRICULUM EXPECTATIONS
Activity 1 **Square Numbers**	• constructing geometric models of square numbers • identifying and extending the pattern of square numbers	• identify, extend, and create patterns in a variety of contexts • analyse and discuss patterning rules • identify, extend, and create patterns that identify changes in terms of two variables • analyse number patterns and state the rule for any relationships
Activity 2 **Figurate Numbers**	• identifying, analyzing, and describing rules for number patterns • identifying similarities and differences among number patterns	• recognize and discuss the mathematical relationships between and among patterns • identify, extend, and create patterns that identify changes in terms of two variables • discuss and defend the choice of a pattern rule • use patterns in a table of values to make predictions
Activity 3 **Arithmetic Short-cuts**	• recognizing and describing algorithmic patterns • identifying number properties that may affect a pattern (e.g., products of 5 will end in 5 or 0)	• analyse and discuss patterning rules • identify, extend, and create patterns that identify changes in terms of two variables • use a calculator to explore pattern • pose and solve problems by applying a patterning strategy • discuss and defend the choice of a pattern rule
Activity 4 **Pascal's Triangle**	• exploring patterns in a number array • relating these patterns to other areas of mathematics such as probability	• identify, extend, and create patterns in a variety of contexts • recognize the relationship between the position of a number and its value • identify and extend patterns to solve problems in meaningful contexts • analyse number patterns and state the rule for any relationships • discuss and defend the choice of a pattern rule
Activity 5 **Tin Cans and Equations**	• using a physical model for equations • solving simple equations using a	• apply patterning strategies to problem-solving situations • pose and solve problems by applying a patterning strategy • determine the value of a missing factor in equations involving multiplication, with and without the use of calculator

Overview

Prerequisites

Aside from basic computation skills, these activities require little previous knowledge.

Logos

The following logos, which are located in the margins, identify segments related to, respectively:

Problem Solving Communication Assessment Use of Technology

Materials

ACTIVITY	MATERIALS
Activity 1 Square Numbers	• Copies of BLMs 1 and 2 for each student or group • Copies of BLMs 3-8 and scissors (optional) • Copies of BLMs 9 and 10 (optional)
Activity 2 Figurate Numbers	• Copies of BLMs 11 and 12 for each student or group • Copies of BLMs 13 and 14 (optional) • Acetate copies of BLMs 11, 12, 13 for overhead projector
Activity 3 Arithmetic Short-cuts	• Copies of BLMs 15, 16, 17 (optional)
Activity 4 Pascal's Triangle	• Copies of BLMs 18 and 19 for each student • Acetate copy of BLM 18 for overhead projector • Copies of BLMs 20 and 21 (optional)
Activity 5 Tin Cans and Equations	• Copies of BLMs 22 and 23 for each student • Small tins (or paper cups) and counters (or paperclips) • Copies of BLM 24 (optional)

Letter to Parents

SCHOOL LETTERHEAD

DATE

Dear Parent(s)/Guardian(s):

For the next week or so, students in our classroom will be participating in a unit titled “Picture the Pattern”. The classroom activities will focus on patterns in square numbers (i.e. 1, 4, 9, ...), and other figurate numbers, patterns that provide short-cuts for arithmetic, Pascal’s Triangle, and a simple physical model for solving equations.

You can assist your child in understanding the relevant concepts by working together to perform simple experiments, and play games, and helping to locate everyday ways patterns are used.

Various family activities have been planned for use throughout this unit. Helping your child with the completion of these will enhance his/her understanding of the concepts involved.

If you work with patterns in your daily work or hobbies, please encourage your child to learn about this so that he/she can describe these activities to his/her classmates. If you would be willing to visit our classroom and share your experience with the class, please contact me.

Sincerely,

Teacher’s Signature

A Note to the Teacher:

If you make use of the suggested Family Activities, it is important to schedule class time for sharing and discussion of results.

Activity 1: Square Patterns

Focus of Activity:
- Constructing geometric models of square numbers
- Identifying and extending the pattern of square numbers

What to Assess:
- Accuracy in extending the patterns
- Validity of the rules identified
- Clarity of description of the pattern(s) identified

Preparation:
- Make copies of BLMs 1 and 2
- Make copies of BLMs 3 to 10 (optional)

Activity:

Present the following problem to students:

"How many squares does it take to make a larger square?"

Illustrate that 4 small squares (use Pattern Blocks or cut-outs from BLM 3) can be put together to make a 2 by 2 square. Ask students for the minimum number of the cut-out small squares they think it will take to make the next largest square. Ask one of students to demonstrate. Once they have decided that 9 squares will be needed, record the data for the first three squares (1 by 1, 2 by 2, 3 by 3) in a chart such as the following.

Excerpt from BLM 1:

Size of Square	1 by 1	2 by 2	3 by 3			
Number of small squares needed	1	4	9			

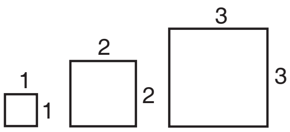
Have students use graph paper or cut-out squares (BLM 3) to determine the sizes of the next three larger squares. Remind students that they want the minimum number of cut-outs in each case.

The advantage of using either of these sources of squares means that students can keep a permanent record, either by drawing on graph paper or by pasting the paper squares together on a larger sheet.

Comments in italics are explanatory, and need not be conveyed to the students.

When students have completed the task, have individuals/pairs/groups illustrate their results to the class, explaining how they arrived at those conclusions.

Activity 1: Square Numbers



When students have all agreed that the next 3 squares are 4 by 4, 5 by 5, and 6 by 6, using, respectively, 16, 25, and 36 small squares, ask how many small squares they think will be needed for a 10 by 10 square. Give them time to consider the problem, then ask how they arrived at the answer.

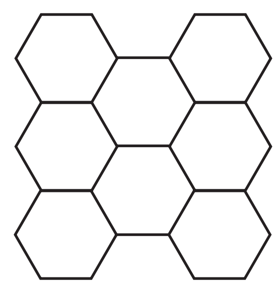
Tell students that the numbers 1, 4, 9, 16, ... are called Square Numbers, and each is the product of a number multiplied by itself. Have students list other square numbers (e.g., 7×7 or 49, 8×8 or 64). Ask students why they think such numbers are called square numbers. Show them how the dimensions and area of a square suggest this term.

Communication

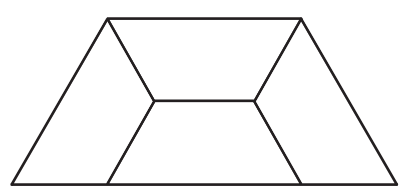
Distribute copies of BLMs 1 and 2, and indicate to students that they have already solved the first problem, and should record their answers. The other problems are similar and involve using other figures (e.g., equilateral triangle, isosceles right triangle) to make larger figures of the same type. Tell them that if their work is accurate there will be a pattern in the table to help them predict numbers for the larger figures, or to state that a pattern does not exist.

When students have completed most of the problems, discuss the results. Children may be surprised to find that the "square numbers" emerge in the tables for both types of triangle tested as well as for the rhombus.

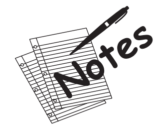
If students attempt #5 on BLM 2 they will find that the regular hexagon on BLM 7 cannot be fitted together to make a larger hexagon.



However, if they try the second challenge, they should find that 4 of the trapezoids from BLM 8 can be fitted together to make a larger, similar trapezoid as shown below. See "Solutions and Notes" for further diagrams.



Activity 1: Square Patterns



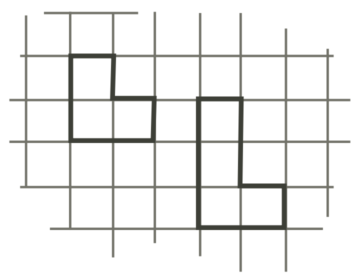
Extensions in Mathematics:

1. *Figures that have the same shape, but not necessarily the same size are said to be "similar". Thus all squares are similar, but not all triangles. The triangles made in problem 2 are all similar because they are all equilateral, but these triangles are not similar to those of problem 4. Although the rhombuses made in problem 3 are all similar, they are not similar to other rhombuses that have different angles.*
   Rectangles are similar to each other if the ratios of their sides are equal.

   Students could explore, using rectangles with a length to width ratio of 2 to 1, to see if these rectagles can be used to build other, similar, rectangles. Students should be able to construct a series of similar rectangles whose table will show the square numbers, as in problems 1 to 4.

Family Activities:

1. Use square dot paper to make non-regular hexagons. (These could be cut out if desired). Can you fit 4 of either of the hexagons on the right to make a larger, similar hexagon? Can you fit 9 of them together to make an even larger similar hexagon? If you can, show how. If you cannot, tell why not.
   Try to design another non-regular hexagon that can be used to illustrate square numbers.
   Have students report their experiences to the class.



non-regular hexagons

Problem Solving

Communication

Other Resources:

For additional ideas, see annotated Other Resources list on page 65, numbered as below.

1. Patterns, Addenda Series, Grades K-6.

Activity 2: Figurate Numbers

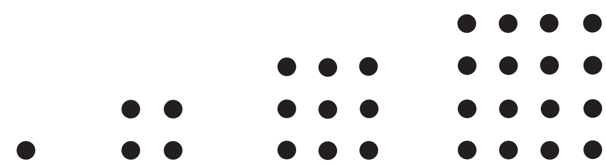
Focus of Activity:
- Identifying, analyzing, and describing rules for number patterns

What to Assess:
- Accuracy in extending patterns
- Clarity and validity of descriptions of the patterns

Preparation:
- Make copies of BLMs 11 and 12
- Make copies of BLM 13 (optional)
- Make copies of BLMs 11, 12, and 13 for use with the overhead projector
- Make copies of BLM 14 (optional)

Activity:

In Activity 1 students illustrated square numbers using various geometric figures. Another illustration uses dots as shown below:



Square numbers: 1 4 9 16

There are other "figurate" numbers that are also often illustrated with dots. Students will explore some of these in this activity.

Distribute copies of BLMs 11 and 12, and direct students to problem #1. Ask if they remember the square numbers from the previous activity. If so, they should write them in the table. Explain that numbers in a sequence are called terms, so the first square number would be the 1st term in that sequence.

Communication

Tell students that, in algebra, mathematicians often wish to refer, not to a specific number, but to "any number". The "*n*" in the last column refers to any term in the sequence. The *n*th square number is found by multiplying the *n*th number by itself, thus giving $n \times n$ as the *n*th square number.

You may wish to give each pair/group of students one or two of the problems on the BLMs, and have each pair/group explain to the rest of the class what they did.

Some students may be interested in attempting to illustrate the 5th figurate number for #4, 5, and 6 on BLM 12.

OPTIONAL:
Note that the problems on BLMs 11 and 12 do not ask students to determine the 100th term or the *n*th term in any sequence. The *n*th square number is given. Some students may wish to try to determine the others, but the expressions for the *n*th term can become rather complicated. (See "Solutions and Notes" for more on these.)

Activity 2: Figurate Numbers

Pursue these problems with the whole class, beginning by asking about the numbers they have already written in the chart. For example,

"Did you need to draw all the triangles to determine the 10th triangular number? Why or why not?"

"Did you use a pattern to determine the 10th pentagonal number? Describe your pattern."

There are two type of 'patterns' students may find in the tables. For triangular numbers, one such pattern (or 'rule' for forming the sequence) is "Add one more each time" (See Figure 2.1). This pattern can be called an "iterative" (or repeating) rule and lets students move from one triangular number to the next. Another rule, called a "functional" rule would allow students to answer questions such as, "What is the 100th term?" without writing down all 100 terms. Students should also be able to identify other patterns in the chart - for example, a pattern of whole numbers in the second column of the chart, or multiples of '3' in the third column.

Comments in italics are explanatory, and need not be conveyed to the students.

Triangular numbers: 1   3   6   10   15

+ 2   + 3   + 4   + 5

Students may say, "Add one more each time" when referring to the addends shown.

Square numbers: 1   4   9   16   25

+ 3   + 5   + 7   + 9

Students may say, "Add the next biggest odd number each time" for these addends.

Figure 2.1

Problem Solving

The first patterns students identify are usually iterative or repeating rules. As well, students will word the rule in various ways. It is worth discussing different iterative rules so that students can see that they are often saying the same thing as a classmate, but in different words. The functional rules are usually not as simple as the iterative rules, but students should be given the opportunity to explore these.

Ask students if they could determine a rule that would give the triangular (or pentagonal or hexagonal ...) number if its position in the sequence is known. For example,

"What is the 20th triangular number? Can you find it without writing down all 20 triangular numbers?"

Activity 2: Figurate Numbers

Extensions in Mathematics:

1. Give students the following pattern to complete; then ask if they recognize the sequence of numbers they have written.

$$
\begin{aligned}
1 &= 1\\
1 + 2 &=\\
1 + 2 + 3 &=\\
1+ 2 + 3 + 4 &=\\
1 + 2 + 3 + 4 + 5 &=
\end{aligned}
$$

   They should recognize the triangular numbers. Ask if they can find a similar pattern for the square numbers.

   Distribute copies of BLM 13. The chart illustrates that all the figurate numbers can be determined by adding, for example, consecutive whole numbers (which gives triangular numbers), or consecutive odd numbers (which gives square numbers).

   Students should identify a pattern in the addends for each column and complete the column to the 7th row. They should then examine the sequence of totals in each column and compare this with the figurate numbers they discovered from BLMs 11 and 12.

   The “Challenge” at the bottom of BLM 13 is one of those interesting connections that exist in mathematics. Students may be interested in looking for other such connections.

Cross-curricular Activities:

1. BLM 14 shows a number of ancient number systems from a variety of civilizations. Students could examine these for patterns, and compare them. For example,
   - the Chinese and Egyptian symbols begin the same way but then vary considerably.
   - the only one with a zero (besides our Arabic system) is the Mayan system.

   Ask students which they think would be easiest to work with, and which would be the hardest.
   If appropriate, have students talk to their parents about number names in different languages.

2. If students are studying French, have them compare the number names in English and French. For example, in English, once a student recognizes the pattern of “twenty, 1, 2, 3, ..., 9, thirty, 1, 2, 3, ..., 9, forty”, he/she can follow this pattern all the way to 99. However, in French, this pattern changes at 70, and again at 80.

Activity 2: Figurate Numbers

Other Resources:

For additional ideas, see annotated Other Resources list on page 65, numbered as below.

1. Patterns, Addenda Series, Grades K-6.
3. Multicultural Math Posters and Activities.
4. Multicultural Mathematics Materials.

Notes

Activity 3: Arithmetic Short-cuts

Focus of Activity:

- Recognizing and describing algorithmic patterns

What to Assess:

- Accuracy of addition
- Ability to extend the pattern

Preparation:

- Make copies of BLMs 15, 16, and 17 (optional)

Activity:

Ask students to calculate the sum $1+2+3+4+5+6+7+8+9+10$ as fast as they can. If students work in pairs, one can add while the other is timing the solution; then they can switch roles. Ask various students how long it took them, recording their times on the blackboard/chart paper/overhead. Have them estimate an average time.

Then ask them to estimate how long it would take, adding at the same speed, to add $1+2+3+4+5+...+98+99+100$. Stress that they are not to do the addition, simply estimate how long it would take them if they were to do the addition. If a student suggests using a calculator to do the addition, have the class time that student while he/she punches in all the numbers.

Comments in italics are explanatory, and need not be conveyed to the students.

Students may be surprised to discover that the calculator doesn't make the computation a great deal shorter, just easier. Point out to the students that most entries (the two-digit numbers) take 3 key strokes (2 digits, and '+').

Ask if any students can suggest a solution that takes much less time than adding the numbers singly. Introduce Frederick Gauss (18th century) who apparently astounded his teacher one day at finding just such a solution. In an effort to keep the class occupied and quiet one day, the teacher set the students an addition something like the following:

$$23,478+23,480+23,483+23,484+...+24,102+24,104+24,106$$

Young Freddie had the solution almost immediately, and, what's more, he was the only one in the whole class who had the correct answer.

Demonstrate Gauss's technique using the problem $1+2+3+...+19+20$. Writing the addends as illustrated below will help students understand the method. Begin by writing the first half of the numbers from left to right; then, beginning with '11' , which is written under the '10', write the remaining numbers from right to left, each one paired with a number in the first row.

1	2	3	4	5	6	7	8	9	10
20	19	18	17	16	15	14	13	12	11

Activity 3: Arithmetic Short-cuts

Ask the students to add the numbers in vertical pairs as shown. They are usually surprised to discover that the totals are all the same. Ask them how many pairs there are, and how the number of pairs is related to the number of addends. Ask how they can find the total if they know that there are ten 21s to add. Most will suggest multiplication.

Ask students to check that using the Gauss method to add the whole numbers from 1 to 10 gives the same answer they calculated earlier.

Give them one or two more problems to practice the technique - for example, all whole numbers from 1 to 30 or all whole numbers from 1 to 50. When they have finished, ask them if they wrote down all the pairs as illustrated above. Ask if it is really necessary to write all the pairs.

"How could you decide what the total will be for each pair without writing down all the pairs?"

"How do you know how many pairs there are?"

Ask them to add all the whole numbers from 1 to 100 to see if using this new method takes as long as they estimated.

Each problem so far has involved an even number of addends. This means, of course, that every number is paired with another number. Dividing the number of addends by 2 gives the number of pairs. Ask the students how the Gauss method could be used with an odd number of addends, such as

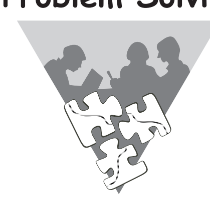
$$1+2+3+4+5+6+7+8+9.$$

Students should understand that they can still divide the number of addends in the problem by 2, but they will have a remainder, indicating a certain number of pairs with one addend left over. Thus, $9 \div 2 = 4$, R1 means that there will be four pairs and one single addend. The single addend will be the middle number, 5.

