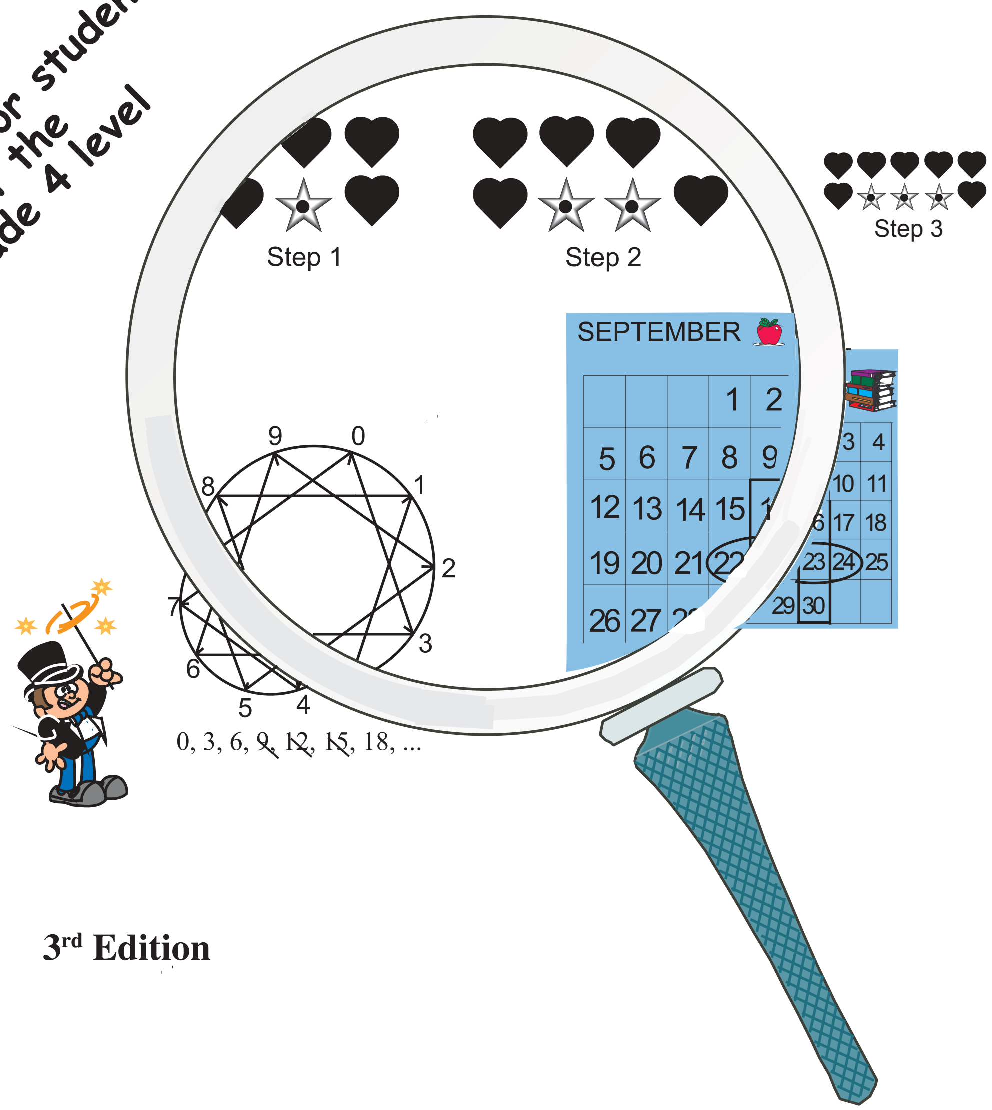
Invitations to Mathematics

Investigations in Patterns and Algebra

"What's the Pattern?"

Suggested for students at the Grade 4 level



3rd Edition

An activity of
The CENTRE for EDUCATION in MATHEMATICS and COMPUTING
Faculty of Mathematics, University of Waterloo
Waterloo, Ontario, Canada N2L 3G1

Preface

The Centre for Education in Mathematics and Computing at the University of Waterloo is dedicated to the development of materials and workshops that promote effective learning and teaching of mathematics. This unit is part of a project designed to assist teachers of Grades 4, 5, and 6 in stimulating interest, competence, and pleasure in mathematics among their students. While the activities are appropriate for either individual or group work, the latter is a particular focus of this effort. Students will be engaged in collaborative activities which will allow them to construct their own meanings and understanding. This emphasis, plus the extensions and related activities included with individual activities/projects, provide ample scope for all students’ interests and ability levels. Related “Family Activities” can be used to involve the students’ parents/care givers.

Each unit consists of a sequence of activities intended to occupy about one week of daily classes; however, teachers may choose to take extra time to explore the activities and extensions in more depth. The units have been designed for specific grades, but need not be so restricted. Activities are related to the Ontario Curriculum but are easily adaptable to other locales.

“Investigations in Patterns and Algebra” is comprised of activities which explore numeric and geometric patterns in both mathematical and everyday settings, including, for example, architectural structures, biology, and experimental data. The recognition, description, and extension of patterns is arguably one of the most fundamental skills needed in mathematics, or for any problem solving situation; the activities in this unit are aimed specifically at developing this skill.

Acknowledgements

Contributing Teachers

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Jan Scully (York Region Board of Education/OISE/University of Toronto)
Mary Thompson (University of Waterloo)

Authors/Co-editors

Bev Marshman (University of Waterloo)
Lorna Morrow (Mathematics Consultant)

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Overview

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Common Beliefs

These activities have been developed within the context of certain beliefs and values about mathematics generally, and pattern and algebra specifically. Some of these beliefs are described below.

Mathematics is the science of patterns. Recognizing, describing and generalizing patterns are thus key to developing students' understanding of mathematics, and also provide powerful tools for problem solving.

Patterns can be investigated through concrete materials, number tables, and experiments, and in a wide variety of contexts, such as music, art, natural science, and architecture. Patterns inherent in cultural artifacts have a relevance for students that helps them realize the importance of patterns, and motivates them to explore patterns further and to make connections between and among patterns. Verbalizing such patterns through descriptions, analysis, and predictions helps lead to the understanding necessary for generalizations expressed in the language of algebra.

As students attempt to justify their choices and patterns they develop a facility with, and understanding of, both the nature of proof and the language of mathematics.

Essential Content

The activities in this unit allow students to discover how patterns arise in a variety of mathematical and everyday contexts, and to establish the rules which govern them. In addition, there are Extensions in Mathematics, Cross-Curricular Activities and Family Activities. These may be used prior to or during the activity as well as following the activity. They are intended to suggest topics for extending the activity, assisting integration with other subjects, and involving the family in the learning process.

During this unit, the student will:

- identify and compare patterns in arrays of numbers, such as calendars and number charts;
- identify and compare geometric patterns formed by skip counting on circles, and curve-stitching;
- identify patterns in fractional parts of a whole, or of a set, and use them in computation;
- use manupulatives to develop and extend patterns;
- use similarities and differences to help define a pattern;
- give verbal descriptions of all of the above types of pattern;
- justify their choice of pattern.

Overview

"Curriculum Expectations" are based on current Ontario curricula.

Curriculum Expectations

ACTIVITY	DESCRIPTION OF THE ACTIVITY	CURRICULUM EXPECTATIONS
Activity 1 **Calendar Patterns**	• identifying patterns in arrays of number, including 100-charts	• recognize and discuss patterning rules • analyse number patterns and state the rule • discuss and defend the choice of a pattern rule
Activity 2 **Circular Patterns**	• analyzing geometric patterns developed by skip counting • analyzing geometric patterns developed from digital sums (i.e.,)	• demonstrate an understanding of mathematical relationships in patterns using drawings • recognize and discuss patterning rules • describe patterns encountered in any context • analyse number patterns and state the rule for any relationships • discuss and defend the choice of a pattern rule
Activity 3 **Fraction Patterns**	• identifying and comparing fractional parts of a whole and of a set • using patterns to identify equivalent fractions	• demonstrate an understanding of mathematical relationships in patterns using drawings • recognize mathematical relationships in patterns • demonstrate equivalence in simple numerical equations using concrete materials • pose and solve problems by applying a patterning strategy
Activity 4 **Building Patterns**	• using manipulatives to build designs and identifying a rule relating one design to the next • identifying similarities and differences among student-described patterns	• recognize and discuss patterning rules • identify, extend, and create patterns by changing two or more attributes • identify and extend patterns to solve problems in meaningful contexts • analyse number patterns and state the rule for any relationships • discuss and defend the choice of a pattern rule
Activity 5 **Picture Patterns**	• identifying and stating characteristics given examples and non-examples • using similarities and differences to establish a pattern rule	• demonstrate an understanding of mathematical relationships in patterns using drawings • apply patterning strategies to problem-solving situations • identify, extend, and create patterns by changing two or more attributes • analyse number patterns and state the rule for any relationship • discuss and defend the choice of a pattern rule

Overview

Prerequisites

Students need very little previous knowledge for these activities, although some experience with slides, flips, and turns will help in Activity 3.

Logos

The following logos, which are located in the margins, identify segments related to, respectively:

Problem Solving Communication Assessment Use of Technology

Materials

ACTIVITY	MATERIALS
Activity 1 **Calendar Patterns**	• Copies of BLMs 1, 2, 3 for each pair/group of students • Copies of BLMs 4, 5, 6 (optional)
Activity 2 **Circular Patterns**	• Copies of BLMs 7 and 8 for all students • Copies of BLM 9 (optional) • Cardboard, needles, and coloured thread for curve stitching (optional)
Activity 3 **Fraction Patterns**	• Copies of BLM 10 for each student • Scissors to cut out fraction strips • Small counters (or bits of paper) • Small cardboard or styrofoam trays (optional) • String, or elastics (optional) • Copies of BLM 11 (optional)
Activity 4 **Building Patterns**	• Copies of BLM 12 for each pair/group of students • Copies of BLMs 13 and 14 and scissors (optional) • Toothpicks for use with BLM 17 (optional) • Copies of BLMs 15, 16, 17 (optional)
Activity 5 **Picture Patterns**	• Copies of BLMs 18, 19, 20 for each student

Notes

Letter to Parents

SCHOOL LETTERHEAD

DATE

Dear Parent(s)/Guardian(s):

For the next week or so, students in our classroom will be participating in a unit titled "What's the Pattern?" The classroom activities will focus on number patterns in calendars and among fractions, and geometric patterns formed from skip-counting around a circle, and constructed using manipulatives. The emphasis will be on describing and extending the patterns found.

You can assist your child in understanding the relevant concepts by working together to perform simple experiments, and play games, and helping to locate everyday ways patterns are used.

Various family activities have been planned for use throughout this unit. Helping your child with the completion of these will enhance his/her understanding of the concepts involved.

If you work with patterns in your daily work or hobbies, please encourage your child to learn about this so that he/she can describe these activities to his/her classmates. If you would be willing to visit our classroom and share your experience with the class, please contact me.

Sincerely,

Teacher's Signature

A Note to the Teacher:

If you make use of the suggested Family Activities, it is important to schedule class time for sharing and discussion of results.

Activity 1: Calendar Patterns

Focus of Activity:

- Identifying patterns in arrays of numbers, such as calendars and 100 charts

What to Assess:

- Verbal description of any pattern discovered
- Accuracy of addition in identifying or testing patterns
- Recognition of differences among the patterns and the reasons for these differences

Preparation:

- Prepare copies of BLMs 1, 2, and 3.
- Prepare copies of BLMs 4, 5 and 6 (optional)

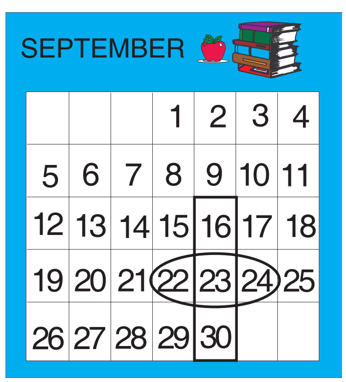
Activity:

Give each student or pair/group of students a copy of BLM 1. Ask students to select a number on Calendar 1 from somewhere in the middle of the array of numbers. Then have them draw an oval around that number, together with the numbers before and after it. They should then draw a rectangle around the first number together with the numbers above and below it, as shown in the diagrams below. The use of a rectangle instead of a second oval simply makes it easier to refer to the two sets of three numbers.

See "Solutions and Notes" for more on the mathematicians whose birth dates are marked on the calendars.

Have students add the three numbers in the oval and the three numbers in the rectangle, and compare the two sums. Then compare these two totals with the middle number that was chosen first. If students need prompting, ask

"What do you notice about the two sums?"
"How are the sums related to the middle number?"



Students should realize that the two totals are the same, and that each total is three times the middle number. Have students experiment by selecting other "middle" numbers, (using the same or another calendar on BLM 1) and marking the two sets of numbers as shown above. Ask students why this happens. Give them a few minutes in pairs or groups to devise an explanation that they can share with the class.

Problem Solving

Notes

Comments in italics are explanatory, and need not be conveyed to the students.

Activity 1: Calendar Patterns

Consider the first number chosen as 'n' (or any other letter, or symbol such as ☐). Then the other numbers in the oval are n - 1 and n + 1. This gives a total in the oval of (n -1) + n + (n + 1) which is 3n or three "ens". The numbers in the rectangle are n - 7, n, and n + 7 giving a total of (n - 7) + n + (n + 7), which is also 3n.

Distribute copies of 100 charts as given on BLM 2, and ask if students think that the oval and rectangle sums will give the same results on these charts as on the calendars. Allow time for students to test the "calendar pattern." Students should find the same number relationships exist here as on the calendars. Ask them why. Note that the numbers above and below the middle number are 10 less or 10 more than the middle number, rather than 7 less or 7 more.

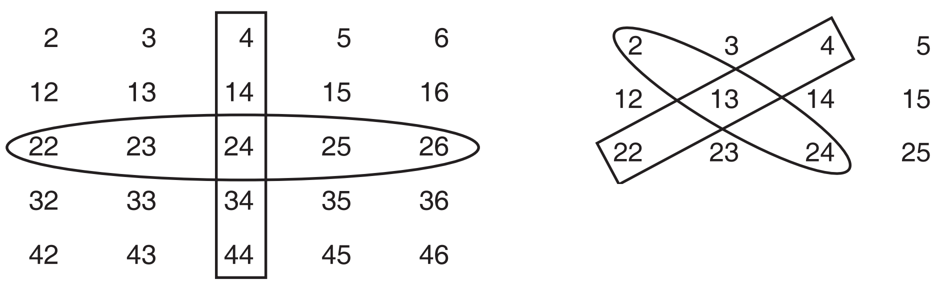
Explore other patterns such as those shown below on the two 100 charts.

Ask

"What happens if the oval and rectangle are larger? (See below)

"What happens if the oval and rectangle are drawn on diagonals?"

Excerpt from BLM



Use of Technology

Some children may want to extend this idea so the loops go from one side of the chart to the other, and from top to bottom. If students become tangled in the addition, allow calculator use to test the pattern.

Hundred charts such as those shown below may also be used. (See BLM 3.) Students may not expect the "calendar patterns" above to apply, and they may even at first refuse to accept these charts as "100 charts" since they do not have rows of 10 numbers each, as in the traditional 100 charts.

Excerpt from BLM

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	. . . .			

1	2	3	4
5	6	7	8
9	10	11	12
13	14	. . . .	

Activity 1: Calandar Patterns

Explore the differences among all these 100 charts.

"How are the hundred charts alike? How are they different?"

Hundred charts contain many other types of patterns. Help students identify some of them with such questions as the following:

> "If you loop all the multiples of 5 (or "If you count by five and loop the numbers you say") what pattern will they make on the first 100 chart (i.e., on BLM 2)? on the second one? on the others?"
>
> "If you mark all the even numbers on the 100 charts given, how will the patterns be alike? How will they be different? Design a 100 chart that will not have the even numbers in straight columns."

The nature of even numbers will arise again in Activity 2.

Extensions in Mathematics:

1. Explore addition and multiplication charts for patterns (see BLM 4). Ask students if the "calendar patterns" apply to addition or multiplication charts. The questions on BLM 4 can be used to direct students to some of the patterns in the charts.

Cross-curricular Activities

1. Lattice multiplication is a way of simplifying "long" multiplication. It was used in many parts of Europe during the Middle Ages. Samples of the technique are given on BLM 5. Give the students copies of the BLM and ask them to find the patterns in the completed multiplications, so that they know how to do "lattice multiplication", and can illustrate the procedure with their own examples.

   The patterns students will be looking for with lattice multiplication and the Russian Peasant Method (see Family Activities below) are of a different type from the patterns they looked for in calendars and 100 charts. They should be looking for ways the examples are alike, such as, "The numbers being multiplied in lattice multiplication are on the top and on the right of the lattice" or "Only one number (i.e., digit) is written in each triangle." Some students will see only superficial patterns, while others will identify aspects of the actual method used.

For further notes on both the Russian Peasant and lattice types of multiplication see "Solutions and Notes" following the BLMs.

Assessment

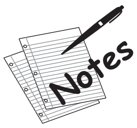
Family Activities:

1. Another interesting technique for multiplication is called the Russian Peasant Method. This is outlined on BLM 6. Students could work with their families to figure out how to use it and to test it on several samples.

Other Resources:

For additional ideas, see annotated Other Resources list on page 69, numbered as below.

1. Patterns, Addenda Series, K-6.
2. Multicultural Mathematics Posters and Activities.



Activity 2: Circular Patterns

Focus of Activity:
- Creating and analyzing circular patterns through skip counting and digital sums

What to Assess:
- Recognition of identical patterns in different problems
- Accuracy of predictions of new patterns
- Clarity of descriptions of patterns

Preparation:
- Make copies of BLMs 7 and 8
- Make copies of BLM 9 (optional)

Activity:

Tell students that the title of to-day's activity is "Circular Patterns" and ask them what this might mean. After some discussion, tell them that they will be discovering patterns using numbered circles and skip counting.

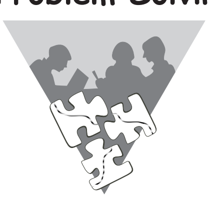
PATTERNS FROM SKIP COUNTING:
First, ask the students to count by 2s, starting at 0, while you record only the ones or units digits. Thus, you would write 0, 2, 4, 6, 8, 0, 2, 4, ... on the blackboard/chart paper/overhead. *It may be clearer for students if you record the tens digits as well, where relevant, and then cross them out: 0, 2, 4, 6, 8, ~~1~~0, ~~1~~2, ~~1~~4.*

Distribute copies of BLM 7. Have students copy your list of digits on the line beneath circle (a). Then have them begin with '0' on circle (a) and join zero to 2 with a straight line. Then join 2 to 4, 4 to 6, and so on until they start to repeat.

Comments in italics are explanatory, and need not be conveyed to the students.

The circles are small enough that students should be able to draw reasonably straight lines without the use of a ruler. Whether or not they use rulers should be up to the individual students, since some will find the manipulation of the ruler the most difficult part of the exercise.

Problem Solving



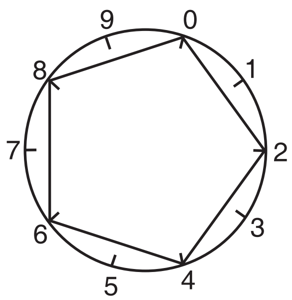
When students have completed the first diagram, ask them to describe the figure formed; it should be a regular pentagon. Ask them why counting by 2s didn't use all the numbers on the circle. This is a good place to review the meaning of "even number", and some of the characteristics of even numbers (e.g., multiplying by an even number always gives an even number) if they were not discussed in Activity 1.

Activity 2: Circular Patterns

Notes

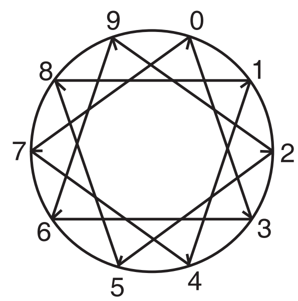
Excerpt from BLM 7:

(a) × 2



0, 2, 4, 6, 8, 10, 12, 14, ...