Use more examples to help students discover how they can determine what the middle number is. (One technique is to add the first and last addends and divide the sum by 2. See "Solutions and Notes" for more on this.)

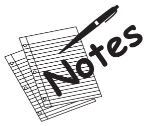
Ask students if they recognize any problem solving techniques in what they have been doing. Two that have been used are "Look for a pattern" and "Make a smaller problem."

Problem Solving



OPTIONAL

Students can now be given some challenges, such as adding only even numbers, or adding a series such as $24+27+30+33+...+43+45$, or these can be used later, or sent home as activities in which the student can teach his/her family members how to use the Gauss Technique.



Activity 3: Arithmetic Short-cuts

Extensions in Mathematics:

1. BLMs 15 and 16 provide some interesting patterns in multiplication. Students will discover that their calculators are not of much use in checking the equations in problems 2 and 3 on BLM 16 because the numbers are too large.

2. BLM 17 provides number patterns of a different kind. These "cyclic" number patterns can be fun to explore and try to analyze. Trying to describe the loops and explain them will call on number sense as well as ablity to compute.

   Further examples of each loop can be found in the Solutions and Notes.

Use of Technology

Cross-curricular Activities:

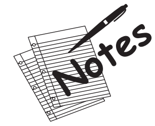
1. Explore books in the school or local library, or look on the web for information about Carl Friedrich Gauss.

Other Resources:

For additional ideas, see annotated Other Resources list on page 65, numbered as below.

5. Math Magic
6. Responses to "What Are the Clues? Patterns in More than One Direction" problem.

Activity 4: Pascal's Triangle



Focus of Activity:

- Exploring patterns in an array of numbers, and relating them to various areas of mathematics

What to Assess:

- Identification of patterns and their interconnections
- Accuracy of descriptions of patterns identified

Preparation:

- Make copies of BLMs 18 and 19, and a transparency of BLM 18
- Make copies of BLMs 20 and 21 (optional)

Activity:

Tell students that this activity deals with a famous set of numbers called Pascal's Triangle. The Triangle with all its patterns was known to Chinese and Persian scholars as early as 1100 A.D. Blaise Pascal studied its patterns and its connections to many areas of mathematics. Pascal published his findings in 1665, and since then these numbers have been called Pascal's Triangle.

Problem Solving

Comments in italics are explanatory, and need not be conveyed to the students.

Distribute copies of BLM 18, and direct students' attention to the first few rows of the Triangle on your transparency. Have them try to find the pattern that generates the numbers in rows 3 to 5, and apply this pattern to complete the numbers in rows 6 to 9.

In order to avoid confusion, "Triangle" with a capital T will refer to Pascal's Triangle.

The pattern is a simple one of addition. Each number is the sum of the two numbers above it to the right and left. Thus, the next rows are

1, 5, 10, 10, 5, 1
1, 6, 15, 20, 15, 6, 1
1, 7, 21, 35, 35, 21, 7, 1
1, 8, 28, 56, 70, 56, 28, 8, 1
1, 9, 36, 84, 126, 126, 84, 36, 9, 1

Have students look for other patterns. For example, looking at diagonals running down to the left, the first diagonal is all ones; the second diagonal is whole numbers in order; and the third diagonal is the triangular numbers explored in Activity 2 (1, 3, 6, 10, 15, 21, ...)

To help students become more familiar with the Triangle, ask a few questions, such as

"What whole number does not appear in the Triangle?" [0]

"What whole number appears only once?" [2]

"Row 1 and Row 2 have only odd numbers. What other rows, if any, have only odd numbers?" [rows 4 and 8]

"Do any rows have only even numbers? Why or why not?"

"If you saw just one row of the Triangle, how would you know if it was an odd-numbered row or an even-numbered row?" {Even-numbered rows have two identical numbers in the centre.]

Activity 4: Pascal's Triangle

Notes

Communication

"How often does '1' appear in the Triangle from row 1 to row 9?" [17 times]

If students completed BLM 16, they would have explored some patterns in #3 multiplying 11 x 11, 111 x 111, etc. In Pascal's Triangle, they can find patterns that come from
11×11 or 11^2, $11 \times 11 \times 11$ or 11^3, 11^4, 11^5, etc.

Have students determine the values of 11^2, 11^3, and 11^4. Then have them examine the rows of Pascal's Triangle to see where their answers can be found. (See Figure 4.1)

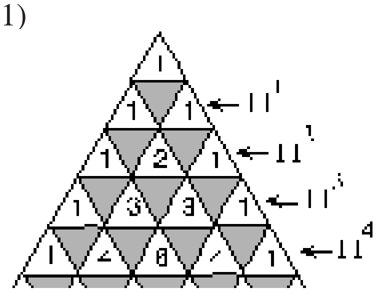


Figure 4.1

	100 000	10 000	1000	100	10	1
Row 6	1	5	10	10	5	1
Row 6 regrouped	1	6	1	0	5	1

Figure 4.2

Problem Solving

Have students calculate 11^5. Compare the answer to row 6 of the Triangle. Ask students why the numbers do not match. In order to explain, one needs to use a place value chart. (See Figure 4.2) The two-digit numbers in the Triangle need to be regrouped. When this is done, it can be seen that the regrouped row 6 of the Triangle matches the value of 11^5. Have students regroup rows 7, 8, and 9 to determine the values of 11^6, 11^7, and 11^8.

Have students work together to complete BLM 18. Students should report their results to the problems on BLM 18 before proceeding to BLM 19, because the results from BLM 18 will be useful for the first problem on BLM 19.

You might wish to have students use scrap paper to record the possible sums for #2 on BLM 19. They can then try to organize these sums in some way before writing them in the given chart.

Since it is unlikely that students will have time to complete the chart on BLM 19 in school, you might wish to send this home as a Family Activity. Alternatively, assign part of each problem to different groups. For example, for sums of 6, have some groups find all the sums that use only ones and two; other groups those that use ones and threes and those that use ones, two, and threes; and other groups those that include fours or fives. (See "Solutions and Notes" for all possibilities.)

You might wish to post chart paper on which students add different sums as they find them. This allows all students to check for duplications.
The problems on BLM 20 relate Pascal's Triangle to a probability problem (the number of heads and tails possible for 2, 3, 4, or 5 coins).

Activity 4: Pascal's Triangle

Extensions in Mathematics:

1. Distribute copies of BLM 21, and have students explore yet another way to form triangular patterns of numbers.

Cross-curricular Activities:

1. Have students examine encyclopedias both at home and at school for more information on Pascal or the history of the triangle.
2. Examine websites for information on the mathematician Blaise Pascal.

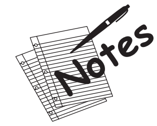
Family Activities:

1. Have students examine encyclopedias both at home and at school for more information on Pascal or the history of the triangle.
2. See the note above regarding BLM 19, problem 2.

Other Resources:

For additional ideas, see annotated Other Resources list on page 65, numbered as below.

2. Patterns and Functions, Addenda Series, Grades 5-8.
3. Multicultural Math Posters and Activities.



These problems are similar to some in another book in this series, "Investigations in Probability", Grade 5. See order form at the back of this book.

Use of Technology

Notes

Activity 5: Tin Cans and Equations

Focus of Activity:
- Picturing and solving simple equations

What to Assess:
- Accuracy of solutions to equations
- Understanding of the role of the "unknown" in an equation as represented by a drawing of a tin or other figure.

Preparation:
- Make copies of BLMs 22 and 23
- Make copies of BLM 24 (optional)

Activity:

This activity will introduce students to the basic algebraic idea of using symbols (usually letters) to represent unknown numbers. Starting with physical models gives students a mental model they can use in solving simple equations.

The model consists of some small containers (tins, paper cups, or small yogurt containers), and counters of any kind (buttons, paper clips).

Illustrate an equation for the students in the following way:

Place four counters in a tin without the students knowing how many are there. Then, display the "equation" in Example 1 below.

Comments in italics are explanatory, and need not be conveyed to the students.

If you display the equations using an overhead projector, the tin will appear as a circle or oval and paper clips make easily identified counters. The equals sign can be written on a sheet of acetate positioned under the materials.

Example 1:

Tell the students that the equation in Example 1 means that the number of paper clips in the tin is equal to the number of paper clips shown. Obviously, this means that there are 4 paper clips in the tin. Once students decide this, you can have one of them count the number of paper clips in the tin as proof that the tin does contain 4 items. Be sure that students understand that it is the contents of the tin that are important, not the tin itself.

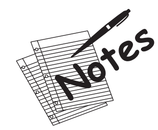
Example 2:

Communication

For Example 2, the tin should contain 3 paper clips. Students should not, of course, know this at the beginning. Present the equation, give a few minutes for thinking, then ask for ideas about how to solve the equation (i.e., calculate how many items are in the tin).

One suggestion students make is that 2 items can be removed from each side of the equation so that what is left shows clearly that the tin must contain 3 items. Other techniques should be explored for validity. Once students have decided on an answer (even if not all students decide on the same answer), have one of them count the items in the tin.

Activity 5: Tin Cans and Equations



The technique just described is analogous to the algebraic situation below, in which the tin is represented by the letter 'T':

$$\begin{aligned} \boldsymbol{T} + 2 &= 5 \\ \boldsymbol{T} + 2 - 2 &= 5 - 2 \qquad \textit{(subtract 2 from both sides of the equation)} \\ \boldsymbol{T} &= 3 \end{aligned}$$

Problem Solving

BLM 22 presents several more equations of this type. You may wish to use some of them as more examples for the whole class before assigning group work. Students should have tins/cups and counters to use to help them decide on answers.

Since BLM 23 presents equations in slightly more abstract form, you may wish to check student work when BLM 22 is completed, before proceeding to BLM 23. Have students share their techniques with the class. If students have difficulty with any of the equations, place the correct number of items in the tin(s) (see "Solutions and Notes") and present those equations as concrete models.

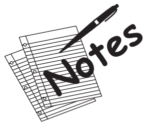
Problem 1 on BLM 23 replaces the tin with a symbol — a square. Students should understand that the equations are solved in the same way as for BLM 22. In fact, some students will feel more comfortable if they continue to use tins and counters.

Problem 2 on BLM 23 presents students with the use of a variable. This is a symbol, usually a letter that represents some number. When that number is determined, the equation is solved.

To maintain a link with the physical model of Problem 1, the variable used is '**S**' which students can think of as "a square".

Assessment

Problem 3 can be viewed as a "short-cut" in writing the equation. Thus 2**S** would mean "2 squares", 3**S** would mean "3 squares", and so on. Whether or not students do this consistently is not important. What is important is that they view equations written with variables as being a short way of writing equations with tins and counters. Thus, for problem 4, some students may still want to model the equations with the materials.



Activity 5: Tin Cans and Equations

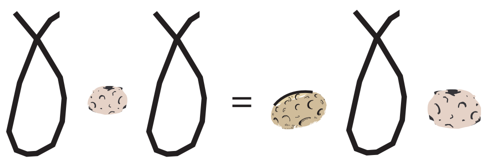
Extensions in Mathematics:

1. Another model for equations was devised by mathematician W.W. Sawyer in his book "Vision in Elementary Mathematics", published in 1964. He used sketches of bags (which were easily drawn) instead of tins, with stones as counters. The examples below are equations from BLM 23.

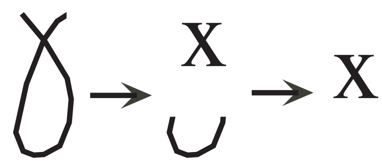
(a)

(c)

(g)



Once the students understand the nature of equations, Sawyer simply cut off the bottom of the bag leaving"x", that favoured variable in algebra.



Show students this model and have them try to develop another model. In order to develop a meaningful model, they will need to understand equations. The type of model a student devises could be a useful assessment tool.

Problem Solving

Family Activities:

1. BLM 24 describes a way to construct a balance that students can use to solve simple equations. You may wish to have students construct the balances at school to be sure they are accurate. Figuring out how to use the model is part of the problem solving involved in this activity. (See "Solutions and Notes" for some analysis.)

Other Resources:

For additional ideas, see annotated Other Resources list on page 65, numbered as below.

7. Algebraic Thinking, Focus Issue TCM.
8. Algebraic Thinking, Opening the Gate, Focus Issue MTMG.

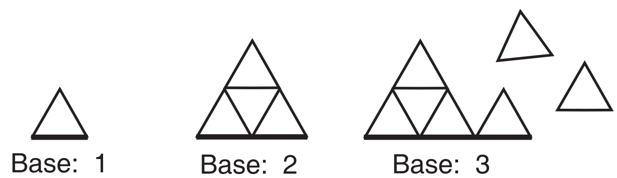
BLM 1: Square Numbers - 1

1. How many small squares does it take to make the next largest square? Complete the table to show your results.

Size of Square	1 by 1	2 by 2	3 by 3				
Number of small squares needed	1	4	9				

How many small squares will it take to make a 10 by 10 square? How do you know?

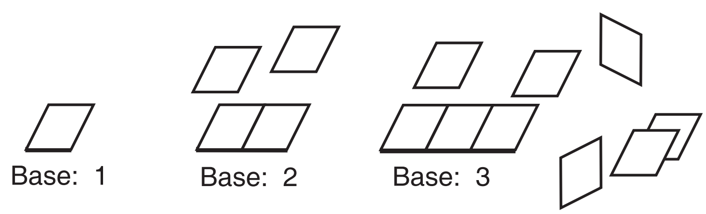
2. How many small equilateral triangles will it take to make larger equilateral triangles? Complete the table to show your results.



Length of base of triangle	1	2					
Number of small triangles needed	1	4					

How many small triangles will be needed to construct a large triangle with a base length of 10 units? How do you know?

3. How many small rhombuses will it take to make larger rhombuses? Complete the table to show your results.

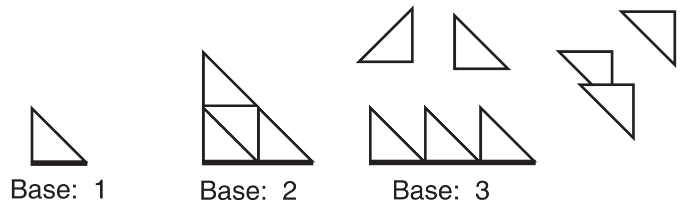


Length of base of rhombus	1	2	3				
Number of small rhombuses needed	1	4	9				

How many small rhombuses will be needed to construct a large rhombus with a base length of 10 units? How do you know?

BLM 2: Square Numbers - 2

4. How many small triangles will it take to make larger triangles if the small triangles and the larger triangles have the shape shown below:



Length of base of triangle	1	2	3				
Number of small triangles needed	1	4	9				

How many of these small triangles will it take to construct a similar triangle with a base length of 10 units? How do you know?

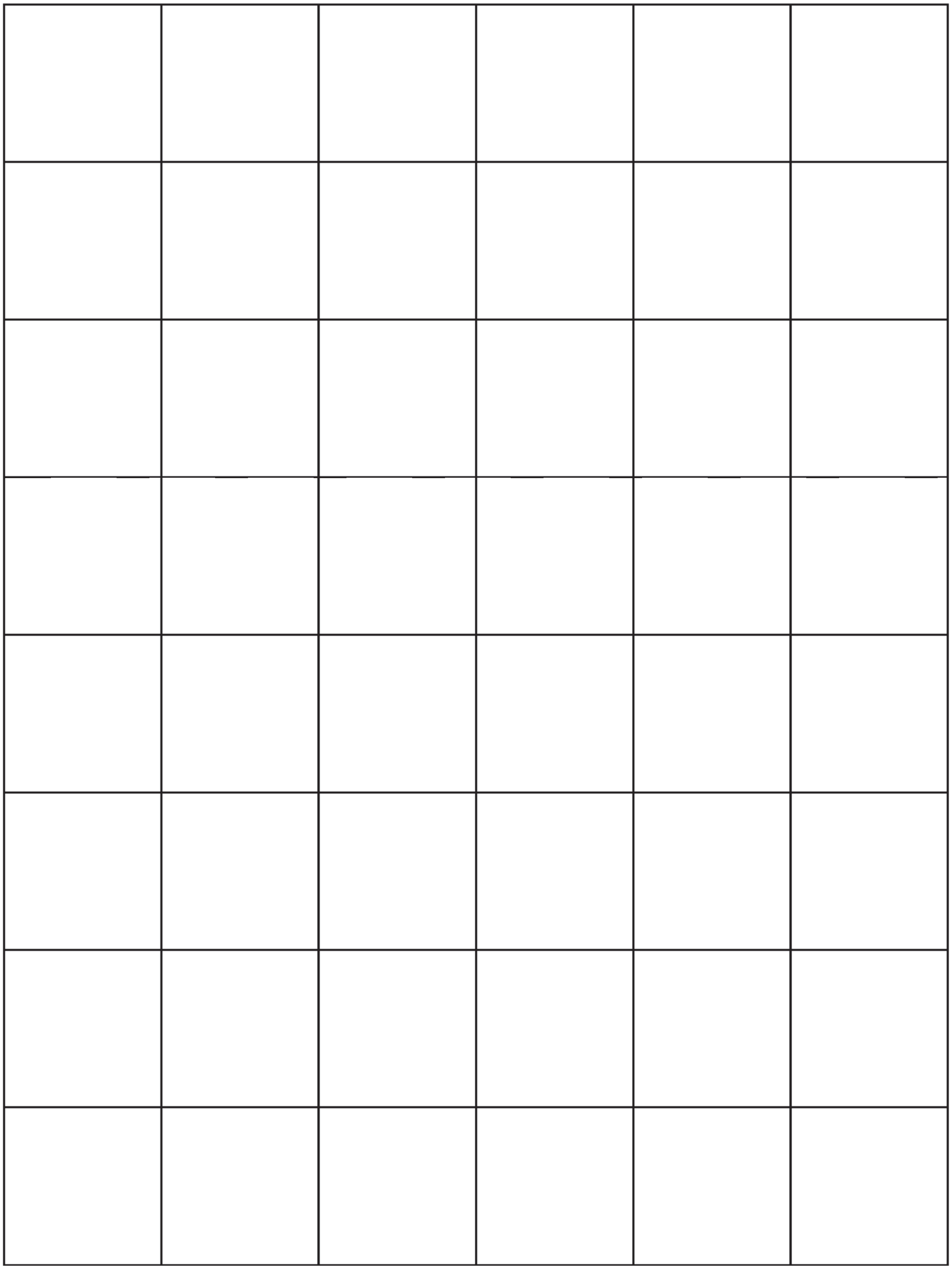
A CHALLENGE:

5. How many small regular hexagons do you think it will take to make a larger hexagon with a side length of 4 units? with a side length of 10 units? Why do you think so? Check your prediction.