(b) × 3



0, 3, 6, 9, 12, 15, 18, ...

Ask students if they think they will get the same design if they count by 3s. Give them a few minutes to think about this before they carry out the test to see if their predictions were correct. For this, and succeeding problems they should begin with zero and list the ones digits for the multiples of 3, of 4, etc., and then join the points in order on the designated circle (or, as suggested above, list the multiples of 3 or 4 etc. and then cross out the tens digits). A 10-pointed star should be the result of this for counting by 3s.

Ask students what they think the results will be if they count by other numbers from 4 to 9 (always beginning with zero). "Do you think the patterns will all be different? Why or why not?"

Allow time for students to complete the circle patterns for counting by 4, 5, 6, 7, 8, and 9. They may be surprised to discover that, for example, 8 and 2 produce the same design/pattern, and that 7 and 3 give the same result.

Communication

Students could try to come up with some explanation. They should certainly be able to make some observations about the patterns. If not, ask them to list the units digits for, say, counting by 8 and counting by 2. Counting by 8 gives 0, 8, 6, 4, 2, 0 and counting by 2 gives 0, 2, 4, 6, 8, 0 — the same digits but in the reverse order. They may notice that 2 and 8 total 10, and there are 10 points on the circle.

Ask students what they think would happen if they counted by numbers greater than 9? Then ask specifically what they think would happen if they counted by 10 or 11.

Some students will see very quickly that counting by ten gives 0, 0, 0, 0, ... and counting by eleven gives 1, 2, 3, 4, 5, ... However, other students will need to write down the multiples of both numbers before they see the patterns.

Some students may wish to pursue the activity with other numbers greater than 9 to see if any of the greater numbers produce the same patterns as they have found for numbers 1 to 9. A second copy of BLM 7 would be useful here.

Activity 2: Circular Patterns

Patterns from Digit Sums:

Another set of patterns will be produced if students use digit sums instead of skip counting. A digit sum is obtained when one adds the individual digits of a number and continues to do this with successive sums until the result is one digit. For example, to determine the digit sum of 6529, calculate 6 + 5 + 2 + 9 to get 22; then calculate 2 + 2 to get 4. The digit sum of 6529 is 4. This can be written as $6529 \rightarrow 6+5+9 \rightarrow 22 \rightarrow 2+2 \rightarrow 4$.

Students may feel that an "equals" sign may be used in place of an arrow. Point out that this usually is incorrect since 6529 does not equal 6 + 5 + 2 + 9, and 22 does not equal 2 + 2. Rather than use some arrows and some equals signs, it is simpler to use arrows throughout.

Select a single digit number such as 3, and write on the blackboard/chart paper/ overhead the numbers used when counting by 3, starting from 3.
Thus: 3, 6, 9, 12, 15, 18, 21, 24, ...
Then write the digit sums for all of these:

3, 6, 9, 3, 6, 9, 3, 6, ...

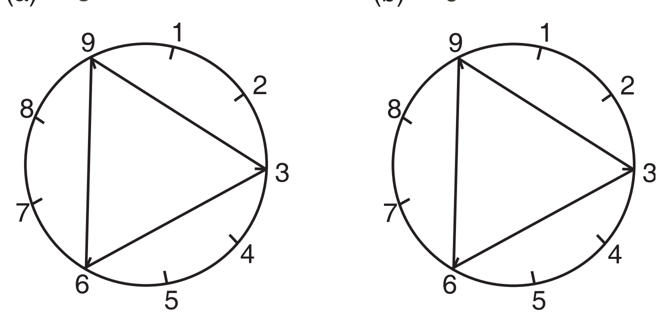
Students will notice the repeating pattern immediately. Distribute copies of BLM 8 and have students draw patterns as they did on BLM 7, but this time joining digit sums.

Notice that, on these circles, there is no zero. If students ask why, ask them to give an example of a number whose digit sum would be zero. They should realize quickly that unless the number itself is all zeros, there will not be a digit sum of zero.

As with the earlier patterns, students will find that some numbers will produce the same patterns/designs. For example, in the digit sum problem, 2 and 7 give the same design/pattern, as do 3 and 6.

Excerpt from BLM 8:

(a) × 3 (b) × 6



3, 6, 9, 3, 6, 9, .. *6, 3, 9, 6, 3, 9, .*

Activity 2: Circular Patterns

OPTIONAL:

The results for 9 can be used to explore patterns in the nine times table. The digit sum of each multiple is 9, which produces only a dot at '9' on the circle. Show students how the multiples of 9 in order have the tens digit decreasing by one and the ones digit increasing by one.

increasing by 1 ↓ | 9, 18, 27, 36, 45, 54, 63, 72, 81, 90 | ↓ decreasing by 1

Ask students how the tens digit of the answer is related to the multiplier. That is, for $3 \times 9 = 27$ how is the '3' related to the tens digit of '27'? Have students check to see if this is a pattern for all the multiples of nine. This makes it easy to remember the nine times table. For example, for 8×9 we know that the tens digit is 7, and the two digits in the answer total 9. Therefore $8 \times 9 = 72$.

This is the basis of an interesting way of multiplying by 9 using one's fingers. See "Solutions and Notes" for more on this.

Extensions in Mathematics:

1. Curve stitching: Distribute copies of BLM 9 and go over #1 with the students. In this exercise students draw straight lines which then appear to form a curve. Students could use the layouts suggested or could draw their own. Graph paper would be helpful in getting them to mark the points on the axes equal distances apart. (This activity could be sent home as an activity to do with family members.)
2. Students could be encouraged to use a suitable computer program to
   (i) draw a "curve-stitching" design
   (ii) mark 12 or more equidistant dots on a circle, join them using an "add 2", "add 3", or "add 5" rule, and compare the designs generated.

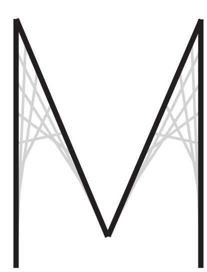
Use of Technology

This is often called "curve stitching" because students can draw one of the layouts given on BLM 9 on cardboard first, poking holes at the numbers along the axes, and stitching with thread to show the design . The use of coloured thread is suggested.

Family Activities:

1. Students could design a family logo using a surname initial as the basis for a curve stitching design.

e.g.,



Notes

Activity 2: Circular Patterns

Other Resources:

For additional ideas, see annotated Other Resources list on page 69, numbered as below.

4. The Wonderful World of Digital Sums, TCM.

Activity 3: Fraction Patterns

Focus of Activity:
- Identifying and comparing fractional parts of a whole and of a set

What to Assess:
- Identification of patterns, leading to algorithms (procedures) for computation of fractional parts
- Use of mathematical language

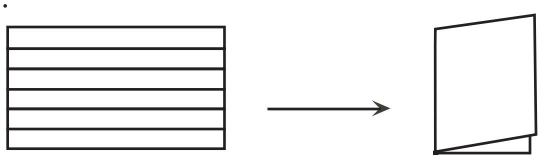
Preparation:
- Make copies of BLM 10
- Make copies of BLM 11 (optional)

Activity:

Distribute copies of BLM 10 to students and instruct them to cut out the large rectangle and the ruler on this page. This 'Ruler' is 24 units long, and will be used later.

FRACTIONS OF A WHOLE

Have students turn the paper so that the long measurement is horizontal. Then ask them to fold the paper in half so that they fold the long side in half, as shown below.



Ask them to open the fold and tell what fraction of the whole strip each part of one strip represents. Have them write "$\frac{1}{2}$" on each piece of the first row as shown in Fig.3.1. Call this strip the "strip of halves".

Have them refold the paper and then fold in half again. Unfold and label each of the four pieces "$\frac{1}{4}$" on each piece of the second row as shown below in Fig. 3.2. Call this strip the "strip of quarters".

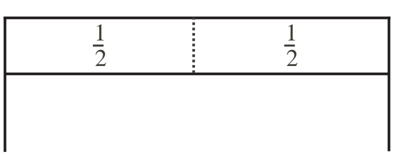


Figure 3.1

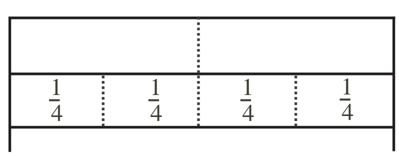


Figure3.2

Fold again to show eighths, and again to show sixteenths, if desired, adding labels for each.

Activity 3: Fraction Patterns

At this point you may want to have students colour the fractional parts to help distinguish them from one another. Colouring them alternately is an effective way. See Figure 3.3.

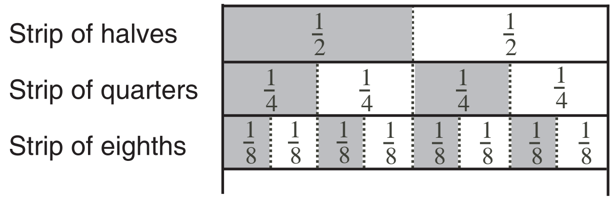


Figure 3.3

Leave the other fraction strips blank for now. They will be used later.

You might want to have students label the strips with cumulative fractions as shown below in figure 3.4. This might make identification of equivalent fractions easier. To keep the fraction strips from becoming too cluttered with numbers, students could label one side as in Figure 3.3 and the other side as in figure 3.4.

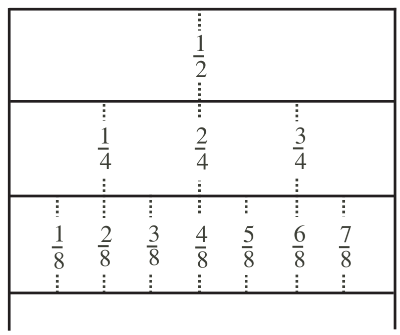


Figure 3.4

Using the fraction strips, explore with students such questions as

"Which is greater, $\frac{1}{2}$ or $\frac{1}{4}$? $\frac{1}{2}$ or $\frac{1}{8}$? $\frac{1}{8}$ or $\frac{1}{4}$?"

"What fraction is equal to $\frac{2}{4}$? to $\frac{6}{8}$?" How can you use the fraction strips to help you answer questions like these?

"What happens to the size of a fraction of the strip when the denominator of the fraction increases?"

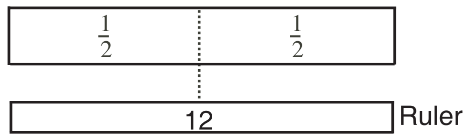
Help students to see that, for example, $\frac{2}{8}$ is twice as long as $\frac{1}{8}$, or $\frac{3}{4}$ is three times as long as $\frac{1}{4}$.

Activity 3: Fraction Patterns

Fractions of a Group

Since the total length of each fraction strips is 24 units, students can use the 'Ruler' from BLM 10 to measure the "$\frac{1}{2}$ segment" to determine $\frac{1}{2}$ of 24. Similarly, they can determine $\frac{1}{4}$ of 24 or $\frac{1}{8}$ of 24, by measuring the appropriate segment.

Notice that we are simply asking students to read a ruler, not to do any computation with fractions.



$\frac{1}{2}$ of 24 is 12

Record the results in a chart and encourage students to find a pattern:

Fraction	Denominator	Fraction of 24 units
$\frac{1}{2}$	2	12 cm
$\frac{1}{4}$	4	6 cm
$\frac{1}{8}$	8	3 cm

Ask students to compare the last two columns of the table. With some help from a calculator, students should see that the denominator times the resulting "fraction of 24" will give 24 in each case. Note that this is true only for unit fractions (i.e., fractions with a numerator of 1).

These fraction strips can be used to determine fractions of other amounts, not just 24. Students will need some small counters, or even bits of paper. Give them, for example, 10, and tell them you will show them how to determine $\frac{1}{2}$ of 10.

Have them distribute the counters on the strip of halves as follows: put a counter in the first section, then a counter in the second section, and repeat until all the counters are evenly distributed. There should, of course, be 5 counters in each section, showing that $\frac{1}{2}$ of 10 is five. Repeat for other numbers and unit fractions such as $\frac{1}{2}$ of 16, $\frac{1}{4}$ of 16, $\frac{1}{4}$ of 12, ...

Throughout this activity, keep the emphasis on counting and sharing, rather than computation.

To determine other fractions, such as $\frac{1}{3}$, have students draw light lines on one of the blank fraction strips where they think the marks for thirds should go. Then take 24 counters and distribute them on this strip of thirds to determine $\frac{1}{3}$ of 24. When they discover that $\frac{1}{3}$ of 24 is 8, they can use the 'Ruler' to measure equal segments of 8 units along the strip and label each of the 3 sections $\frac{1}{3}$. In a similar way, they could determine $\frac{1}{6}$ of 24 and mark the final strip into six equal segments of 4 units, and label each section "$\frac{1}{6}$".

OPTIONAL:

You may wish to extend this to problems such as $\frac{1}{2}$ of 5, where the solution would not be a number of whole counters. Students may suggest that one of the counters could be cut in half so that each section on the strip of halves contains $2\frac{1}{2}$ counters. Thus $\frac{1}{2}$ of 5 is $2\frac{1}{2}$.

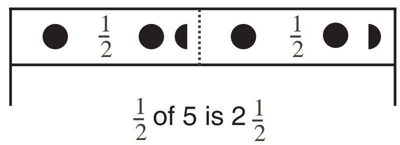


Figure 3.6

Cash register tapes cut to lengths of 20 or 30 cm could be used to illustrate fifths and tenths.

Extensions in Mathematics:

1. An alternative format for sharing counters to determine a fraction of a number uses a rectangular piece of paper or cardboard or a small styrofoam tray, and some pieces of string, or elastic bands large enough to go around the trays easily. For example, Fig. 3.7 shows the results of finding $\frac{1}{4}$ of 12. The strings/elastics do not need to be exactly in the correct spots, but they should be good estimates. Figure 3.8 shows how students might begin to determine $\frac{1}{8}$ of 16.

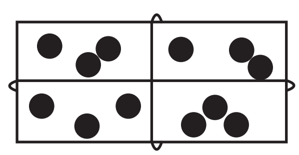


Figure 3.7

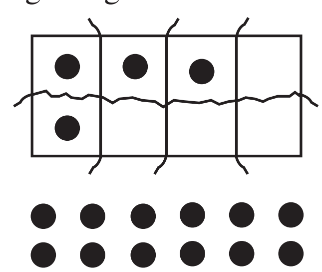


Figure 3.8

It is important for students to realize that the denominator of the fraction tells the number of approximately equal spaces to be formed with the strings.

Notice that once students have determined $\frac{1}{8}$ of 16, they can easily tell $\frac{2}{8}$ of 16 or $\frac{3}{8}$ of 16, etc., simply by finding the total number of counters in the appropriate number of sections. Recording results in charts (as shown below) will help them identify a pattern that may give them a technique for eventually calculating a fraction of a number without using fraction strips or counters.

Activity 3: Fraction Patterns

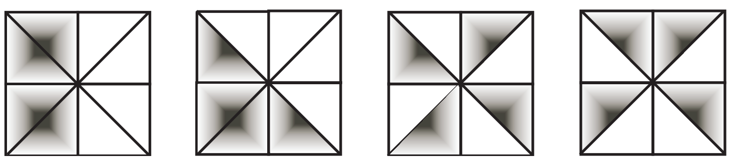
Fractions of 12	No. of counters
$\frac{1}{4}$	3
$\frac{2}{4}$	(2 x 3 =) 6
$\frac{3}{4}$	(3 x 3 =) 9

Fractions of 16	No. of counters
$\frac{1}{8}$	2
$\frac{2}{8}$	(2 x 2 =) 4
$\frac{3}{8}$	(3 x 2 =) 6
$\frac{5}{8}$	(5 x 2 =) 10

This technique reinforces the idea of, for example, $\frac{3}{8}$*, as 3 pieces of* $\frac{1}{8}$ *each or* $\frac{1}{8}+\frac{1}{8}+\frac{1}{8}$.

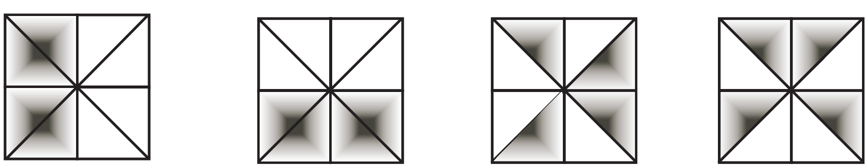
Have students write how they would calculate, say, $\frac{6}{8}$ of 16 or $\frac{3}{8}$ of 16.

2. Using BLM 11, ask students to colour $\frac{1}{2}$ of one square. When they are finished, ask them to colour $\frac{1}{2}$ of another square in a different way. Then ask them to colour $\frac{1}{2}$ of the square in as many different ways as they can. Some are given below:



Examples of colouring $\frac{1}{2}$ of the square

The word "different" in this case, should be interpreted to mean "non-congruent". For example, Squares A and B below are considered the same, as are Squares C and D.



Square A Square B Square C Square D

When students have coloured half the square in as many different ways as they think possible, have them share their results with other students/pairs/groups and have them try to explain to each other why they think they have found all the possibilities.

Notes

Communication

Problem Solving

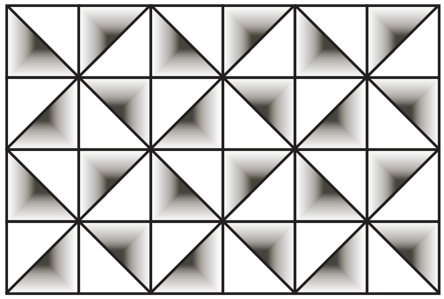
Assessment

For similar ideas using slides, flips and turns, see Investigations in Geometry: Grade 4: Pattern and Colour. Ordering information is given on final the pages of this booklet.

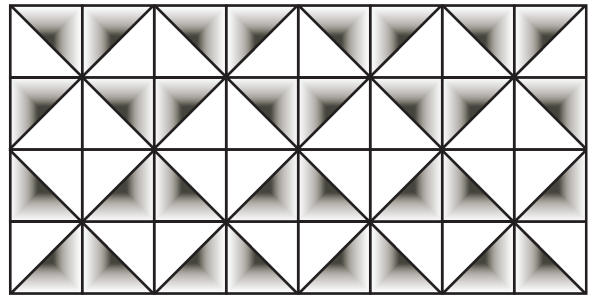
Activity 3: Fraction Patterns

Cross-curricular Activities:

1. Have students select their favourite design from Extension 2 and use it with slides, flips, and turns to make a wallpaper-type design. Two examples are given below:



This design uses slides.

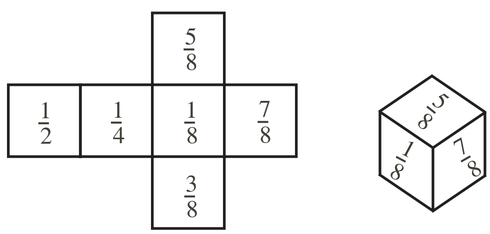


This design uses flips.

Once the designs are completed, students can examine each other's for symmetries.
Students could involve their families in developing designs for class discussions. Students and their families can examine fabrics, wallpaper, and wrapping paper for similar patterns.

Family Activities:

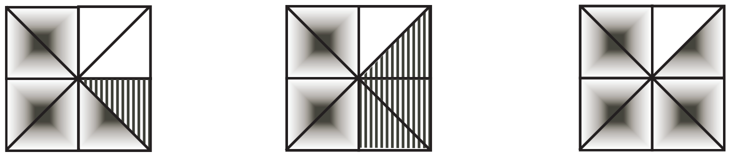
1. The board for a game, "Counting to One" is given on BLM 11. A template for the cube is also given. To play the game, students will need a die marked as shown below. Alternatively, they could write the fractions on cards and draw from a face-down pile; for this method, make several of each fraction, and mix them well.



Make sure the students know how many pieces of one of the squares on the game board give $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{3}{8}$, $\frac{5}{8}$, and $\frac{7}{8}$.