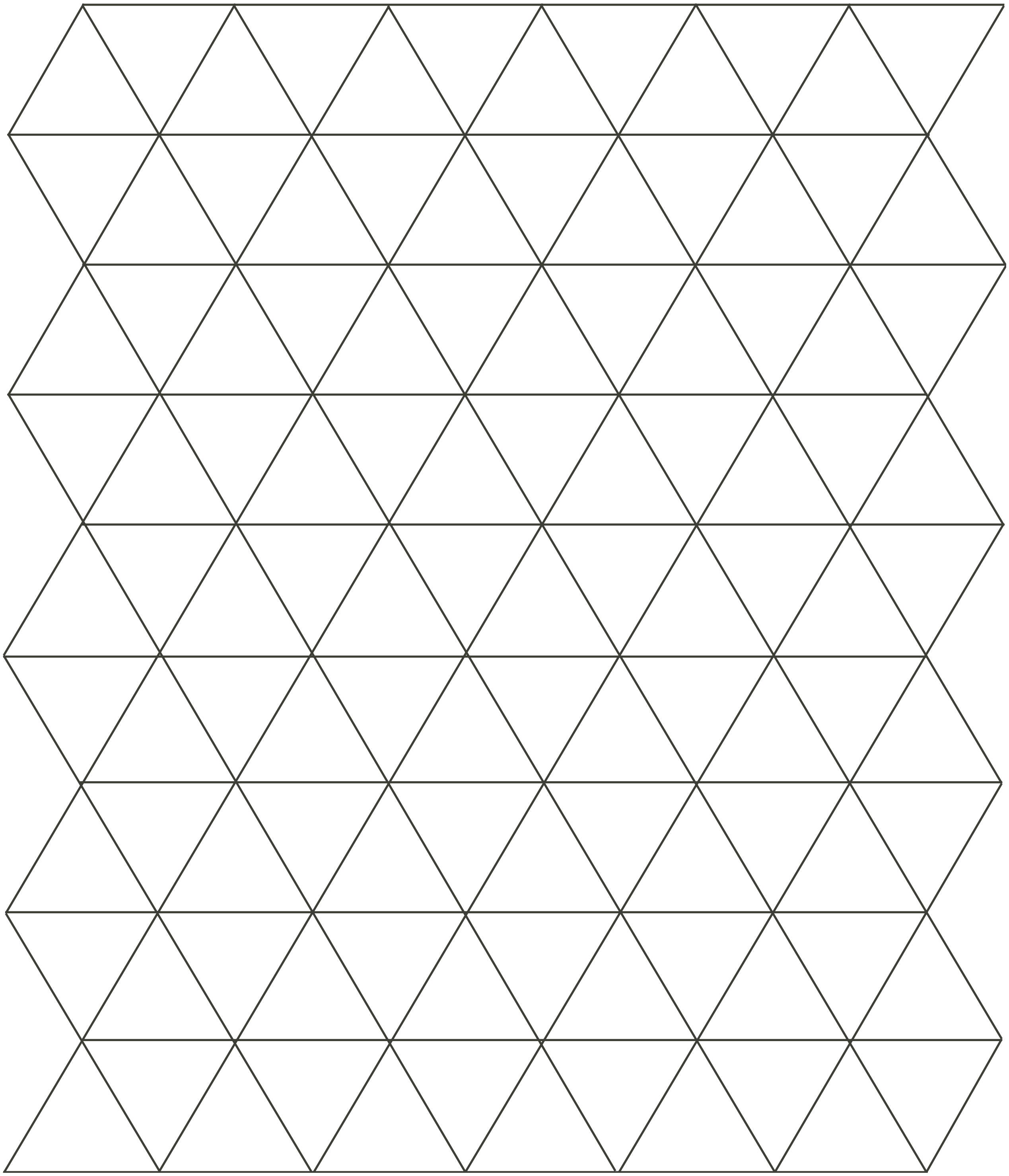
A CHALLENGE:

6. Do you think it is possible to fit 4 trapezoids together to construct a larger trapezoid? Why or why not? Check your prediction.