In turn, players roll the die (or draw a card), and colour that fraction of one of the squares. On his/her second turn, the player tries to continue colouring the same square, but if the roll of the die gives a fraction too great for that, he/she begins colouring another square. On any turn, the fraction rolled must be applied to only one square. The aim is to be the first to have 3 completely coloured squares. It is possible that a student may partially colour all eight squares on one game board before he/she is able to complete the colouring on 3 squares. A partial game is shown on the next page.

Activity 3: Fraction Patterns



1st roll: $\frac{5}{8}$ 2nd roll: $\frac{1}{2}$ 3rd roll: $\frac{7}{8}$

4th roll: $\frac{1}{8}$ 5th roll: $\frac{3}{8}$

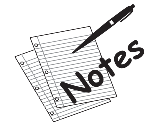
Note that, on the 4th roll, the student could have added the $\frac{1}{8}$ to the third square, and hence completely coloured that square. Thus there is an element of decision-making in this game.

OPTIONAL:
Have students use different colours to show the different fractions used to cover the whole square. Then they can record their results in equations for each square. For example, if the 6th roll for the game shown above was 1/8, the student could complete either the second square, recording $\frac{1}{2}+\frac{3}{8}+\frac{1}{8}=1$ or the third square, recording $\frac{7}{8}+\frac{1}{8}=1$.

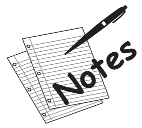
Other Resources

For additional ideas, see annotated Other Resources list on page 69, numbered as below.

1. Patterns, Addenda Series, Grades K-6.



Problem Solving



Activity 4: Building Patterns

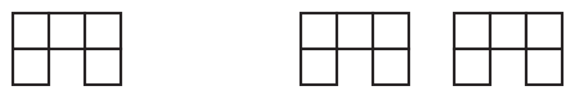
Focus of Activity:
- Using manipulatives to develop patterns and predict extensions

What to Assess:
- Clarity of descriptions of patterns
- Use of mathematical language

Preparation:
- Make copies of BLM 12
- Make copies of BLMs 13, 14, 15, 16, and 17 (optional)

Activity:

Present the following problem to students, illustrating with squares or cubes. One bridge uses 5 squares/cubes. How many squares/cubes do 2 bridges use?



1 bridge 2 bridges

How many cubes will 3 bridges use? Record in a table, adding numbers for 4 and 5 bridges:

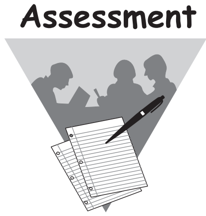
Number of bridges	1	2	3	4	5	10	100
Number of cubes	5	10	15				

Have students justify their answers.

"Why do you think 3 bridges will need 15 cubes?"
"What pattern are you using to help you find the answers?"
"Can you find any other patterns?"

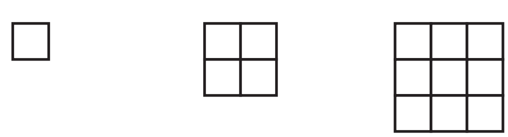
Discuss various patterns identified and ask students to use their patterns to determine the number of cubes needed to build 10 bridges.

Assessment



Some will find this answer counting by 5, the obvious pattern in the second line of the table. However, it would be time consuming to count by 5 for 100 bridges. Students should recognize that they can find the number of cubes for any number of bridges by multiplying the number of bridges by 5.

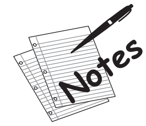
Present a similar problem, using squares, and record the first bits of data in a table:



Square 1 Square 2 Square 3

Activity 4: Building Patterns

Square Number	1	2	3	4	5	10	100
Number of squares	1	4	9				



Communication

Ask students to complete the table, and to explain, in writing, how they determined the number of squares for Square Number 100. If students work in pairs or small groups, they should try to write an explanation with which everyone in the group agrees.

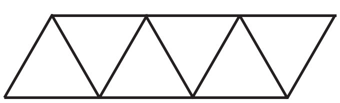
For more experience with this kind of thinking, distribute copies of BLM 12 to all pairs/groups of students. There are four problems here. You may wish to assign each problem to only 1 or 2 groups, or assign 2 of the problems to each group, but not always the same problems, or you may wish all students to have experience with each problem. In any case, students should write explanations as to how they answered each question, and be prepared to give these explanations to the class, and to justify them. Cubes, squares, triangles, and toothpicks should be available for those students who need a concrete representation of the various items (For squares and triangles, see BLMs 13 and 14.)

Problem Solving

Review the method used for the two introductory problems and suggest that students use one or more of the following problem-solving strategies:

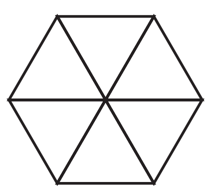
- make a smaller problem
- make a table
- look for a pattern

Be sure students understand that the pattern suggested by the first few items for each problem should be extended to solve the problem. For example, if students use a linear pattern for #4 on BLM 12, as shown in Fig. 4.1, the pattern of numbers of logs will be relatively easy to discover, but if they construct 6 pens as shown in Figure 4.2, they may have difficulties identifying a pattern.



6 pens, 13 logs

Figure 4.1



6 pens, 12 logs

Figure 4.2

Communication

Students will likely identify a number of different patterns in order to solve the problems. It is important to examine the patterns with the students to determine which are different patterns, and which are merely different ways of describing the same pattern. Students should not only describe patterns they see, but should be prepared to defend them.

Notes

Activity 4: Building Patterns

Extensions in Mathematics:

1. Other problems can be found on BLMs 15 and 16. These could be used as classroom activities or as problems-of-the-day or problems-of-the-week.

Family Activities:

1. Problems from BLM 17 could be sent home for families to work out together. The problems require only toothpicks as materials. Students should bring their solutions to school prepared to describe how they and their families solved the problems.

Other Resources:

For additional ideas, see annotated Other Resources list on page 69, numbered as below.

5. Algebraic Thinking, Focus Issue of TCM.

Activity 5: Picture Patterns

Focus of Activity:
- Using similarities and differences to help define a pattern

What to Assess:
- Reasonableness of definitions
- Accuracy and consistency in applying these definitions

Preparation:
- Make copies of BLMs 18, 19, and 20

Notes

Activity:

In the identification of pattern, it is necessary to identify similarities and differences among components of a pattern, and how they are connected. This activity gives students practice in recognizing these features.

Comments in italics are explanatory, and need not be conveyed to the students.

Distribute copies of BLM 18 to students and direct their attention to the first problem. Tell students that "wurffle" is a nonsense word and they aren't expected to know what a wurffle is. They should instead try to figure out what a wurffle is by looking at both the wurffles and the "not-wurffles". Give them a few minutes to do this, and then ask several children what they think a wurffle is, and why. Accept different definitions. Then, using two or three of these definitions, ask students to identify the wurrfles in the third row of the problem. Students should realize

(a) that there could be more than one valid definition, and

(b) that a definition that seems valid might prove invalid when applied.

For example, looking at the first line only, a student may say "A wurffle is less than 100". However, the "not-wurffles" include numbers less than 100, so the definition produces a conflict.

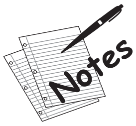
Problem Solving

Some students may still need to be reassured that "wurffle" and "podlap" and other names are just contrived or nonsense names and students are not expected to know these words before starting the problem. Rather, they are going to be defining the meanings of these words for themselves.

The use of hyphenated negatives such as "not-wurffle" or "not-beedop" has been found to be easy for students to use, and makes the instruction "Write two more not-beedops" less wordy than it might otherwise have been.

Communication

Working in pairs or small groups allows students to clarify their thinking by discussion with others. Before writing a definition for one of the problems, all students in the group should agree on it.



Activity 5: Picture Patterns

BLM 19 contains similar problems, but these call on a student's spatial sense rather than his/her number sense. The first problem should be fairly straightforward, since only one characteristic is necessary to define "gurb". For example, a gurb has curved lines only. Later problems use 2 or more characteristics to define the terms. For example, wibbles are made up of straight lines and are simple polygons (i.e. have only one inside space, i.e., have no intersecting sides).

Problem Solving

BLM 20 contains a different kind of problem, but, again, with more than one possible solution. However, in these problems, students need to compare each item with each other item, identify differences, and then decide which difference excludes two of the items. For example, in problem 2, students may note that box A has 4 triangles shaded; but so do B and C. In fact, this is true for all boxes. Since there is no box that does not belong according to this criterion, the student must look for other characteristics. See "Solutions and Notes" for possible solutions.

Problems like these frequently occur in aptitude or intelligence tests. It is possible to help students develop the kind of thinking needed to be successful with these and similar problems.

Extensions in Mathematics:

1. Have students examine the format of the problems on BLMs 18, 19 and 20 and make up problems of their own.
   This may prove to be a difficult task, except for those students who construct such an easy problem that solutions are obvious. Some samples of student work are given in "Solutions and Notes".

Assessment

   Discuss students' first efforts with the class and compare them with BLMs 18, 19, and 20 for clarity and difficulty. Have students then create a second problem. Observing the growth in understanding provides a good opportunity for assessment.

Cross-curricular Activities:

1. Examine the names of the entities on BLM 18. Ask students why the words sound real. [*The words are made up of common letter combinations and sounds in English.*] Ask students if words like "azjebque" or "mnaptl" look or sound real, and why they don't. Have students make up some "real-sounding" names and have them explain why. For example, "wurffle" sounds like "waffle"; "garnobs" is partly from "garden" and partly from "knobs".

2. Most of the nonsense words on BLM 18 have simple plurals, formed by adding an "s". However, two of them (dawnlies and lundies) have plurals formed by "changing the 'y' to 'i' and adding 'es'." Explore other plural forms with students. As they make up nonsense words, have them use different plural forms.

Activity 5: Picture Patterns

Family Activities:

1. Students can work with their families to make up some nonsense words in other languages, and can share these with their classmates, explaining how the nonsense words are like real words in the language used.

Other Resources:

For additional ideas, see annotated Other Resources list on page 69, numbered as below.

1. Patterns, Addenda Series, Grades K-6.

Notes

Communication

BLM 1: Calendar Patterns

MARCH – 1596

	1	2	3	4	5	6
7	8	9	10	11	12	13
14	15	16	17	18	19	20
21	22	23	24	25	26	27
28	29	30	31*			

* birthdate of Rene Descartes

JUNE – 1623

1	2	3	4	5	6	7
8	9	10	11	12	13	14
15	16	17	18	19*	20	21
22	23	24	25	26	27	28
29	30					

* birthdate of Blaise Pascal

MAY – 1718

				1	2	3
4	5	6	7	8	9	10
11	12	13	14	15	16*	17
18	19	20	21	22	23	24
25	26	27	28	29	30	31

* birthdate of Marie Gaetana Agnesi

APRIL – 1776

	1*	2	3	4	5	6
7	8	9	10	11	12	13
14	15	16	17	18	19	20
21	22	23	24	25	26	27
28	29	30				

* birthdate of Marie Sophie Germain

APRIL – 1777

		1	2	3	4	5
6	7	8	9	10	11	12
13	14	15	16	17	18	19
20	21	22	23	24	25	26
27	28	29	30*			

* birthdate of Carl Friederich Gauss

OCTOBER – 1811

		1	2	3	4	5
6	7	8	9	10	11	12
13	14	15	16	17	18	19
20	21	22	23	24	25*	26
27	28	29	30	31		

* birthdate of Évariste Galois

BLM 2: Hundred Charts

Chart 1:

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Chart 2:

0	1	2	3	4	5	6	7	8	9
10	11	12	13	14	15	16	17	18	19
20	21	22	23	24	25	26	27	28	29
30	31	32	33	34	35	36	37	38	39
40	41	42	43	44	45	46	47	48	49
50	51	52	53	54	55	56	57	58	59
60	61	62	63	64	65	66	67	68	69
70	71	72	73	74	75	76	77	78	79
80	81	82	83	84	85	86	87	88	89
90	91	92	93	94	95	96	97	98	99

BLM 3: Variations on Hundred Charts

Chart 1:

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30
31	32	33	34	35	36
37	38	39	40	41	42
43	44	45	46	47	48
49	50	51	52	53	54
55	56	57	58	59	60
61	62	63	64	65	66
67	68	69	70	71	72
73	74	75	76	77	78
79	80	81	82	83	84
85	86	87	88	89	90
91	92	93	94	95	96
97	98	99	100		

Chart 2:

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16
17	18	19	20
21	22	23	24
25	26	27	28
29	30	31	32
33	34	35	36
37	38	39	40
41	42	43	44
45	46	47	48
49	50	51	52
53	54	55	56
57	58	59	60
61	62	63	64
65	66	67	68
69	70	71	72
73	74	75	76
77	78	79	80
81	82	83	84
85	86	87	88
89	90	91	92
93	94	95	96
97	98	99	100

Chart 3:

1	11	21	31	41	51	61	71	81	91
2	12	22	32	42	52	62	72	82	92
3	13	23	33	43	53	63	73	83	93
4	14	24	34	44	54	64	74	84	94
5	15	25	35	45	55	65	75	85	95
6	16	26	36	46	56	66	76	86	96
7	17	27	37	47	57	67	77	87	97
8	18	28	38	48	58	68	78	88	98
9	19	29	39	49	59	69	79	89	99
10	20	30	40	50	60	70	80	90	100

BLM 4: Addition and Multiplication Patterns

Addition Chart

+	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	10
2	2	3	4	5	6	7	8	9	10	11
3	3	4	5	6	7	8	9	10	11	12
4	4	5	6	7	8	9	10	11	12	13
5	5	6	7	8	9	10	11	12	13	14
6	6	7	8	9	10	11	12	13	14	15
7	7	8	9	10	11	12	13	14	15	16
8	8	9	10	11	12	13	14	15	16	17
9	9	10	11	12	13	14	15	16	17	18

first row →

↑ first column

Look for patterns in the addition chart. These questions might help you.

1. Compare the first row of the chart with the first column.
2. Where are the 6s in the chart? Where are the 8s? the 15s?
3. One of the diagonals starts with 0, 2, 4, 6, ... What numbers are added to give these sums?
4. Describe other patterns you find.

Multiplication Chart

x	0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9
2	0	2	4	6	8	10	12	14	16	18
3	0	3	6	9	12	15	18	21	24	27
4	0	4	8	12	16	20	24	28	32	36
5	0	5	10	15	20	25	30	35	40	45
6	0	6	12	18	24	30	36	42	48	54
7	0	7	14	21	28	35	42	49	56	63
8	0	8	16	24	32	40	48	56	64	72
9	0	9	18	27	36	45	54	63	72	81

first row →

↑ first column

Look for patterns in the multiplication chart. Use these questions to help.

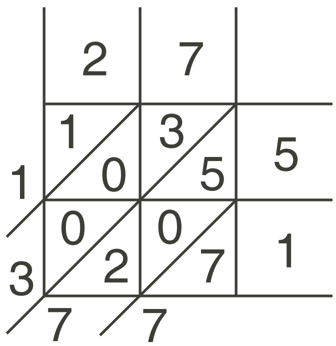
5. Compare the first row and the first column. How are they different from the first row and column of the addition chart? Why?
6. Where are the 6s? the 8s? the 15s? Compare this with your answer to #2 above.
7. One of the diagonals starts with 0, 1, 4, 9, ... What numbers are multiplied to give these products? Compare with the addition chart.

BLM 5: Lattice Multiplication

A method of multiplication used in Europe in the Middle Ages (about 1000 - 1400 A.D.) was called Lattice Multiplication.

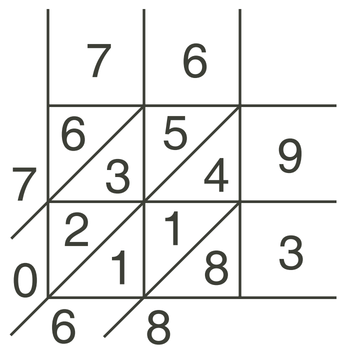
Three examples are given below. Study these examples, to discover how this method works. Some questions are given below the examples to help you understand the method.

Example 1:



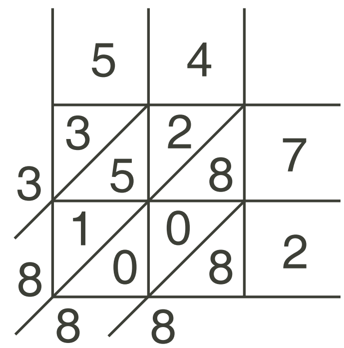
This shows
27 x 51 = 1377

Example 2:



This shows
76 x 93 = 7068

Example 3:

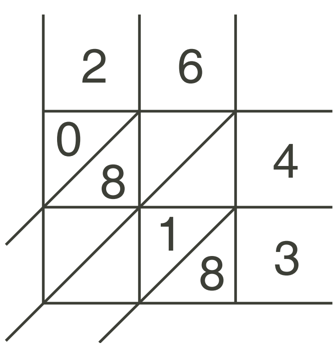


This shows
54 x 72 = 3888

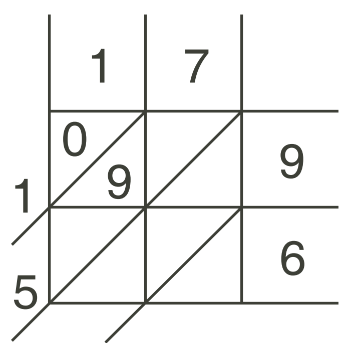
1. What patterns can you see in these lattice multiplications?
2. You know that 7 x 5 = 35. Where is this shown in Example 1?
3. You know that 7 x 1 = 7. Why is this written as "07" in the lattice in Example 1?
4. The answer in Example 1 is 1377. Where is this written in Example 1? What numbers are added to give each of these digits?

5. Complete each of the following lattice multiplications. Check with your calculator.

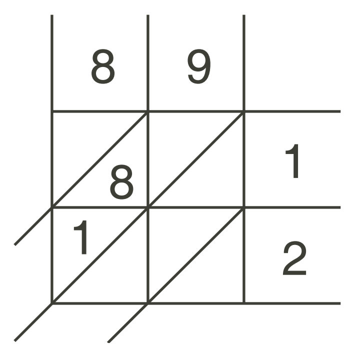
(a)



(b)



(c)



BLM 6: Russian Peasant Multiplication

The type of multiplication known as Russian Peasant Multiplication was easy for people to use because they did not have to remember all the multiplication facts. To use this method, you need to be able to divide by 2, multiply by 2, and add.

Three examples are given below. See what patterns you can find. The questions below the examples are given to help you figure out how to use the Russian Peasant method.

Example 1: 96 x 8 = 768

Step 1

96	8
48	16
24	32
12	64
6	128
3	256
1	512

Step 2

~~96~~	~~8~~
~~48~~	~~16~~
~~24~~	~~32~~
~~12~~	~~64~~
~~6~~	~~128~~
3	256
1	512
	768

Example 2: 86 x 9 = 774

Step 1

86	9
43	18
21	36
10	72
5	144
2	288
1	576

Step 2

~~86~~	~~9~~
43	18
21	36
~~10~~	~~72~~
5	144
~~2~~	~~288~~
1	576
	774

1. In one of the columns, the number is doubled over and over. Which column is it?

2. In the other column the numbers are divided by 2, but the answers are not all exactly correct. For instance, in Example 1, 3 ÷ 2 is not 1. What is the remainder that has been discarded? Where does this happen in Example 2?

3. In Step 2, certain numbers in both columns are crossed out. Look at the numbers in the first column that are not crossed out. What kind of numbers are these? What is alike about all the numbers in the first column that are crossed out?