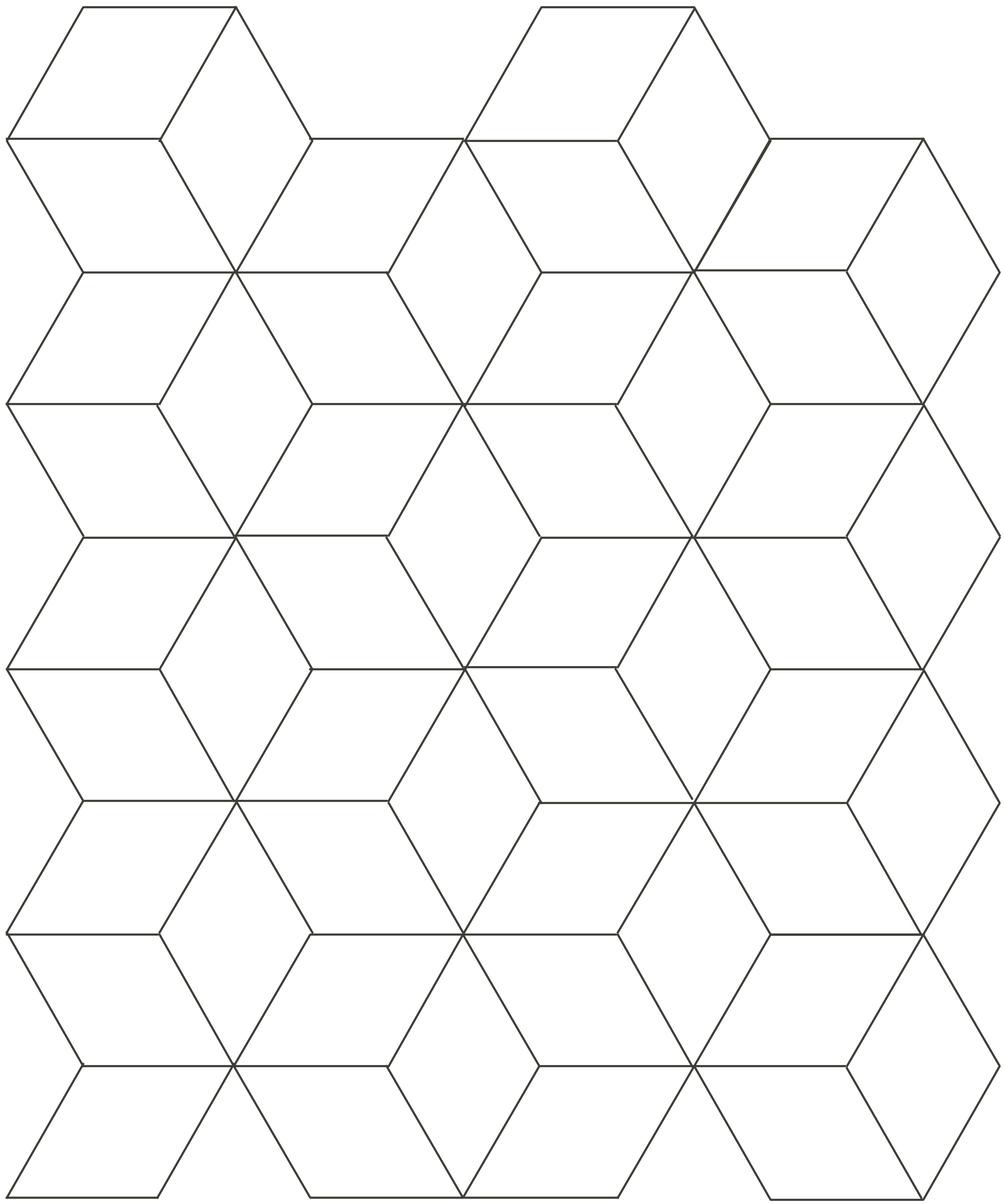
BLM 3: Squares



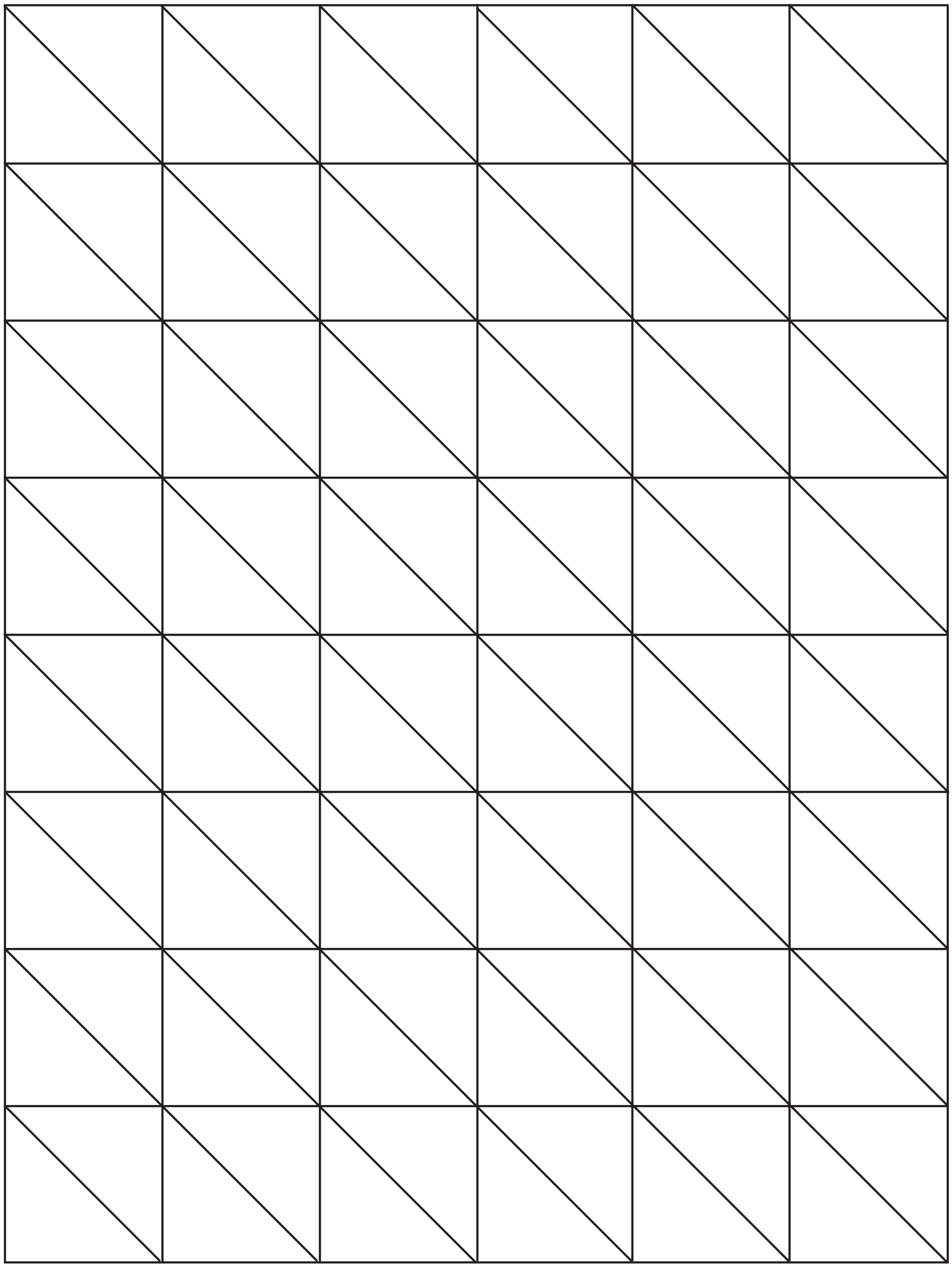
BLM 4: Triangles



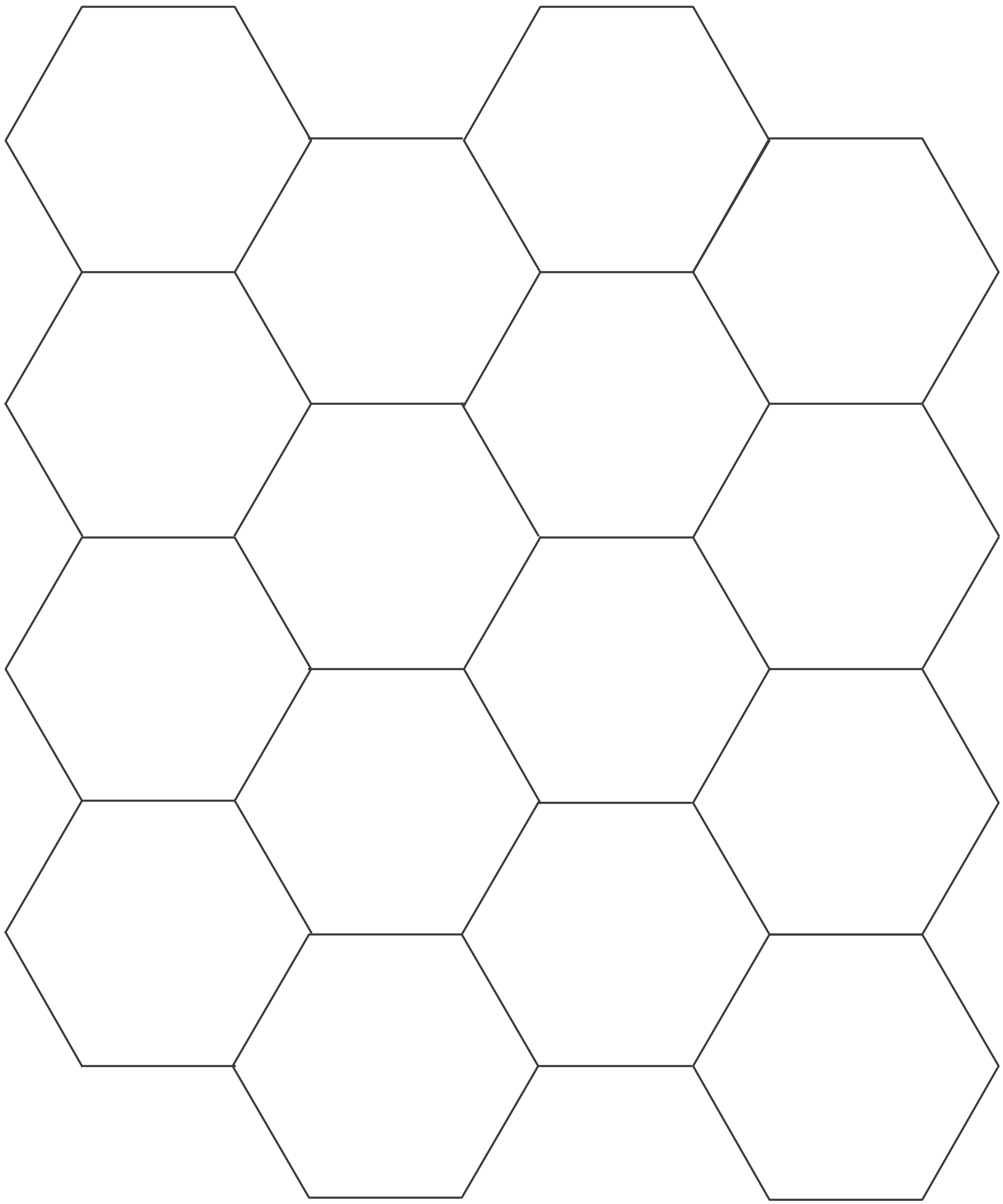
BLM 5: Rhombuses



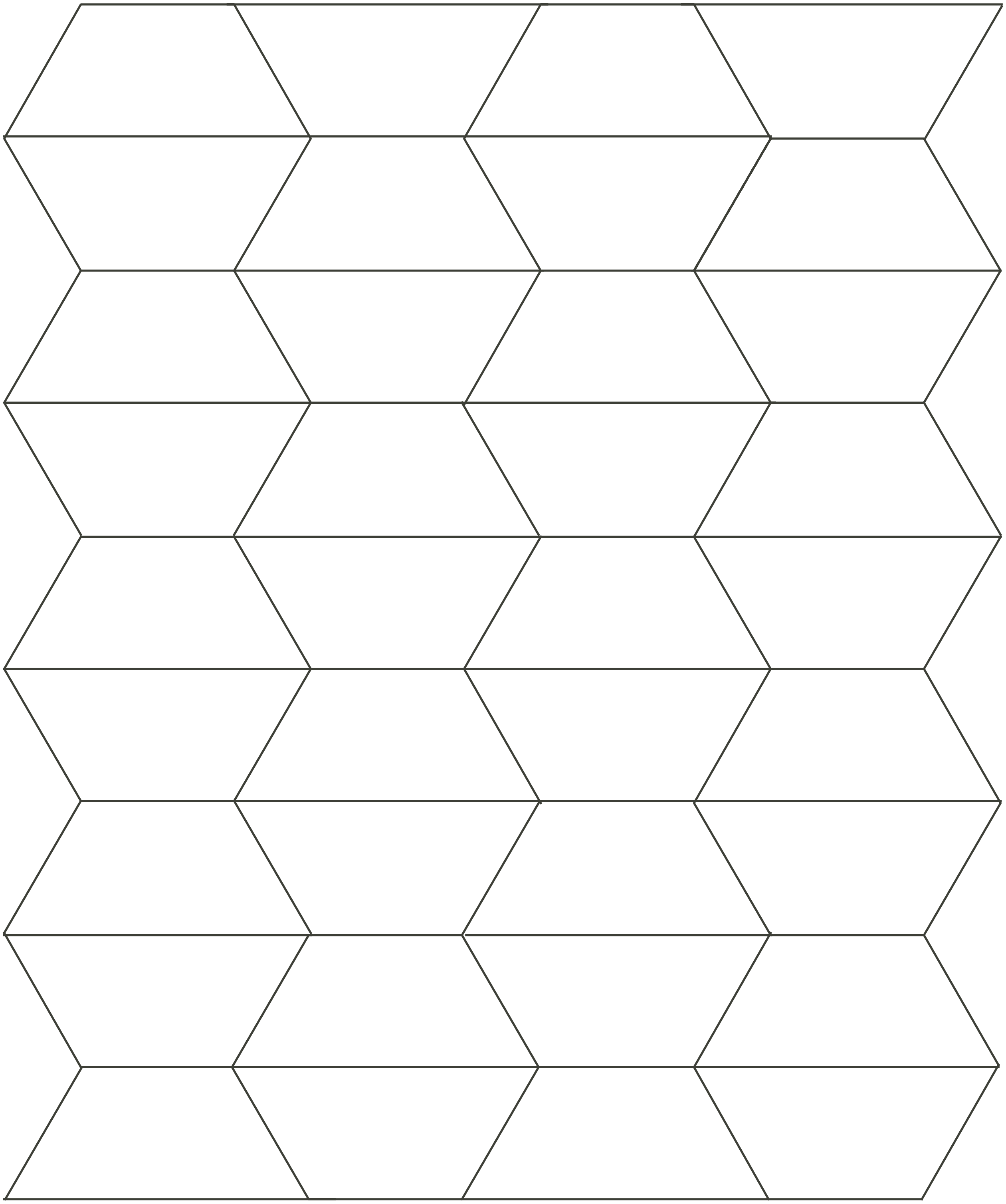
BLM 6: Isosceles Right Triangles



BLM 7: Hexagons

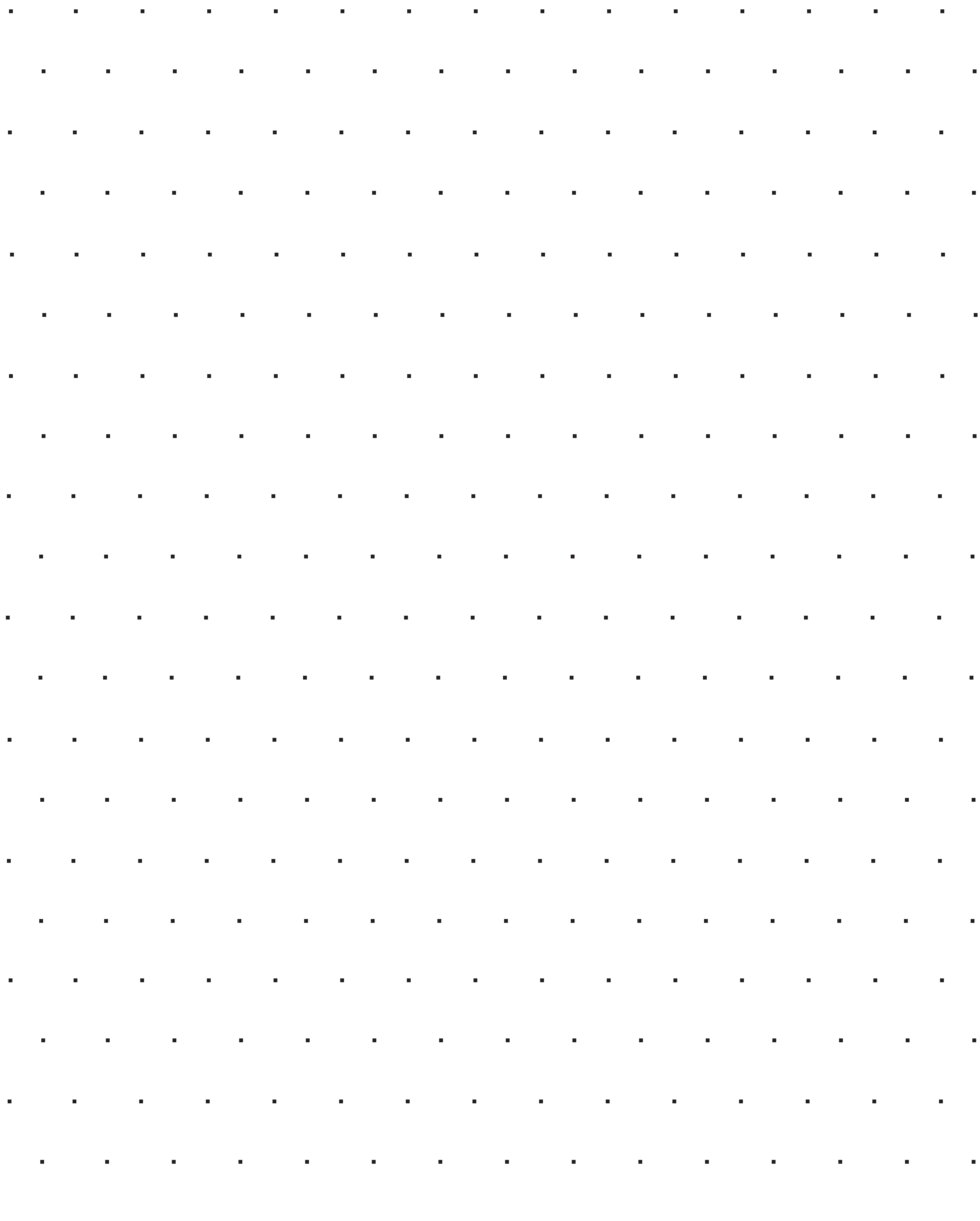


BLM 8: Trapezoids



BLM 9: Square Dot Paper

BLM 10: Triangle Dot Paper



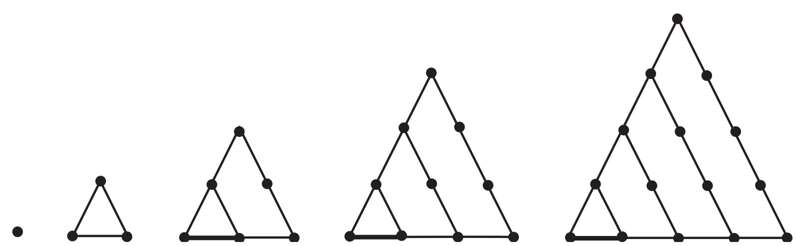
BLM 11: Shapes of Numbers - 1

1. There are different kinds of numbers named for the shapes they can be arranged in. For example, you found square numbers in the previous activity. Write the 5th and the 10th square numbers in the chart.

For problems 2 to 6, record the numbers in the following chart.

Term ——>	1st	2nd	3rd	4th	5th	10th	100th	nth
Triangular numbers	1	3						
Square numbers	1	4	9	16				n x n
Pentagonal numbers								
Hexagonal numbers								
Heptagonal numbers								
Octagonal numbers								

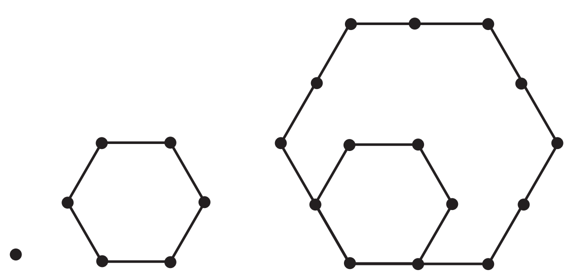
2. The first few triangular numbers are given below. Add these to the chart.



1 3 ___ ___ _____

Look for a pattern to help you determine what the 10th triangular number will be. How can you check this?

3. The first hexagonal numbers are given below. Add these to the chart.



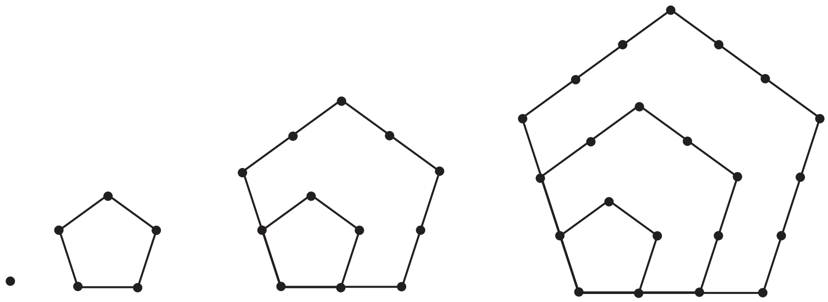
1 6 15 28

Use triangle dot paper to help you determine the next two hexagonal numbers. Add them to the chart. Look for patterns to help you predict what the 10th hexagonal number will be. How can you check this?

BLM 12: Shapes of Numbers - 2

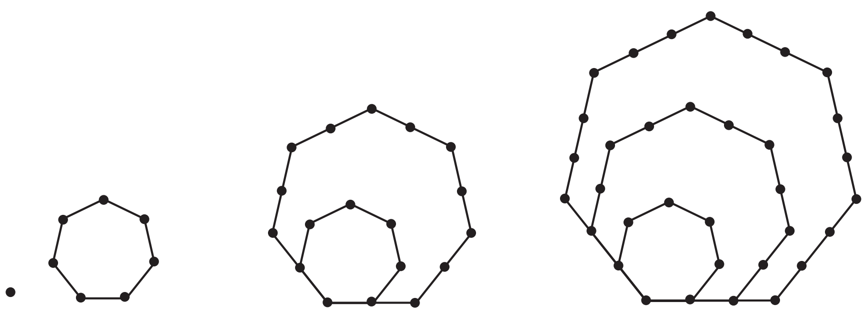
For the sequences of numbers below, find a pattern to help you complete the first 5 columns of the chart, and then to predict the 10th number for each sequence.

4. The first four pentagonal numbers are illustrated below. Add them to the chart.



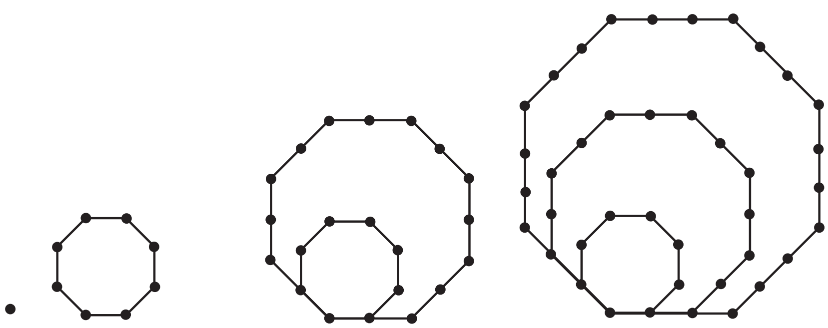
1 5 12 ________ 35

5. The first four heptagonal numbers are illustrated below. Add them to the chart.



1 7 18 ________ 55

6. The first four octagonal numbers are illustrated below. Add them to the chart.



1 8 21 ________ 65

BLM 13: Adding to Figurate Numbers

1. In each column, find a pattern in the addends to help you complete the column.

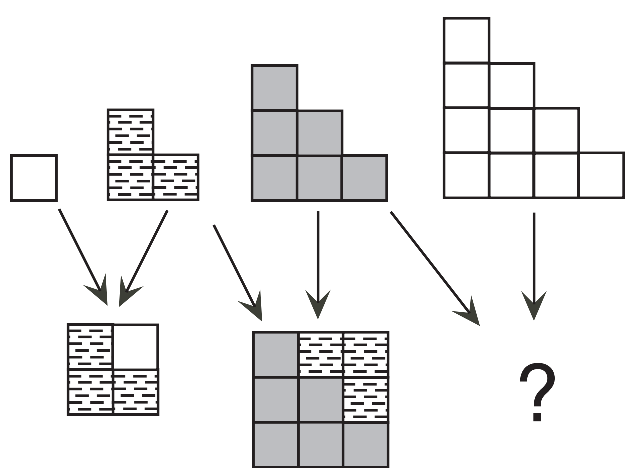
Term	Triangular Numbers	Square Numbers	Pentagonal Numbers	Hexagonal Numbers	Heptagonal Numbers
1st	1 = 1	1 = 1	1 = 1	1 = 1	1 = 1
2nd	1 + 2 = 3	1 + 3 = 4	1 + 4 = 5	1 + 5 = 6	1 + 6 = 7
3rd	1 + 2 + 3 = 6	1 + 3 + 5 = 9	1 + 4 + 7 = 12	1 + 5 + 9 = 15	1 + 6 + 11 = 18
4th	1 + 2 + 3 + 4 = 10	1 + 3 + 5 + 7 = 16	1 + 4 + 7 + 10 = 22	1 + 5 + 9 + 13 = 28	1 + 6 + 11 + 16 = 34
5th					
6th					
7th					

2. Compare the results of each column with the others. What similarities can you find? What differences are there?

3. A Challenge: Square Building

The triangular numbers:

1 3 6 10



The square numbers: 4 9 16

The triangular numbers: (1 + 3) (3 + 6) (6 + 10)

Will two consecutive triangular numbers (two triangular numbers in a row) always have a sum equal to a square number? Why or why not?

BLM 14: Other Number Systems

Arabic	Roman	Chinese	Egyptian	Mayan
0				𝋠
1	I	𝍠	|	•
2	II	𝍡	||	••
3	III	𝍢	|||	•••
4	IIII or IV	𝍣	||||	••••
5	V	𝍤	|||||	—
6	VI	𝍥	||||||	• —
7	VII	𝍦	|||||||	•• —
8	VIII	𝍧	||||||||	••• —
9	IX	𝍨	|||||||||	•••• —
10	X	𝍩	∩	═
15	XV		∩|||||	≡
20	XX	𝍪	∩∩	• 𝋠
50	L	𝍭	∩∩∩∩∩	
100	C		𓍢	— 𝋠
200	CC		𓍢𓍢	
300	CCC			
500	D			
1000	M		𓆼	
10 000			𓂭	
100 000			𓆐	
1 000 000			𓁨	

BLM 15: Short-cuts to Calculation

Look for a pattern to help you determine the solution to each problem.

1. Multiplying fives
   (a) Calculate the answers to the following:

$$\begin{array}{r} 15 \\ \underline{\times 15} \end{array} \qquad \begin{array}{r} 25 \\ \underline{\times 25} \end{array} \qquad \begin{array}{r} 35 \\ \underline{\times 35} \end{array} \qquad \begin{array}{r} 45 \\ \underline{\times 45} \end{array}$$

   (b) Compare the answers in (a) with the numbers you multiplied. Look for a pattern to help you determine the products for the following. Then predict answers for the following.

$$\begin{array}{r} 55 \\ \underline{\times 55} \end{array} \qquad \begin{array}{r} 65 \\ \underline{\times 65} \end{array} \qquad \begin{array}{r} 75 \\ \underline{\times 75} \end{array} \qquad \begin{array}{r} 85 \\ \underline{\times 85} \end{array} \qquad \begin{array}{r} 95 \\ \underline{\times 95} \end{array}$$

   (c) Check the answers in (b) using a calculator. Did your pattern help you get the right answers?

   (d) Do you think you could apply your pattern to give correct answers to any of the following? Explain.

$$\begin{array}{r} 105 \\ \underline{\times 105} \end{array} \qquad \begin{array}{r} 75 \\ \underline{\times 65} \end{array} \qquad \begin{array}{r} 225 \\ \underline{\times 225} \end{array} \qquad \begin{array}{r} 65 \\ \underline{\times 45} \end{array} \qquad \begin{array}{r} 1005 \\ \underline{\times 1005} \end{array}$$

2. Units of Ten
   (a) Calculate answers for the following.

$$\begin{array}{r} 27 \\ \underline{\times 23} \end{array} \qquad \begin{array}{r} 14 \\ \underline{\times 16} \end{array} \qquad \begin{array}{r} 63 \\ \underline{\times 67} \end{array} \qquad \begin{array}{r} 72 \\ \underline{\times 78} \end{array}$$

   (b) Compare your answers with the numbers multiplied. The title of these calculations is “Units of Ten”. What is alike about the units digits in each multiplication, and how are they related to 10?

   (c) Predict the answers to the following using any pattern you found.

$$\begin{array}{r} 92 \\ \underline{\times 98} \end{array} \qquad \begin{array}{r} 34 \\ \underline{\times 36} \end{array} \qquad \begin{array}{r} 81 \\ \underline{\times 89} \end{array} \qquad \begin{array}{r} 67 \\ \underline{\times 63} \end{array} \qquad \begin{array}{r} 44 \\ \underline{\times 46} \end{array}$$

   (d) Check your answers with a calculator. Did your pattern help you get the right answers?

   (e) How are the multiplication patterns in #1 and #2 alike?

BLM 16: Patterns in Equations

1. Reversals
   (a) Check to find if the following equations are true.

   $12\times 42 = 21\times 24$ $13\times 62 = 31\times 26$ $23\times 96 = 32\times 69$

   (b) How are the numbers on one side of the equation like the numbers on the other side of the equation? How are they different?

   (c) Which of the following do you think are true? Why? Now check your predictions using a calculator.

   $12\times 84 = 21\times 48$ $24\times 63 = 42\times 36$ $37\times 54 = 73\times 45$

   $14\times 82 = 41\times 28$ $15\times 95 = 51\times 59$ $23\times 64 = 32\times 46$

   (d) Were your predictions correct? If so, describe your pattern.
   If not, try to find a pattern that will predict which equations in (c) are correct.

2. Backwards Nines
   (a) Examine the equations on the right carefully. Are they true?

   $$1089\times 9 = 9801$$
   $$10989\times 9 = 98901$$
   $$109989\times 9 = 989901$$
   $$1099989\times 9 = 9899901$$

   (b) Do you think this pattern will continue forever? Why or why not?

   (c) Predict the next two equations.

   (d) How can you check your predictions? Will a calculator help? Explain.

3. Lots of Ones
   (a) Are the equations on the right true or false?

   $$1\times 1 = 1$$
   $$11\times 11 = 121$$
   $$111\times 111 = 12321$$
   $$1111\times 1111 = 1234321$$

   (b) Describe any pattern you see.

   (c) Do you think this pattern will continue forever? Why or why not?

   (d) Predict the next two equations in the sequence.

   (e) Can you use a calculator to check your predications? Explain.

BLM 17: Number Loops

1. (a) Start with any number. Follow these two rules:
   If the number is even, divide by 2.
   If the number is odd, add one.

   (b) Apply the two rules to the new number.

   (c) Apply the rules again and again as long as you can.

   (d) Did the numbers start to repeat themselves?
   The example above that starts with 12 becomes a "loop" going from 2 to 1 and back again to 2.
   What will happen to the example that starts at 13?

Examples:

$12 \to 6$ or $13 \to 14$

$12 \to 6 \to 3$ $\quad$ $13 \to 14 \to 7$

$12 \to 6 \to 3 \to 4$ $\quad$ $13 \to 14 \to 7 \to 8$

$12 \to 6 \to 3 \to 4 \to 2 \to 1 \to 2 \to 1 \to 2 \to 1$

   (e) Do you think there is a number you could start with that won't end in a loop?
   Start with 25; with 112; with 88; with 101. Write each sequence of numbers using small arrows as in the examples above.

   (f) Did you end with a loop in all of the numbers from (e)? Did they all end with the same loop? Why do you think that happened?

2. (a) Use the different rules below to see if all numbers will still end in loops.
   Start with any number.
   If the number is even, divide by 2.
   If the number is odd, add 1 and then multiply by 2.

Example:

$38 \to 19 \to 40 \to 20 \to 10 \to 5 \to 12 \to 6 \to 3 \to 8 \to 4 \to ...$

   (b) Choose your own four numbers to test, and write out each sequence.

   (c) Did you end in a loop? Was it always the same loop? Was it like the loop in #1? Why or why not?

3. Use these rules:
   Start with any number.
   If the number has more than one digit, add the digits, (e.g.,$134 \to 1+3+4 \to 8$).
   If the number has only one digit, multiply it by 2.

Example:

$37 \to 10 \to 1 \to 2 \to 4 \to 8 \to 16 \to 7 \to 14 \to 5 \to 10 \to 1...$

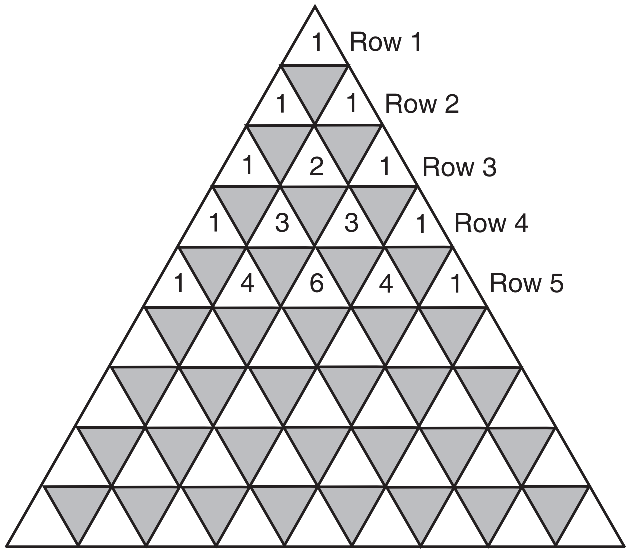
   Choose numbers of your own to start with. Did they all end with a loop? Did they all end with the same loop? Why do you think this happened?

4. Repeat #3 but with these rules:
   Start with any number.
   If the number has more than one digit, multiply the digits (e.g., $46 \to 4 \times 6 \to 24$).
   If the number has only one digit, multiply it by itself (e.g., $7 \to 7 \times 7 \to 49$).

   Describe what happened when you tested your own chosen numbers.

BLM 18: Pascal's Triangle

1. Pascal's Triangle is a triangular array of numbers as shown below. The numbers follow a pattern. Find the pattern and continue the Triangle.



2. (a) Calculate the sum of the numbers in each row from row 1 to row 7, and record these in the table below.

Row no.	1	2	3	4	5	6	7
Sum	1	2	__	__	__	__	__

(b) What do you think will be the sum of the numbers in row 10? Why?

3. You know that 2^2 means 2×2 and equals 4, and that 2^3 means $2\times2\times2$ and equals 8.

(a) Calculate the value of each of the following.

(i) 2^4 (ii) 2^5 (iii) 2^6

$$2^2 = 2\times2 = 4$$
$$2^3 = 2\times2\times2 = 8$$
$$2^4 = 2\times2\times2\times2 = ?$$

(b) Compare these numbers with the numbers you wrote in the table in #2.

(c) Which of the following will give you the sum of the numbers in row 10? How do you know?

(i) 2^9 (ii) 2^{10} (iii) 2^{11}

BLM 19: Pascal and Sums

1. Instead of calculating the sum for just one row, calculate the sum of several rows.

Rows:	1 and 2	1, 2, and 3	1 to 4	1 to 5	1 to 6	1 to 7
Sum:	3	7	____	____	____	____

Keep those numbers in mind as you work on #2 below.

2. There are 3 ways to write the number ‘3’ as a sum of whole numbers (without using zero.) These are $1+1+1$, $1+2$, $2+1$.

(a) In how many ways can the number ‘4’ or ‘5’ or ‘6’ be written as the sum of whole numbers (without zero)? Some are already in the table. Write the others. (Different orders of the same numbers count as different ways. That is, $1+1+2$ is different from $1+2+1$ or $2+1+1$.)

‘4’ as the sum of whole numbers	‘5’ as the sum of whole numbers	‘6’ as the sum of whole numbers
$1+1+1+1$ $1+1+2$ $1+2+1$ $2+1+1$		
There are _______ different ways	There are _______ different ways	There are _______ different ways

(b) How are the numbers in the last row of the table above like the sums from #1?

BLM 20: Pascal and Heads and Tails

This problem shows how Pascal's Triangle is linked to probability. If you were working out the probability that you would get 2 heads and 1 tail by tossing 3 coins, you would need to know all possible ways 3 coins can land.

1. Start with a smaller problem and write all possible outcomes with just 2 coins. For example, both could be heads (HH), both could be tails (TT), or one could be heads and the other tails (HT and TH). Why are 'HT' and 'TH' different? How many different outcomes are there?

2. If you tossed 3 coins, how many ways would there be of getting
   (i) 3 heads? (ii) 2 heads and 1 tail? (iii) 1 head and 2 tails? (iv) 3 tails?

3. Compare your answers to 2 with Row 4 of Pascal's Triangle.

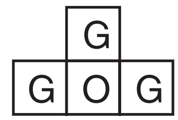
4. What row of Pascal's Triangle do you think might give you the answers for the problem in the box on the right? Check to see if your prediction is correct.

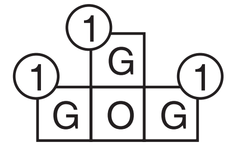
If you tossed 4 coins, how many ways would there be of getting
(i) 4 heads?
(ii) 3 heads and 1 tail?
(iii) 2 heads and 2 tails?
(iv) 1 head and 3 tails?
(v) 4 tails?

5. Use Pascal's triangle to tell the number of outcomes if you tossed 5 coins.

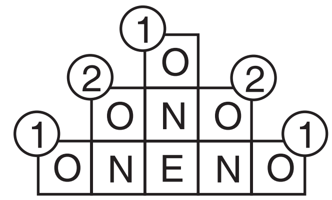
BLM 21: Pascal and Pathways

1. In the first diagram on the right, the word "GO" can be spelled straight down, or from left to right, or from right to left. In the second diagram, the '1' indicates that the word "GO" can be spelled once from each letter 'G'.





(a) In the diagram at right, the word "ONE" can be spelled in several different ways, moving from one letter to another, either down or right or left. The numbers indicate the number of times "ONE" can be spelled starting at that 'O'. Before going on to #3, be sure that you can find all the ways that "ONE" can be spelled in this diagram.



(b) In figure 4.3, look for ways to spell "FOUR", and put numbers in the circles to show how many ways you can spell "FOUR" starting at that 'F'.

(c) In figure 4.4 look for ways to spell "EIGHT", and complete the diagram with numbers in the circles. Use Pascal's Triangle to help you.

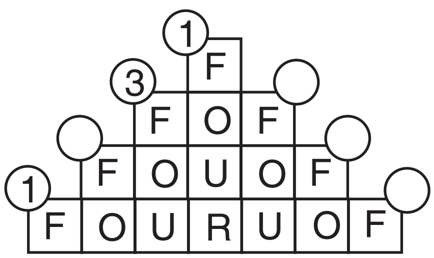


Figure 4.3

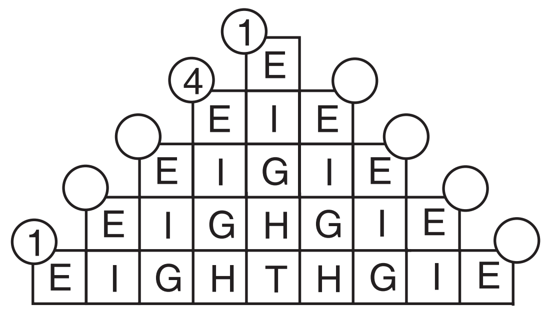
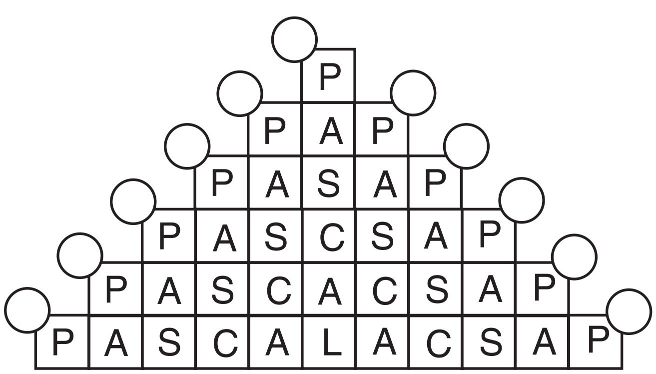


Figure 4.4

(d) How did Pascal's Triangle help you?

2. (a) Use Pascal's Triangle to tell you the total number of times you can spell "PASCAL" using the diagram below.

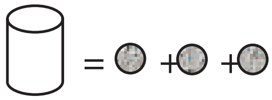
(b) Try to find them all.

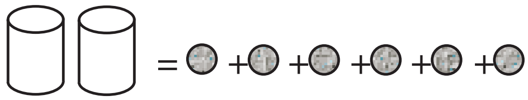


BLM 22: Tins and Counters

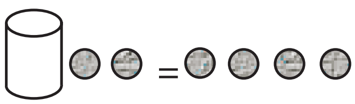
For each of the problems below, each tin in the problem will contain the same number of counters.

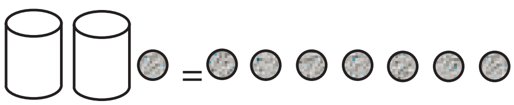
For example, if one tin holds (is equal to) 3 counters, then two tins will hold (be equal to) 6 counters.

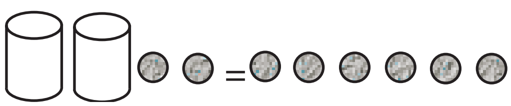


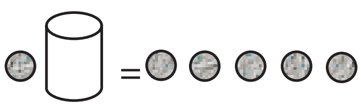


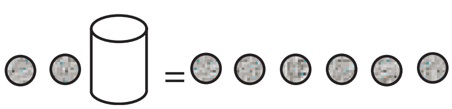
For each of these problems, tell how many counters must be in each tin.

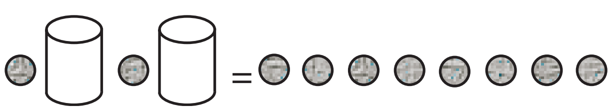
1. 

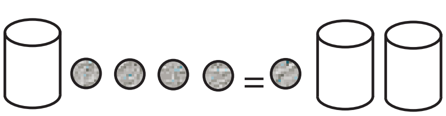
2. 

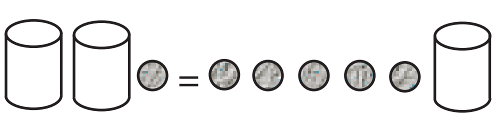
3. 

4. 

5. 

6. 

7. 

8. 

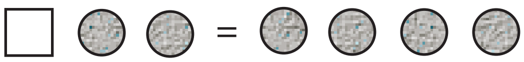
BLM 23: Counters and Equations

1. For each of these problems, the tin used on BLM 22 has been replaced with a square counter. This means that for each of the problems, each square in the problem will represent the same number of counters.

   For each of these problems, tell how many counters are respresented by each square.

(a) 

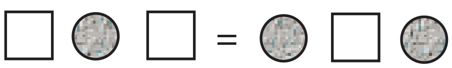
(b) 

(c) 

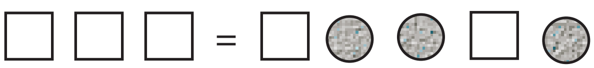
(d) 

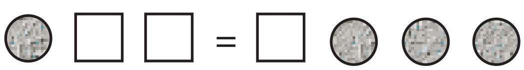
(e) 

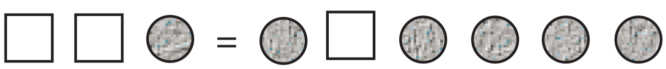
(f) 

(g) 

(h) 

(i) 

(j) 

(k) 

2. Instead of using squares and counters, in algebra we often use letters. For example, if **S** stands for a square, and each counter stands for 1, we could write problem (a) above as ***S*** + ***S*** = 8, meaning that 2 squares are equal to 8 counters.

   The equations below match problems (b), (c), (d), and (e), but they are not in the right order. Match each equation with its problem.

   (i) **S** + **S** + 1 = 9 (ii) 1 + **S** + **S** = 7 (iii) **S** + **S** + **S** = 6 (iv) **S** + 2 = 4

3. Instead of writing **S** + 1 + **S** = 1 + **S** + 1 for problem (g), we could write 2**S** + 1 = **S** + 2.
   For (f) we could write 2**S** = **S** + 5 instead of 2**S** = 1 + **S** + 4.
   Write equations using **S** for problems (h) to (k).

4. For each of these equations, tell how many counters **S** represents.

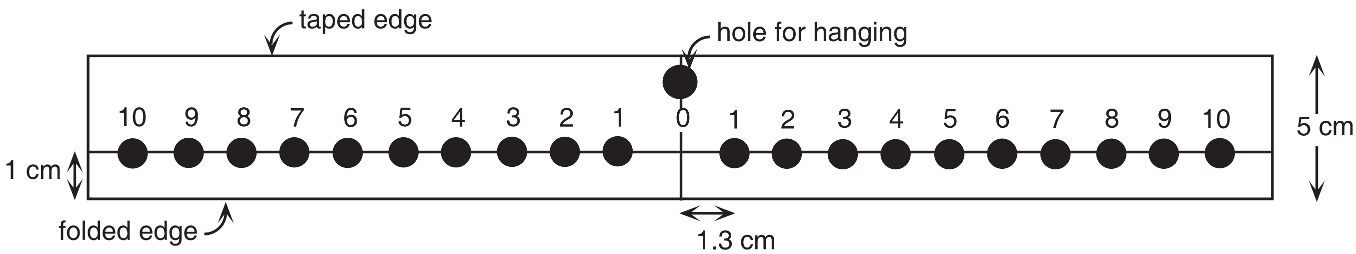
   (a) 2**S** + 2 = 8 (b) 2**S** +1 = **S** + 7 (c) 3**S** + 4 = **S** + 14 (d) 3**S** + 5 = 2**S** + 9

BLM 24: Balanced Equations

You will need:

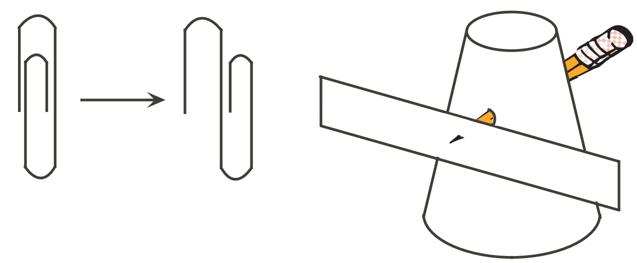
- a strip of bristol board or light cardboard 27 cm by 10 cm
- a styrofoam cup
- several paper clips
- a paper punch

Fold the bristol board in half lengthwise so it measures 27 cm by 5 cm, and tape the edges together. Punch holes in it at carefully measured intervals (1.3 cm apart, centre to centre) as shown below. Number the holes as shown.



Punch one extra hole at the centre of the strip. This will be used to hang the strip.
Invert a styrofoam cup and poke a pen or pencil through it.
Then push the pencil through the 'hanging' hole. The strip should be balanced and be horizontal.