4. Some incomplete examples are given below. Try to complete them using the Russian Peasant method.

(a) 26 x 15

26	15
13	30
6	60
3	
	240

(b) 54 x 35

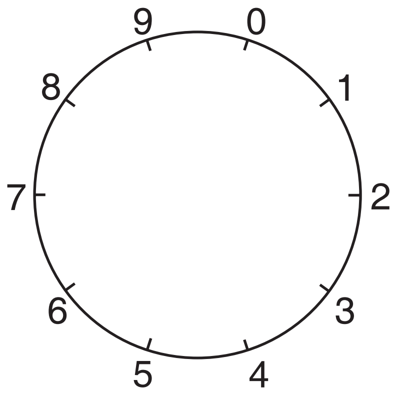
54	35
27	70
6	
	560
1	

(c) 325 x 31

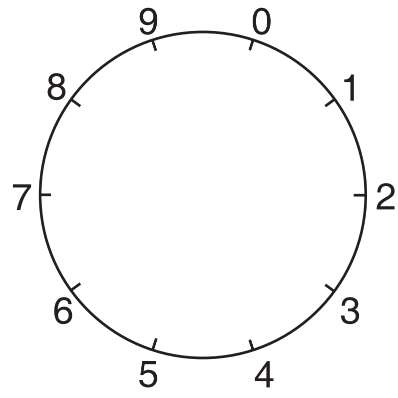
325	31
81	
	248
	1984
2	
1	

BLM 7: Skip Counting Patterns

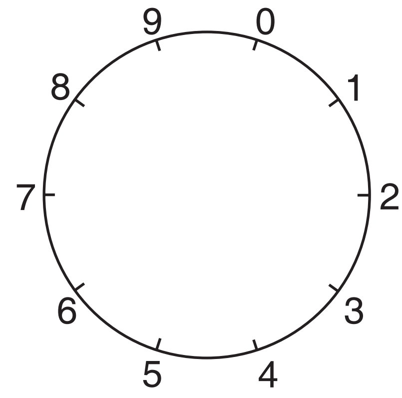
(a) **x 2**



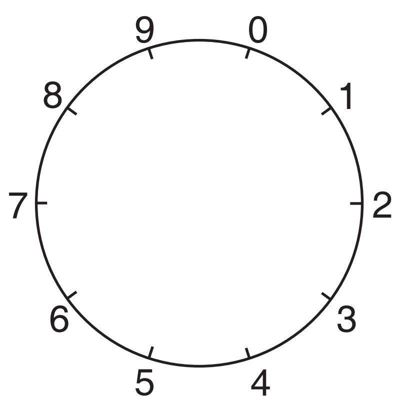
(b) **x 3**



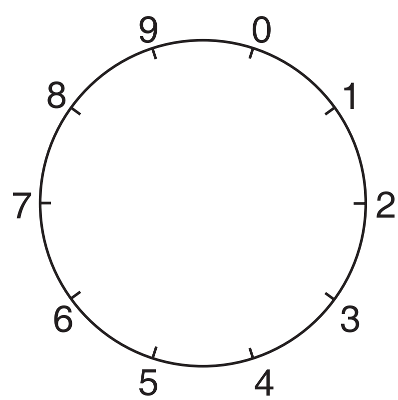
(c) **x 4**



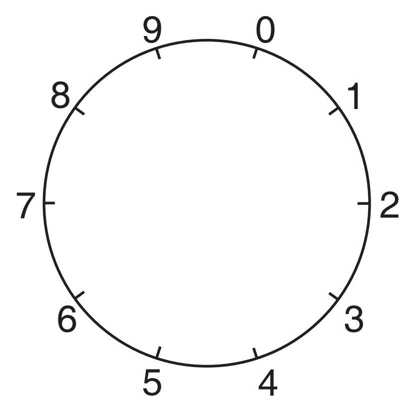
(d) **x 5**



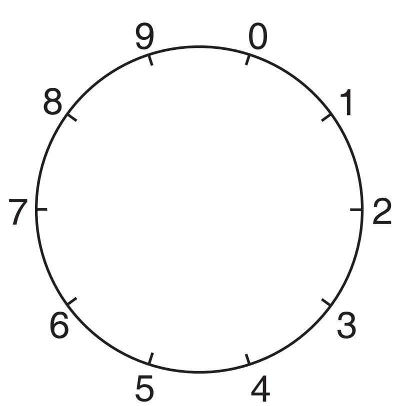
(e) **x 6**



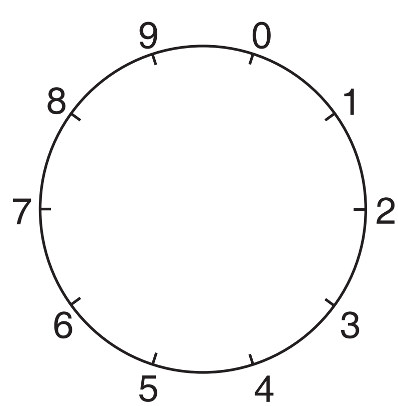
(f) **x 7**



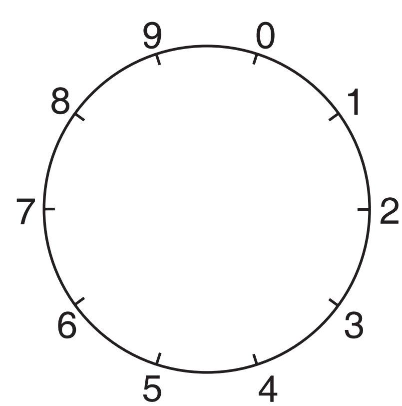
(g) **x 8**



(h) **x 9**

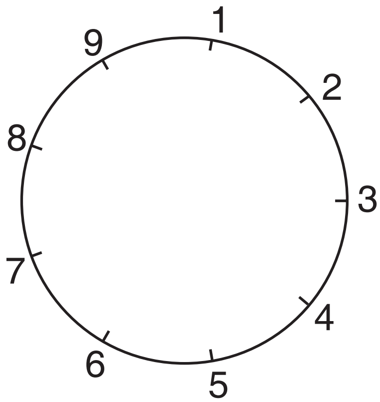


(i)

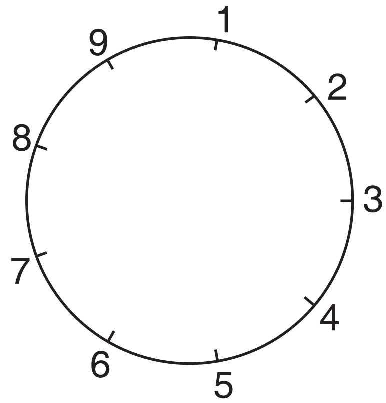


BLM 8: Digit Sums Patterns

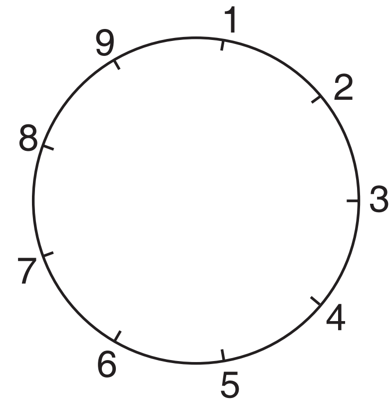
(a) **x 2**



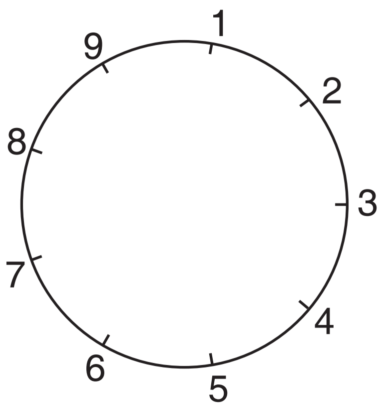
(b) **x 3**



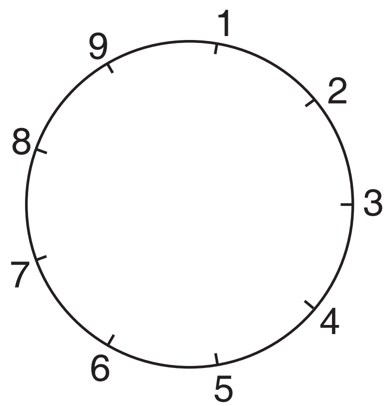
(c) **x 4**



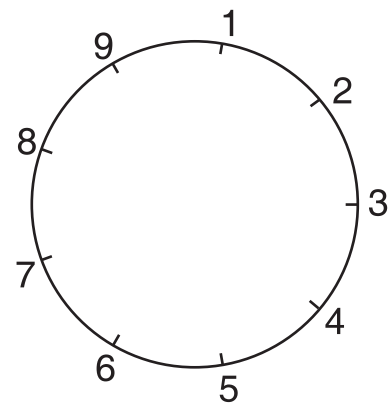
(d) **x 5**



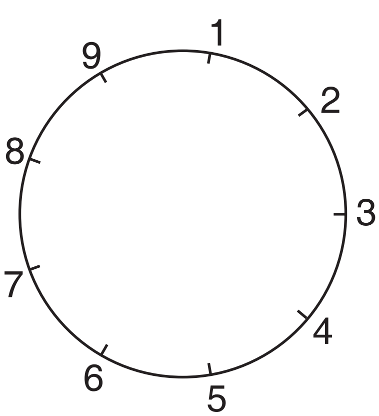
(e) **x 6**



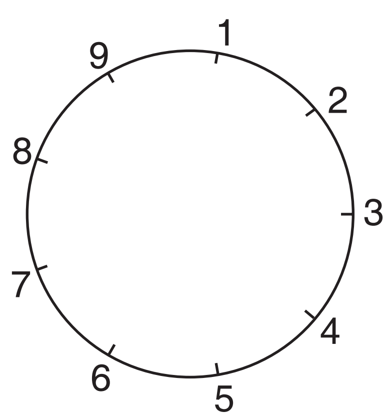
(f) **x 7**



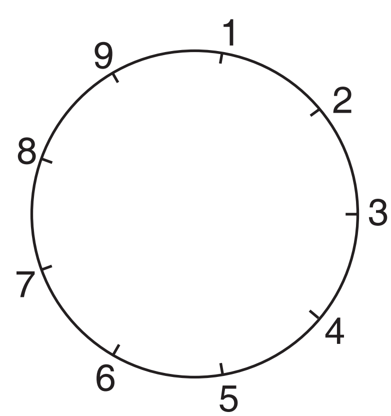
(g) **x 8**



(h) **x 9**



(i)



BLM 9: Curve Stitching

1. In Figure 1 below there are two heavy lines, one vertical, and one horizontal.
   The '10' on the vertical line is joined to the '4' on the horizontal line. What is the sum of 10 and 4?
   The 6 on the vertical line is joined to the 8 on the horizontal line. What is the sum of 6 and 8?
   Find other pairs of numbers that total 14. One number should be on the vertical line and one on the horizontal line. Join the two numbers with a straight line. Join all such numbers that you can find. Describe the result.