Twist open about a dozen paper clips so they can hang from the holes on your balance.



Test your balance. Hang a clip from 8 on one side and hang two clips from 4 on the other. does your strip balance? If it does, you have just illustrated that 8 = 4 + 4, or that $8 = 2 \times 4$.

1. Test your balance on these equations:
   (a) $6 = 3 + 3$ (b) $2 + 2 = 4$ (c) $2 + 3 = 5$ (d) $6 + 3 = 4 + 5$
   (e) $2 \times 3 = 6$ (f) $2 \times 5 = 10$ (g) $9 = 3 \times 3$ (h) $3 \times 4 = 12$ (How can you show 12?)

2. If your balance is working properly, you are now ready to solve some equations. Find the value of S in each equation.
   (a) $3 + S = 8$ (b) $9 = 7 + S$ (c) $5 + 7 = S + 9$ (d) $6 + 7 = 3 \times S + 1$
   (e) $3 + 5 = S + 2$ (f) $7 + S = 5 + 8$ (g) $6 \times S = 2 \times 9$ (h) $2 \times S = 3 \times T$ (Find S <u>and</u> T)

   Did you find more than one answer for (h)?

3. Find as many answers as you can for each of these.
   (a) $7 + S = T + 9$ (b) $2 \times 3 = S + T$ (c) $8 + S = 3 \times T$ (d) $S \times 9 = 3 \times T$
   (e) $6 \times S = 3 \times T$ (f) $S + 4 = 6 + T$ (g) $2 \times S = T + 7$ (h) $6 \times S = T + 10$

   Look for patterns in your answers for each equation.

Solutions & Notes

Activity 1: Square Numbers

BLMs 1 & 2: Square Numbers

1.

Size of square	1×1	2×2	3×3	4×4	5×5	6×6	7×7
Number of small squares needed	1	4	9	16	25	36	49

Students should realize that multiplying the dimensions of a given square will give the number of small squares needed. A 10×10 square would therefore need 100 squares.

Students may be surprised to discover that tables generated in #2, 3, 4, and 6 give the same set of "square numbers", as shown below.

2.

Length of base of triangle	1	2	3	4	5	6	7	10
Number of small triangles needed	1	4	9	16	25	36	49	100

3.

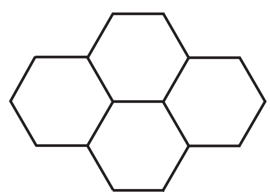
Length of base of rhombus	1	2	3	4	5	6	7	10
Number of small rhombuses needed	1	4	9	16	25	36	49	100

4.

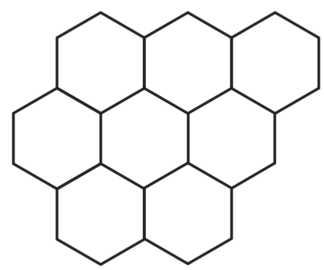
Length of base of triangle	1	2	3	4	5	6	7	10
Number of small triangles needed	1	4	9	16	25	36	49	100

5. As mentioned in the Activity notes, regular hexagons do not fit together to form other hexagons and will therefore not produce a square number pattern.

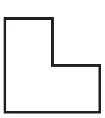
4 hexagons



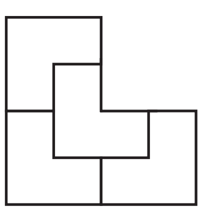
9 hexagons



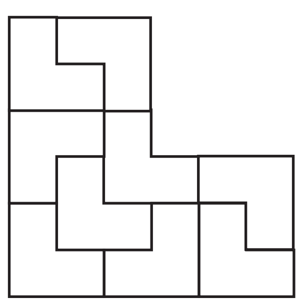
In the Activity notes for Family Activities there is an example of a non-regular hexagon that can produce the square numbers as shown below.



1



4



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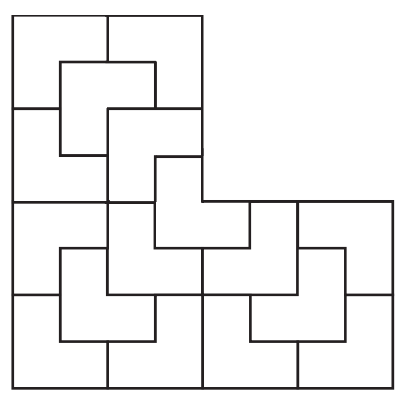
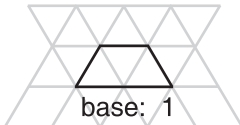


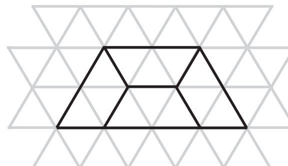
16

Solutions & Notes

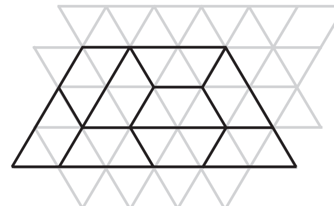
6. Trapezoids can be fitted together to make similar trapezoids as shown below. This is probably the most difficult of the problems.



4 trapezoids



9 trapezoids



Length of base of trapezoid	1	2	3	4	5	6	7	10
Number of small trapezoids needed	1	4	9	16	25	36	49	100

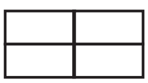
To illustrate 16 trapezoids making the next largest trapezoid, first construct 4 examples of the "4-trapezoid" figure above; then treat each of these as a single trapezoid and arrange these four "4-trapezoid" trapezoids together.

Throughout this activity it is important to give the students opportunities to describe, explain, and justify their number patterns and conclusions.

Extensions in Mathematics

1. 

1



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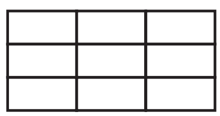

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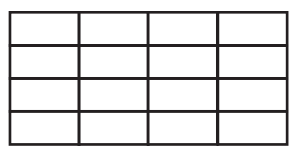
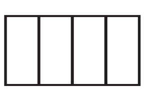
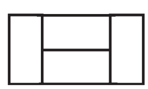


16

The length of each rectangle is twice the width:
1×2 2×4 3×6 4×8

Students may orient the original 1×2 rectangles differently. For example,

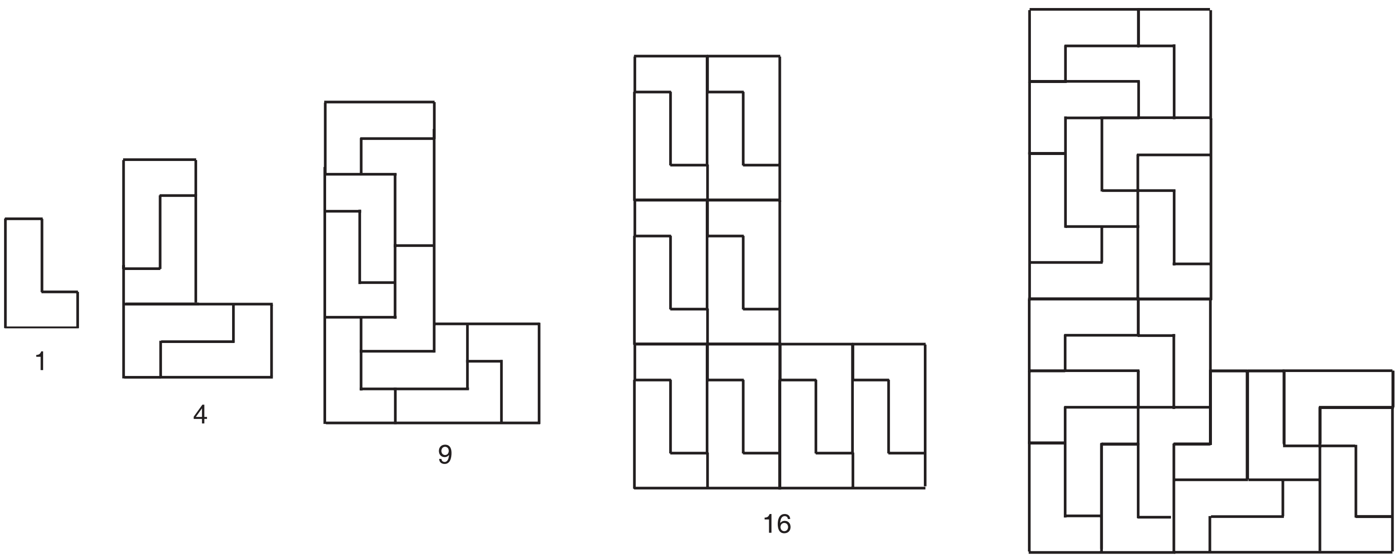
 or  or 

However, each of these "4-rectangle" rectangles is 2×4, and uses 4 of the original rectangles.

Family Activities

1. (See notes on previous page under BLMs 1 and 2, #5 for the solution for the first non-regular hexagon.) The second hexagon can also illustrate the square numbers, but it may be more difficult to fit the hexagons together to illustrate '9' and '25'. This is, however, an obvious pattern for '16'. Students might be interested in exploring '36' to see if there is a pattern in the way pieces fit together. (There is, in fact, a pattern, similar to the one for '16'.)

Solutions & Notes



1 4 9 16 25

Activity 2: Figurate Numbers

BLMs 11 & 12: Shapes of Numbers

1.

Term ——>	1st	2nd	3rd	4th	5th	10th	100th	nth
Triangular numbers	1	3	6	10	15	55	5050	$\frac{n(n+1)}{2}$
Square numbers	1	4	9	16	25	100	10000	$n \times n$
Pentagonal numbers	1	5	12	22	35	145	14 950	$\frac{3n^2 - n}{2}$
Hexagonal numbers	1	6	15	28	45	190	19 900	$2n^2 - n$
Heptagonal numbers	1	7	18	34	55	235	24 850	$\frac{5n^2 - 3n}{2}$
Octagonal numbers	1	8	21	40	65	280	29 800	$3n^2 - 2n$

The key to each row is to look for the pattern in the numbers added to obtain the next number. For example, in the pentagonal numbers, 4 is added to 1 to give 5, 7 is added to 5 to give 12, 10 is added to 12 to give 22, etc. So the pattern of added numbers is 4, 7, 10, …, i.e. they increase by 3 at each step.

As well as patterns leading across the rows, students might identify patterns in each column. For example, in the third column, the numbers go up by 3s; in the 5th column they go up by 10s, and in the 10th column they go up by 45s.

Solutions & Notes

BLM 13: Adding to Figurate Numbers

1.

Term	Triangular Numbers	Square Numbers	Pentagonal Numbers	Hexagonal Numbers	Heptagonal Numbers
1st	1 = 1	1 = 1	1 = 1	1 = 1	1 = 1
2nd	1 + 2 = 3	1 + 3 = 4	1 + 4 = 5	1 + 5 = 6	1 + 6 = 7
3rd	1 + 2 + 3 = 6	1 + 3 + 5 = 9	1 + 4 + 7 = 12	1 + 5 + 9 = 15	1 + 6 + 11 = 18
4th	1 + 2 + 3 + 4 = 10	1 + 3 + 5 + 7 = 16	1 + 4 + 7 + 10 = 22	1 + 5 + 9 + 13 = 28	1 + 6 + 11 + 16 = 34
5th	1 + 2 + 3 + 4 + 5 = 15	1 + 3 + 5 + 7 + 9 = 25	1 + 4 + 7 + 10 + 13 = 35	1 + 5 + 9 + 13 + 17 = 45	1 + 6 + 11 + 16 + 21 = 55
6th	1 + 2 + 3 + ... + 6 = 21	1 + 3 + 5 + ... + 11 = 36	1 + 4 + 7 + ... + 16 = 51	1 + 5 + 9 + ... + 21 = 66	1 + 6 + 11 + ... + 26 = 81
7th	1 + 2 + 3 + ... + 7 = 28	1 + 3 + 5 + ... + 13 = 49	1 + 4 + 7 + ... + 19 = 70	1 + 5 + 9 + ... + 25 = 91	1 + 6 + 11 + ... + 31 = 112

Possible student identified patterns:

- The numbers you add to get triangular numbers are always one more, but the numbers you add to get the square numbers are two more each time.
- You add whole numbers to get triangular numbers, but you add odd numbers to get square numbers.
- The triangular numbers go up by 2, then 3, then 4, then 5; the square numbers go up by 3, then 5, then 7, then 9; the pentagonal numbers go up by 4, then 7, then 10, then 13; and so on.
- The triangular numbers are odd, odd, even, even, odd, odd, …; the square numbers are odd, even, odd, even, odd, even, …; the pentagonal numbers are odd, odd, even, even, odd, odd, even, …; and so on.

Such patterns may be included in students' descriptions of similarities and differences for #2 on BLM 13.

A Challenge:

Two consecutive triangular numbers *will* always have a sum equal to a square number. The diagrams given are designed so that students can see this more readily.

Solutions & Notes

Activity 3: Addition Short-Cuts

Extensions in Mathematics

BLM 15: Short-Cuts to Calculation

1. Multiplying fives
   (a) Each answer ends in ‘25’.
   The rest of the product can be determined by multiplying the tens digit of the multiplier by one more than itself.
   Students may not identify both aspects of the pattern immediately, and may need to try the examples in (b) before determining the “short-cut”.

$$\begin{array}{r} 15 \\ \times 15 \\ \hline 225 \end{array} \qquad \begin{array}{r} 25 \\ \times 25 \\ \hline 625 \end{array} \qquad \begin{array}{r} 35 \\ \times 35 \\ \hline \end{array} \qquad \begin{array}{r} 45 \\ \times 45 \\ \hline 2025 \end{array}$$

‘2’ is 1×2 ‘6’ is 2×3 ‘12’ is 3×4 ‘20’ is 4×5

   (b)

$$\begin{array}{r} 55 \\ \times 55 \\ \hline 3025 \end{array} \qquad \begin{array}{r} 65 \\ \times 65 \\ \hline \end{array} \qquad \begin{array}{r} 75 \\ \times 75 \\ \hline 5625 \end{array} \qquad \begin{array}{r} 85 \\ \times 85 \\ \hline 7225 \end{array} \qquad \begin{array}{r} 95 \\ \times 95 \\ \hline 9025 \end{array}$$

   (c) Students should realize that this short-cut is valid only if multiplier and multiplicand (both factors) are identical and have ‘5’ in the ones’ column.
   Thus, the short-cut cannot be used for the second and fourth examples.

$$\begin{array}{r} 105 \\ \times 105 \\ \hline 11025 \end{array} \qquad \begin{array}{r} 75 \\ \times 65 \\ \hline 4875 \end{array} \qquad \begin{array}{r} 225 \\ \times 225 \\ \hline 50625 \end{array} \qquad \begin{array}{r} 65 \\ \times 45 \\ \hline 2925 \end{array} \qquad \begin{array}{r} 1005 \\ \times \quad 1005 \\ \hline 1010025 \end{array}$$

‘110’ is 10×11 ‘506’ is 22×23 ‘10100’ is 100×101

2. Units of 10
   (a) This pattern is similar to the one above.
   (b) Only the ones digits differ, but in each case their total is 10.
   Simply multiply them to get the two right-hand digits of the answer.
   Determine the rest of the answer as above.

$$\begin{array}{r} 27 \\ \times 23 \\ \hline \underbrace{6}_{2\times 3}\underbrace{21}_{3\times 7} \end{array} \qquad \begin{array}{r} 14 \\ \times 16 \\ \hline \underbrace{2}_{1\times 2}\underbrace{24}_{6\times 4} \end{array} \qquad \begin{array}{r} 63 \\ \times\ 67 \\ \hline \underbrace{42}_{6\times 7}\underbrace{21}_{7\times 3} \end{array} \qquad \begin{array}{r} 72 \\ \times\ 78 \\ \hline \underbrace{56}_{7\times 8}\underbrace{16}_{8\times 2} \end{array}$$

   (c)

$$\begin{array}{r} 92 \\ \times\ 98 \\ \hline 9016 \end{array} \qquad \begin{array}{r} 34 \\ \times\ 36 \\ \hline 1224 \end{array} \qquad \begin{array}{r} 81 \\ \times\ 89 \\ \hline 7209 \end{array} \qquad \begin{array}{r} 67 \\ \times\ 63 \\ \hline 4221 \end{array} \qquad \begin{array}{r} 44 \\ \times\ 46 \\ \hline 2024 \end{array}$$

Solutions & Notes

BLM 16: Patterns in Equations

1. Reversals
   (a) All the equations are true.
   (b) The digits in each number are reversed.
   (c) Two of these equations are not true. For such an equation to be true, it is essential that the products of the tens' digits and the ones' digits are the same, so that when they are reversed, the product does not change.

   For example, 12×84 or 24×63

   $1 \times 8 = 2 \times 4$ $\qquad$ $2 \times 6 = 4 \times 3$

   The two incorrect equations are $15 \times 95 = 51 \times 59$ and $37 \times 54 = 73 \times 45$

   $1 \times 9 \neq 5 \times 5$ $\qquad$ $3 \times 5 \neq 7 \times 4$

2. Backward Nines
   The problem here lies in the fact that most four-function calculators have an 8-digit display, so as students try to test their predictions they will discover that they will either have to abandon their calculators or find a way to break the question down so they can use their calculators.

   For example, the equation $109999989 \times 9 = 989999901$ will overload most calculators, so it can't be shown right or wrong. However, if the multiplication is broken down into, say, 9999989×9 and $100\,000\,000 \times 9$, then we can determine that 9999989×9 equals 89 999 901, and we can multiply the other part mentally and get 900 000 000.
   Then the solution is

   $$\begin{array}{r} 900\,000\,000 \\ +\ 89\,999\,901 \\ \hline 989\,999\,901 \end{array}$$

   so the equation is true.

3. Lots of Ones
   The calculator will overload on the next equation in the sequence after the ones given on BLM 15.
   One way it could be broken down is shown below.

   11111×11111
   $= 11111 \times (11000 + 111)$
   $= 11111 \times 11000 + 11111 \times 11 \leftarrow$ Use the calculator for the second part
   $= 11111 \times 1 \times 1000 + 1233321 \leftarrow$ Use the calculator for 11111×11 and multipy by 1000 mentally
   $= 122221 \times 1000 + 1233321$
   $= 122221000 + 1233321$
   $= 123454321$

Solutions & Notes

BLM 17: Number Loops

Cyclic number patterns are a way of exploring numbers, while practicing basic skills and applying some number sense.
The analysis of the problems will be a challenge for late junior students, but they should be encouraged to try.
There is only one 'loop' for #1 and 2, but there are 3 loops for #3 and 4.
In #4, two of the loops are trivial, consisting simply of 0 or 1 repeated.
For example, $5 \to 25 \to 10 \to 0 \to 0 \to 0$.

1. (d) Both 12 and 13 end in the loop $4 \to 2 \to 1 \to 2 \to 1$ which can be written differently to show the "loop": $4 \to 2 \to 1$.

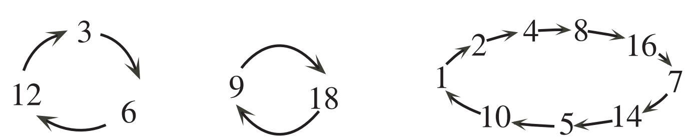
   (e) $25 \to 26 \to 13^* \to 14 \to 7 \to 8 \to 4 \to 2 \to 1 \to 2 \to 1$
   $112 \to 56 \to 28 \to 14^* \to 7 \to 8 \to 4 \to 2 \to 1 \to 2 \to 1$
   $88 \to 44 \to 22 \to 11 \to 12^* \to 6 \to 3 \to 4 \to 2 \to 1 \to 2 \to 1$
   $101 \to 102 \to 51 \to 52 \to 26 \to 13^* \to 14 \to 7 \to 8 \to 4 \to 2 \to 1 \to 2 \to 1$

   * Students should be able to predict the results from this point, based on the examples given.

2. Further examples are given below:
   $38 \to 19 \to 40 \to 20 \to 10 \to 5 \to 12 \to 6 \to 3 \to 8 \to 4 \to 2 \to 1 \to 4 \to 2 \to 1 \ldots$
   $70 \to 35 \to 72 \to 36 \to 18 \to 9 \to 20^* \to 10 \to 5 \to 12 \to 6 \to 3 \to 8 \to 4 \to 2 \to 1 \to 4 \to 2 \to \ldots$
   $134 \to 67 \to 136 \to 68 \to 34 \to 17 \to 36^* \to 18 \to 9 \to 20 \to 10 \to 5 \to 12 \to 6 \to 3 \to 8 \to 4 \to 2 \to 1 \to \ldots$
   $98 \to 49 \to 100 \to 50 \to 25 \to 52 \to 26 \to 13 \to 28 \to 14 \to 7 \to 16 \to 8^* \to 4 \to 2 \to 1 \to 4 \to \ldots$
   $200 \to 100^* \to 50 \to 25 \to 52 \to 26 \to 13 \to 28 \to 14 \to 7 \to 16 \to 8 \to 4 \to 2 \to 1 \to 4 \to \ldots$

   Some students may notice that the rules in #1 and #2 are almost the same. For example,
   Rules for #1:
   $$38 \xrightarrow{\text{divide by 2}} 19 \xrightarrow{\text{add 1}} 20 \xrightarrow{\text{divide by 2}} 10 \xrightarrow{\text{divide by 2}} 5 \xrightarrow{\text{add 1}} 6$$
   Rules for #2:
   $$38 \xrightarrow{\text{divide by 2}} 19 \xrightarrow{\text{add 1 and multiply by 2}} 40 \xrightarrow{\text{divide by 2}} 20 \xrightarrow{\text{divide by 2}} 10 \xrightarrow{\text{divide by 2}} 5 \xrightarrow{\text{add 1 and multiply by 2}} 12 \xrightarrow{\text{divide by 2}} 6$$
   There is an extra step for the #2 rules when the number is an odd number.

3. Further examples are given below:
   $96 \to 15 \to 6 \to 12 \to 3 \to 6 \to 12 \to 3 \to \ldots$
   $39 \to 12 \to 3 \to 6 \to 12 \to 3 \to 6 \to 12 \to \ldots$
   $41 \to 5 \to 10 \to 1 \to 2 \to 4 \to 8 \to 16 \to 7 \to 14 \to 5 \to 10 \to 1 \to 2 \to 4 \to 8 \to \ldots$
   $90 \to 9 \to 18 \to 9 \to 18 \to 9 \to 18 \to \ldots$
   These examples show all three possible loops:



   For analysis, see notes following #4.

Solutions & Notes

4. Further examples are given below:

$63 \to 18 \to 8 \to 64 \to 24 \to 8 \to 64 \to 24 \to 8 \to \ldots$

$37 \to 21 \to 2 \to 4 \to 16 \to 6 \to 36 \to 18 \to 8 \to 64 \to 24 \to 8 \to 64 \to \ldots$

$89 \to 72 \to 14 \to 4 \to 16 \to 6 \to 36 \to 18 \to 8 \to 64 \to 24 \to \ldots$

$49 \to 36 \to 18 \to 8 \to 64 \to 24 \to 8 \to \ldots$

$54 \to 20 \to 0 \to 0 \to 0 \to \ldots$

Ask students why a 'zero loop' occurs. Obviously whenever a multiple of 10 appears, the string of numbers will end in a loop of zero. Backing up one step from that, we see that we are going to multiply by 5 at some point in order to get that multiple of ten. Students may wish to pursue the sequence backwards a bit further to try to decide what kinds of numbers will end in a zero loop. Similarly, students may explore the numbers that lead to a loop of 1. In order to get a loop of 1, we must first get to 1 as the product of digits. Since this is impossible (with one exception), the only number that will lead to a loop of 1 is 1 itself.

This leads to a technique for determining the number of possible loops. Notice that in all of the sequences, the loops include single digit numbers. That means we can check every single-digit number (0, 1, 2, …, 9) to determine what loop it leads to. This should give us the number of different loops. For example, in #4, such a procedure shows that 1 loops to itself, 5 and 0 loop to zero, and 2, 3, 4, 6, 7, 8, 9 all end in the 8, 64, 24 loop.

For #3, there is a zero loop if we start at 0.
Starting at 1 gives $1 \to 2 \to 4 \to 8 \to 16 \to 7 \to 14 \to 5 \to 10 \to 1 \to 2 \to 4 \to \ldots$
Since 1, 2, 4, 8, 7, and 5 are part of this loop, they will obviously all lead to this second loop.
For 3, we get $3 \to 6 \to 12 \to 3 \to 6 \to 12 \to \ldots$
This loop will be the same for 6.
For 9, we get $9 \to 18 \to 9 \to 18 \to 9 \to \ldots$

Thus, if we include the 'zero loop', there are four different loops generated by the #3 rules.

If you wish to extend the activity further, give students the following problem:
Begin with any number. Count the number of letters in its name. That number is the second number in the sequence. Repeat.
Example, 7 (seven) $\to$ 5 (five) $\to$ 4 (four) $\to$ 4 (four) $\to$ 4 …
Students will enjoy exploring this one, being convinced that if they begin with a long number name they will not end at 4. However, …

e.g. $137 \to 21 \to 9 \to 4$

The interesting thing is, that minor spelling mistakes may change the sequence but not the result.

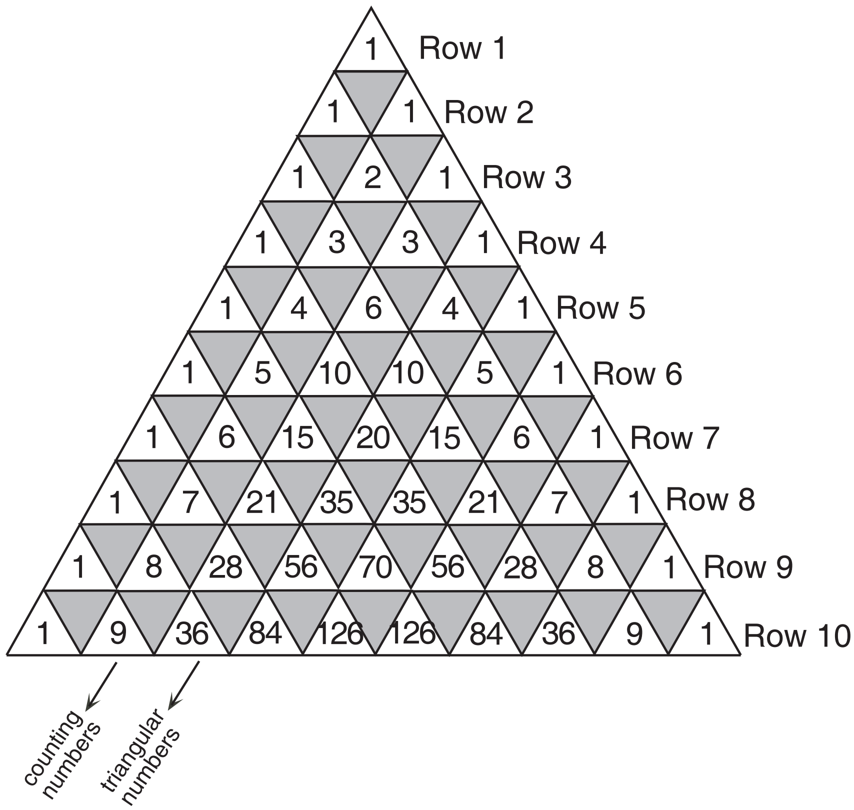
e.g. 42 (fourty-two) $\to 9 \to 4$ vs 42 (forty-two) $\to 8 \to 5 \to 4$

Solutions & Notes

Activity 4: Pascal's Triangle

BLM 18: Pascal's Triangle

1.



Sum of Each Row (#2, BLM 18)	Cummultive Sums of Rows (#1, BLM 19)
1	1
2	3
4	7
8	15
16	31
32	63
64	127
128	255
256	511
512	1023

Notice the sum for row n is one more than the cummulative total for row $n - 1$.

2.

Row No.	1	2	3	4	5	6	7	8	9	10
Sum	1	2	4	8	16	32	64	128	256	512

3. (a), (b)

2^1	2^2	2^3	2^4	2^5	2^6	2^7	2^8	2^9
2	4	8	16	32	64	128	256	512

3(b) Compare these two rows.

(c) (i) 2^9 gives the sum for Row 10

(ii) 2^{10} gives the sum for Row 11

(iii) 2^{11} gives the sum for Row 12

As a generalization, 2^n gives the sum for Row $(n + 1)$.

Solutions & Notes

BLM 19: Pascal and Sums

1. (See the table given for BLM 18 #1 above.)

Rows	1 and 2	1, 2, and 3	1 to 4	1 to 5	1 to 6	1 to 7
Sum	3	7	15	31	63	127

2.

'4' as the sum of whole numbers	'5' as the sum of whole numbers	'6' as the sum of whole numbers
1 + 1 + 1 + 1 1 + 1 + 2 1 + 2 + 1 2 + 1 + 1 2 + 2 1 + 3 3 + 1	1 + 1 + 1 + 1 + 1 1 + 1 + 1 + 2 1 + 1 + 2 + 1 1 + 2 + 1 + 1 2 + 1 + 1 + 1 1 + 1 + 3 1 + 3 + 1 3 + 1 + 1 1 + 2 + 2 2 + 1 + 2 2 + 2 + 1 2 + 3 3 + 2 1 + 4 4 + 1	1 + 1 + 1 + 1 + 1 + 1 1 + 1 + 1 + 1 + 2 1 + 1 + 1 + 2 + 1 1 + 1 + 2 + 1 + 1 1 + 2 + 1 + 1 + 1 2 + 1 + 1 + 1 + 1 1 + 1 + 1 + 3 1 + 1 + 3 + 1 1 + 3 + 1 + 1 3 + 1 + 1 + 1 1 + 1 + 4 1 + 4 + 1 4 + 1 + 1 1 + 5 5 + 1 2 + 2 + 2 2 + 4 4 + 2 3 + 3 1 + 2 + 3 1 + 3 + 2 2 + 3 + 1 2 + 1 + 3 3 + 1 + 2 3 + 2 + 1 1 + 1 + 2 + 2 1 + 2 + 1 + 2 1 + 2 + 2 + 1 2 + 1 + 1 + 2 2 + 1 + 2 + 1 2 + 2 + 1 + 1
There are 7 different ways	There are 15 different ways	There are 31 different ways

Solutions & Notes

BLM 20: Pascal and Heads and Tails

1. For these problems, students may find it easier to picture sets of different coins. For example, if they consider flipping a penny and a nickel, in order,
   then HT will mean heads for the penny, and tails for the nickel.
   while TH will mean tails for the penny, and heads for the nickel.

2.

(i) 1 way	(ii) 3 ways	(iii) 3 ways	(iv) 1 way
HHH	HHT	HTT	TTT
	HTH	THT	
	THH	TTH	

3. Students should see that the numbers correspond. They should also note that Row 3 gives the outcomes for two coins.

4. Row 5 will give the results for 4 coins.

(i) 1 way	(ii) 4 ways	(iii) 6 ways	(iv) 4 ways	(v) 1 way
HHHH	HHHT	HHTT	HTTT	TTTT
	HHTH	HTHT	THTT	
	HTHH	HTTH	TTHT	
	THHH	THHT	TTTH	
		THTH		
		TTHH		
↑	↑	↑	↑	↑
Compare to sums of 4 with four 1s (i.e. four identical addends)	Compare to sums of 5 with three 1s and one 2.	Compare to sums of 6 with two 1s and two 2s.	Compare to sums of 5 with three 1s and one 2.	Compare to sums of 4 with four 1s.

   Row 6 will give the results:
   1 way of getting 5 heads;
   5 ways of getting 4 heads, 1 tail;
   10 ways of getting 3 heads, 2 tails;
   10 ways of getting 2 heads, 3 tails;
   5 ways of getting 1 head, 4 tails;
   1 way of getting 5 tails.

Solutions & Notes

BLM 21: Pascal and Pathways

1. (b)

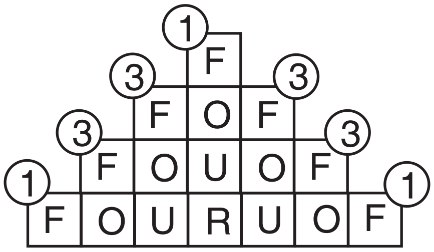


Figure 4.3

1. (c)

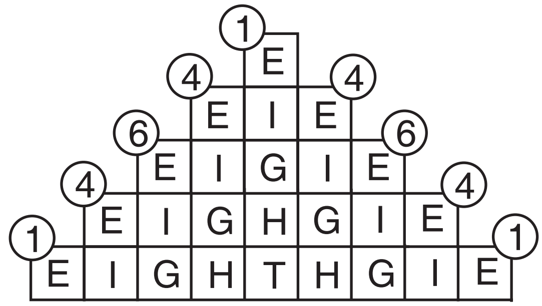
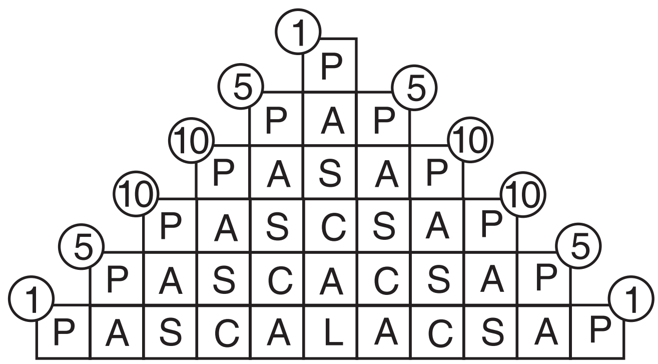


Figure 4.4

(d) The numbers in Figure 4.3 are from Row 4 of Pascal's Triangle. The numbers in Figure 4.4 are from Row 5.

2. (a) Since there are 6 rows in the table, the numbers needed will come from Row 6 of Pascal's Triangle. Adding all the numbers in the circles gives 63. There are 63 ways of spelling PASCAL by going only to the right or left or down.



Activity 5: Tin Cans and Equations

BLM 22: Tins and Counters

The number of counters in each tin is given below.

1. 2 counters
2. 3 counters
3. 2 counters
4. 4 counters
5. 4 counters
6. 3 counters
7. 3 counters
8. 4 counters

Solutions & Notes

BLM 23: Counters and Equations

The number of counters represented by each square is

1. (a) 4 (d) 4 (g) 1 (j) 2
   (b) 2 (e) 3 (h) 5 (k) 4
   (c) 2 (f) 5 (i) 3

2. (i) matches (d): □ □ ● = ● ● ● ● ● ● ● ● ●

   $S + S + 1 = 9$

   (ii) matches (e): ● □ □ = ● ● ● ● ● ● ●

   $1 + S + S = 7$

   (iii) matches (b): □ □ □ = ● ● ● ● ● ●

   $S + S + S = 6$

   (iv) matches (c): □ ● ● = ● ● ● ●

   $S + 2 = 4$

3. (h) ● □ □ = ● □ ● ● ● ● ● $1+2S=1+S+5$ or $2S+1=S+6$ or $2S+1=6+S$

   (i) □ □ □ = □ ● ● □ ● $3S=S+2+S+1$ or $3S=2S+3$ or $3S=3+2S$

   (j) ● □ □ = □ ● ● ● $1+2S=S+3$ or $2S+1=S+3$ or $1+2S=3+S$

   (k) □ □ ● = ● □ ● ● ● ● $2S+1=1+S+4$ or $2S+1=S+5$ or $2S+1=5+S$

There are several possible answers for each equation. It may be necessary to check them individually. It is worthwhile to record several variations for each so students can see that there is no one correct way to write an equation.

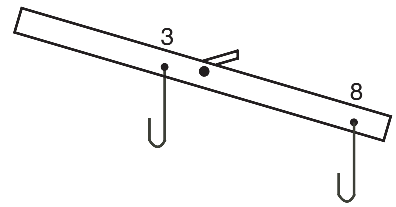
4. (a) $S=3$ (b) $S=6$ (c) $S=5$ (d) $S=4$

Solutions & Notes

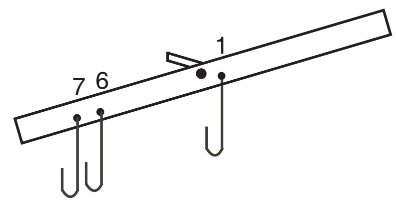
BLM 24: Balanced Equations

If students measure the distances between the holes accurately, they will find they have built a balance that will help them solve simple equations.

2. To solve (a), a student should hang a clip at ‘3’ on one side of the balance, and another clip at ‘8’ on the other side of the balance. To solve for ‘S’ the student must decide where to hang a clip so that the strip will balance.



To solve for (d), a student should hang clips at ‘6’, ‘7’ and ‘1’ as illustrated. To solve for ‘S’ the student must hang three clips ($3S$) at one spot so that the strip is balanced.



(a) $S = 5$	(b) $S = 2$	(c) $S = 3$	(d) $S = 4$	
(e) $S = 6$	(f) $S = 6$	(g) $S = 3$	(h) $S = 6$, $T = 4$	or $S = 9$, $T = 6$

3. One solution for each equation is given here. Students should be prepared to justify their solutions.
   (a) $S = 3$, $T = 1$. The difference between S and T ($S - T$) should always be 2.
   (b) $S = 1$, $T = 5$. The sum of S and T should always be 6.
   (c) $S = 1$, $T = 3$. The value of S and T must be both odd or both even (if S and T are whole numbers).
   (d) $S = 2$, $T = 6$. The value of T should be three times the value of S.
   (e) $S = 1$, $T = 2$. The value of T should be twice the value of S.
   (f) $S = 3$, $T = 1$. The difference $S - T$ should be 2.
   (g) $S = 4$, $T = 1$. If S and T are to have whole number values, then T must be odd.
   (h) $S = 2$, $T = 2$. If S and T are to have whole number values, then T must be even.

Suggested Assessment Strategies

Investigations

Investigations involve explorations of mathematical questions that may be related to other subject areas. Investigations deal with problem posing as well as problem solving. Investigations give information about a student's ability to:

- identify and define a problem;
- make a plan;
- create and interpret strategies;
- collect and record needed information;
- organize information and look for patterns;
- persist, looking for more information if needed;
- discuss, review, revise, and explain results.

Journals

A journal is a personal, written expression of thoughts. Students express ideas and feelings, ask questions, draw diagrams and graphs, explain processes used in solving problems, report on investigations, and respond to open-ended questions. When students record their ideas in math journals, they often:

- formulate, organize, internalize, and evaluate concepts about mathematics;
- clarify their thinking about mathematical concepts, processes, or questions;
- identify their own strengths, weaknesses, and interests in mathematics;
- reflect on new learning about mathematics;
- use the language of mathematics to describe their learning.

Observations

Research has consistently shown that the most reliable method of evaluation is the ongoing, in-class observation of students by teachers. Students should be observed as they work individually and in groups. Systematic, ongoing observation gives information about students':

- attitudes towards mathematics;
- feelings about themselves as learners of mathematics;
- specific areas of strength and weakness;
- preferred learning styles;
- areas of interest;
- work habits — individual and collaborative;
- social development;
- development of mathematics language and concepts.

In order to ensure that the observations are focused and systematic, a teacher may use checklists, a set of questions, and/or a journal as a guide. Teachers should develop a realistic plan for observing students. Such a plan might include opportunities to:

- observe a small number of students each day;
- focus on one or two aspects of development at a time.

Suggested Assessment Strategies

Student Self-Assessment

Student self-assessment promotes the development of metacognitive ability (the ability to reflect critically on one's own reasoning). It also assists students to take ownership of their learning, and become independent thinkers. Self-assessment can be done following a co-operative activity or project using a questionnaire which asks how well the group worked together. Students can evaluate comments about their work samples or daily journal writing. Teachers can use student self-assessments to determine whether:

- there is change and growth in the student's attitudes, mathematics understanding, and achievement;
- a student's beliefs about his or her performance correspond to his/her actual performance;
- the student and the teacher have similar expectations and criteria for evaluation.

Resources for Assessment

"For additional ideas, see annotated Other Resources list on page 72, numbered as below."

1. The Ontario Curriculum Grades 1-8: Mathematics, Ministry of Education and Training, 1997.

2. "Linking Assessment and Instruction in Mathematics: Junior Grades" by OAME/OMCA, Crompton et al, 1996.
   The document provides a selection of open-ended problems tested in grades 4, 5, and 6. Performance Rubrics are used to assess student responses (which are included) at four different levels. Problems could be adapted for use at the Junior Level. Order from OAME/AOEM, P.O. Box 96, Rosseau, Ont., P0C 1J0. Phone/Fax 705-732-1990.

3. Mathematics Assessment: Myths, Models, Good Questions, and Practical Suggestions, by Jean Karr Stenmark (Ed.), NCTM, 1991.
   This book contains a variety of assessment techniques and gives samples of student work at different levels. Order from Frances Schatz, 56 Oxford Street, Kitchener, Ont., N2H 4R7. Phone 519-578-5948; Fax 519-578-5144. email: frances.schatz@sympatico.ca

4. "Assessment", Arithmetic Teacher Focus Issue, February 1992, NCTM.
   This copy of NCTM's journal for elementary school addresses several issues dealing with assessment. It also includes suggested techniques and student activities.

5. How to Evaluate Progress in Problem Solving, by Randall Charles et al., NCTM, 1987.
   Suggestions for holistic scoring of problem solutions include examples of student work. Also given are ways to vary the wording of problems to increase/decrease the challenge. A section on the use of multiple choice test items shows how these, when carefully worded, can be used to assess student work.

Suggested Assessment Strategies

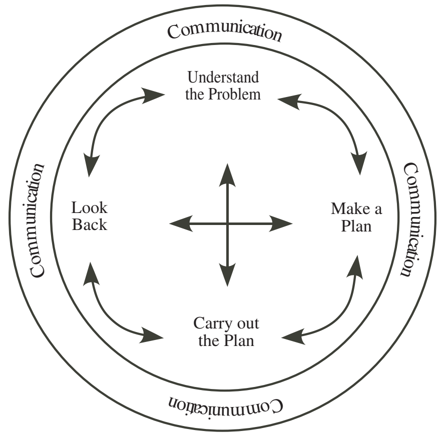
A GENERAL PROBLEM SOLVING RUBRIC

This problem solving rubric uses ideas taken from several sources. The relevant documents are listed at the end of this section.

"US and the 3 R's"

There are five criteria by which each response is judged:
Understanding of the problem,
Strategies chosen and used,
Reasoning during the process of solving the problem,
Reflection or looking back at both the solution and the solving, and
Relevance whereby the student shows how the problem may be applied to other problems, whether in mathematics, other subjects, or outside school.

Although these criteria can be described as if they were isolated from each other, in fact there are many overlaps. Just as communication skills of one sort or another occur during every step of problem solving, so also reflection does not occur only after the problem is solved, but at several points during the solution. Similarly, reasoning occurs from the selection and application of strategies through to the analysis of the final solution. We have tried to construct the chart to indicate some overlap of the various criteria (shaded areas), but, in fact, a great deal more overlap occurs than can be shown. The circular diagram that follows (from OAJE/OAME/OMCA "Linking Assessment and Instruction in Mathematics", page 4) should be kept in mind at all times.



There are four levels of response considered:

Level 1: Limited identifies students who are in need of much assistance;

Level 2: Acceptable identifies students who are beginning to understand what is meant by 'problem solving', and who are learning to think about their own thinking but frequently need reminders or hints during the process.

Level 3: Capable students may occasionally need assistance, but show more confidence and can work well alone or in a group.

Level 4: Proficient students exhibit or exceed all the positive attributes of the **Capable** student; these are the students who work independently and may pose other problems similar to the one given, and solve or attempt to solve these others.

Suggested Assessment Strategies

LEVEL OF RESPONSE →

CRITERIA FOR ASSESSMENT ↓

	Level 1: Limited	Level 2: Acceptable	Level 3: Capable	Level 4: Proficient
UNDERSTANDING	• requires teacher assistance to interpret the problem • fails to recognize all essential elements of the task	• shows partial understanding of the problem but may need assistance in clarifying	• shows a complete understanding of the problem	• shows a complete understanding of the problem
STRATEGIES	• needs assistance to choose an appropriate strategy	• identifies an appropriate strategy	• identifies an appropriate strategy	• identifies more than one appropriate strategy
	• applies strategies randomly or incorrectly • does not show clear understanding of a strategy[1] • shows no evidence of attempting other strategies	• attempts an appropriate strategy, but may not complete it correctly[2] • tries alternate strateges with prompting	• uses strategies effectively • may attempt an inappropriate strategy, but eventually discards it and tries another without prompting	• chooses and uses strategies effectively[3] • recognizes an inappropriate strategy quickly and attempts others without prompting
REASONING	• makes major mathematical errors • uses faulty reasoning and draws incorrect conclusions • may not complete a solution	• may present a solution that is partially incorrect	• produces a correct and complete solution, possibly with minor errors	• produces a correct and complete solution, and may offer alternative methods of solution
	• describes[4] reasoning in a disorganized fashion, even with assistance • has difficulty justifying[5] reasoning even with assisstance	• partially describes[4] a solution and/or reasoning or explains fully with assistance • justification[5] of solution may be inaccurate, incomplete or incorrect	• is able to describe[4] clearly the steps in reasoning; may need assistance with mathematical language • can justify[5] reasoning if asked; may need assistance with language	• explains reasoning in clear and coherent mathematical language • justifies[5] reasoning using appropriate mathematical language
REFLECTION	• shows no evidence of reflection or checking of work • can judge the reasonableness of a solution only with assistance	• shows little evidence of reflection or checking of work • is able to decide whether or not a result is reasonable when prompted to do so	• shows some evidence of reflection and checking of work • indicates whether the result is reasonable, but not necessarily why	• shows ample evidence of reflection and thorough checking of work • tells whether or not a result is reasonable, and why
RELEVANCE	• unable to identify similar[6] problems	• unable to identify similar[6] problems	• identifies similar[6] problems with prompting	• identifies similar[6] problems, and may even do so before solving the problem
	• unlikely to identify extensions[7] or applications of the mathematical ideas in the given problem, even with assistance	• recognizes extensions[7] or applications with prompting	• can suggest at least one extension[7], variation, or application of the given problem if asked	• suggests extensions[7], variation, or applications of the given problem independently

Suggested Assessment Strategies

Notes on the Rubric

1. For example, diagrams, if used, tend to be inaccurate and/or incorrectly used.

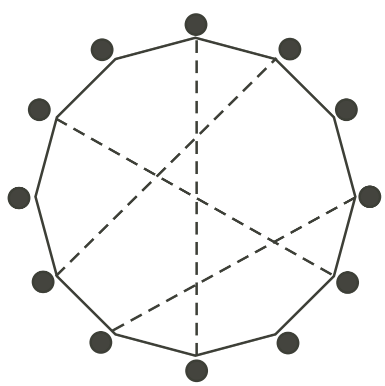
2. For example, diagrams or tables may be produced but not used in the solution.

3. For example, diagrams, if used, will be accurate models of the problem.

4. To *describe* a solution is to tell *what* was done.

5. To *justify* a solution is to tell *why* certain things were done.

6. *Similar* problems are those that have similar structures, mathematically, and hence could be solved using the same techniques.
   For example, of the three problems shown below right, the better problem solver will recognize the similarity in structure between Problems 1 and 3. One way to illustrate this is to show how both of these could be modelled with the same diagram:



Problem 1: There were 8 people at a party. If each person shook hands once with each other person, how many handshakes would there be? How many handshakes would there be with 12 people? With 50?

Problem 2: Luis invited 8 people to his party. He wanted to have 3 cookies for each person present. How many cookies did he need?

Problem 3: How many diagonals does a 12-sided polygon have?

   Each dot represents one of 12 people and each dotted line represents either a handshake between two people (Problem 1, second question) or a diagonal (Problem 3).

   The weaker problem solver is likely to suggest that Problems 1 and 2 are similar since both discuss parties and mention 8 people. In fact, these problems are alike only in the most superficial sense.

7. One type of extension or variation is a “what if...?” problem, such as “What if the question were reversed?”, “What if we had other data?”, “What if we were to show the data on a different type of graph?”.

Suggested Assessment Strategies

Suggested Adapted Rubric for Activity 2, BLMs 11, 12, 13

The rubric below has been adapted for the problems on BLMs 11, 12, 13 for Activity 2: Figurate Numbers. This rubric considers the understanding of the problem, the selection and application of strategies, reasoning, and reflection.

Level 1: Limited	Level 2: Acceptable	Level 3: Capable	Level 4: Proficient
• needs assistance to understand the problem	• shows partial understanding of the problem but may need assistance with some aspect of the problem	• shows a complete understanding of the problem	• shows a complete understanding of the problem
• identifies patterns in one or two columns, but not patterns for each row	• identifies patterns in most columns, and in rows for the first 5 numbers	• identifies several patterns in the completed table; identifies patterns in most rows	• identifies several patterns in the table, both in the columns and the rows
• patterns identified are random or incorrect	• patterns identified are valid for the first few numbers only	• identifies patterns for most rows that help to identify the 10th and 100th numbers	• patterns identified for all rows will be valid and help to identify the 10th and 100th numbers
• uses faulty reasoning and draws incorrect conclusions	• patterns identified may have minor errors	• produces a correct and complete solution	• produces a correct and complete solution; may offer more than one pattern for each set of numbers
• describes patterns with assistance; is unable to justify them	• describes patterns with minor assistance; has difficulty justifying them	• describes patterns clearly; may need some assistance in justifying them attempts to use mathematical language	• describes and justifies patterns using appropriate mathematical language
• does not attempt to determine the 100th or the *n*th numbers	• may try to write the 100th term of some sequences	• attempts to determine the 100th number for most sequences; may try to determine an expression for the *n*th number for some sequences	• completes the 100th number column; determines some of the expressions for the *n*th numbers

Other Resources

1. *Patterns, Addenda Series, K-6*, Miriam A Leiva (Ed.), 1993, NCTM

   Detailed notes for activities include patterns on a hundred chart, patterns built with manipulatives, patterns on the calculator, 'What's My Rule?' games, and tiling patterns. Three or more activities for each grade are given. Some Black Line Masters are included.

2. *Patterns and Functions, Addenda Series, Grades 5-8*, Frances R. Curcio (Ed.), 1991, NCTM

   This booklet contains problems for grades 5 to 8 dealing with various aspects of pattern and algebra. Topics include counting patterns, measurement and geometric patterns, patterns with fractions and repeating decimals, and the graphing of patterns.

3. *Multicultural Math Posters and Activities*, Wayne Peterson (Ed.), 1984, NCTM

   This package contains eighteen different 28 cm by 42 cm posters along with a book of suggestions for use. Some of the patterns explored are patterns in early number systems, Japanese and Arabic geometric patterns , Pascal's Triangle, and Russian Peasant multiplication (See Gr. 4 *Investigations in Pattern and Algebra*)

4. *Multicultural Mathematics Materials*, Marina C. Krause, 1983, NCTM

   This booklet contains Black Line Masters and Teacher Notes concerning patterns from several different areas, including Egypt, Japan, China, the Philippines, Norway, Peru, and ancient Rome, as well as several from the American Indians of south-west U.S.A.

5. "Math Magic", by Hari P. Koirala and Phillip M. Goodwin, pp 562-566, *Mathematics Teaching in the Middle School* , May 2000, NCTM

   The article describes grade 5 and 6 students' reasoning about problems such as "Think of a number; add 5; multiply by 3; subtract 3; divide by 3: subtract your original number" and how they used algebraic notation to explain the "magic" in such problems.

6. "Responses to the "What Are the Clues? Patterns in More Than One Direction Problem", pp115-117, *Teaching Children Mathematics*, October 1999, NCTM

   The article presents responses to a computation problem using patterns that was presented in an earlier version of the journal. Teachers have described the variety of ways their students dealt with the problem and how they extended the problem further.

7. "Algebraic Thinking", Focus Issue of *Teaching Children Mathematics*, February 1977, NCTM

   Articles describe the use of spreadsheets to study patterns, explore ways students of different ages deal with a problem using square tiles to develop a pattern, illustrate the use of Logo and function machines, and suggest a way to introduce to students the use of variables (e.g., the letter 'x' or 'n' to represent a number).

Other Resources

8. "Algebraic Thinking: Opening the Gate", Focus Issue of *Mathematics Teaching in the Middle Grades*, February 1997, NCTM

   This volume contains articles on Exploring Patterns in Nonroutine Problems, Building Equations using M&Ms, and Exploring Algebraic Patterns through Literature (e.g., *Anno's Magic Seeds*)

9. "Using Spreadsheets" by Michael T. Battista and Caroline Van Auken Borrow, pp 470-8, *Teaching Children Mathematics,* April 1998, NCTM

   The article describes the results when problems similar to the "Math Magic" type problem described elsewhere on this page were given to grade 5 students, who used spreadsheets to determine the patterns and use algebraic notation to write generalizations.