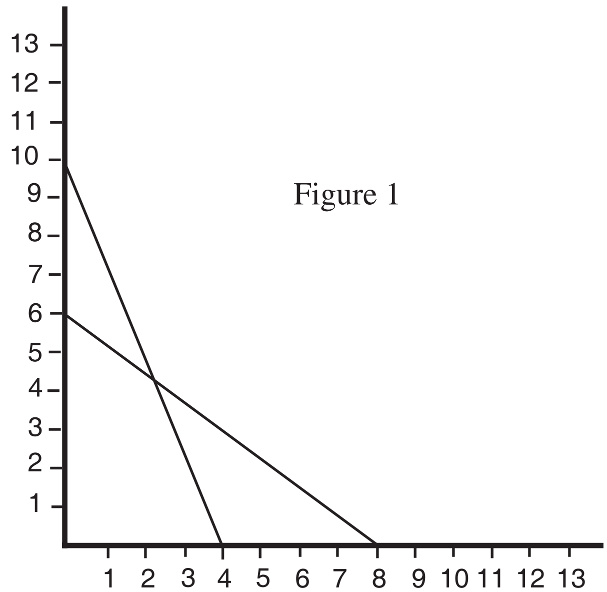
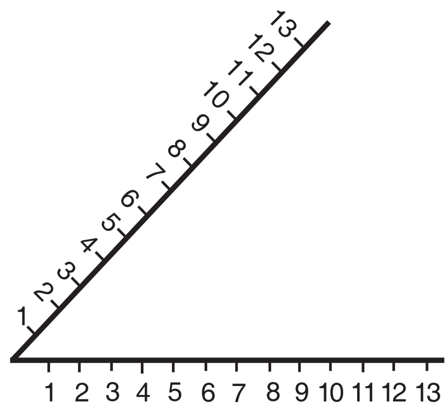
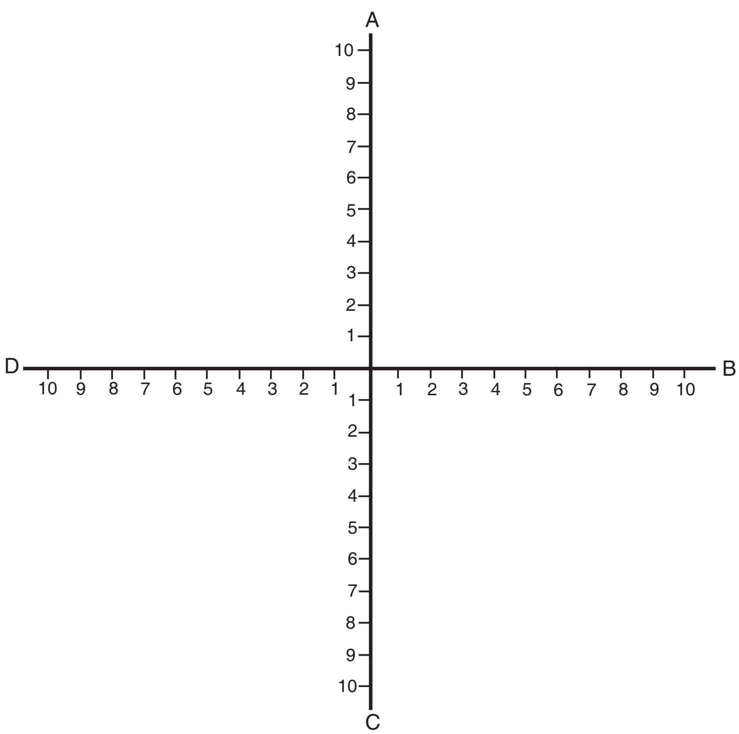


Figure 1

2. Repeat #1 using the figure below.
   Describe the result.

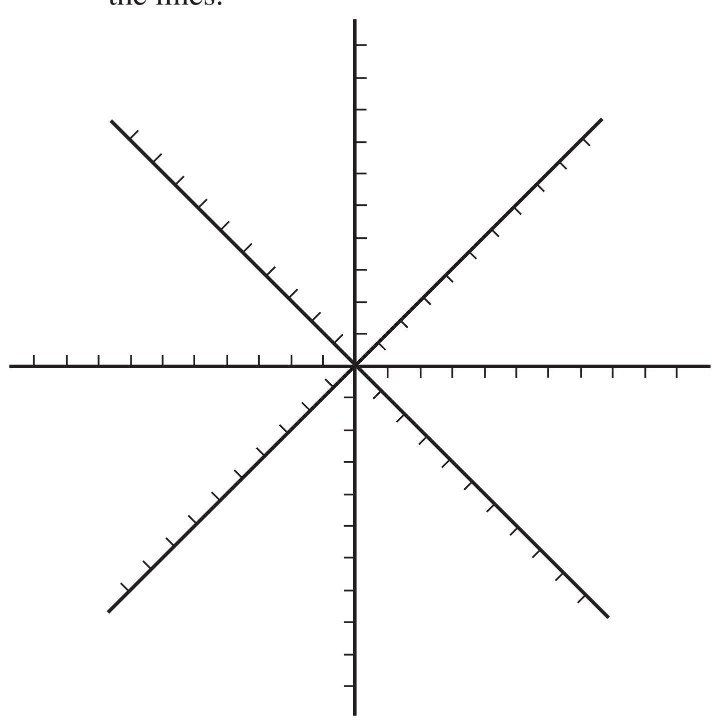


3. (a) In the figure below, draw lines from a number on line A to a number on line B so their sum is 11.
   (b) Repeat (a) for lines A and D.

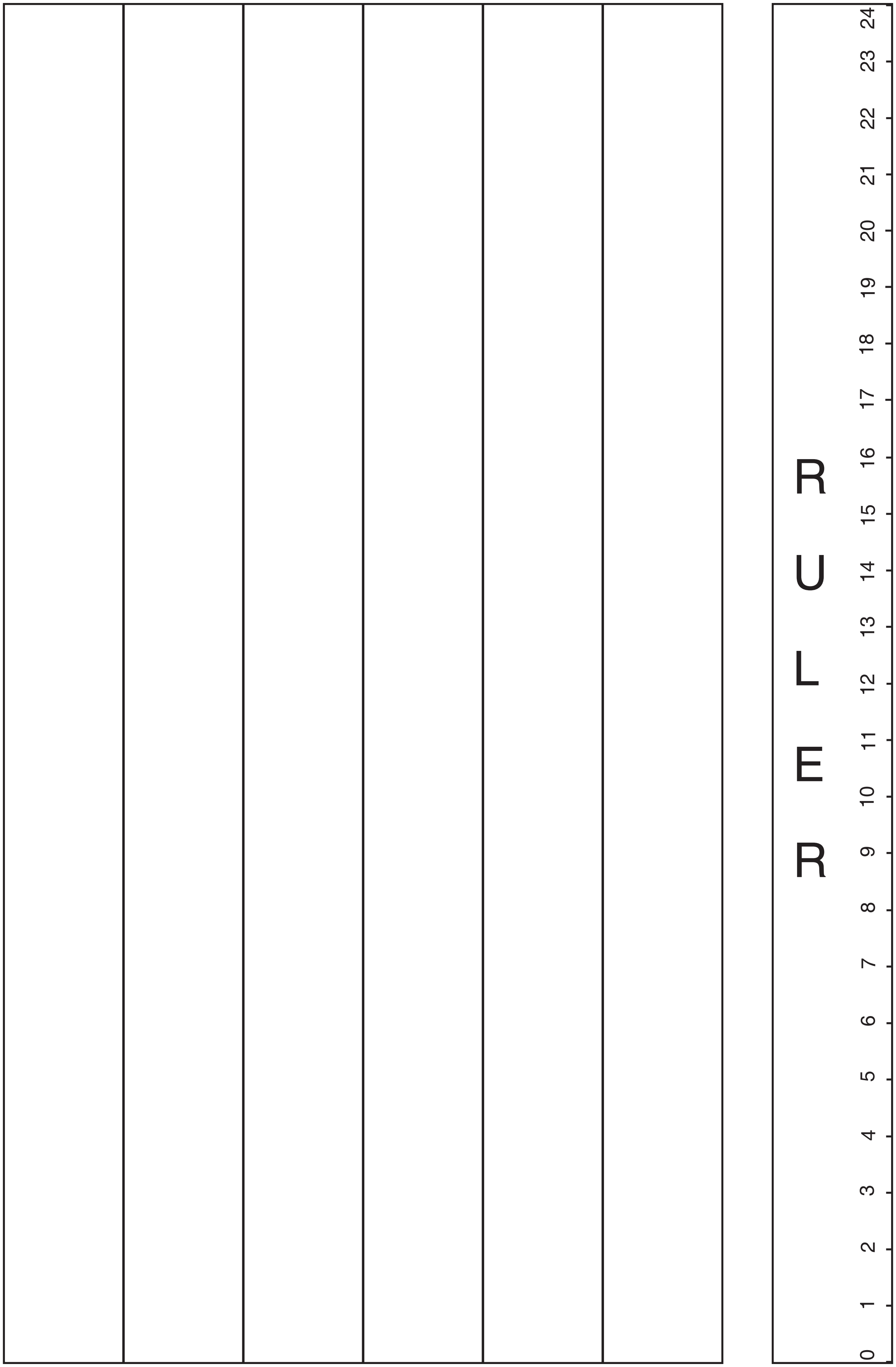


   (c) Repeat (a) for lines D and C.
   (d) Repeat (a) for lines B and C.

4. In the figure below, there are equally spaced points on each line. try to draw a design without putting the numbers on the lines.

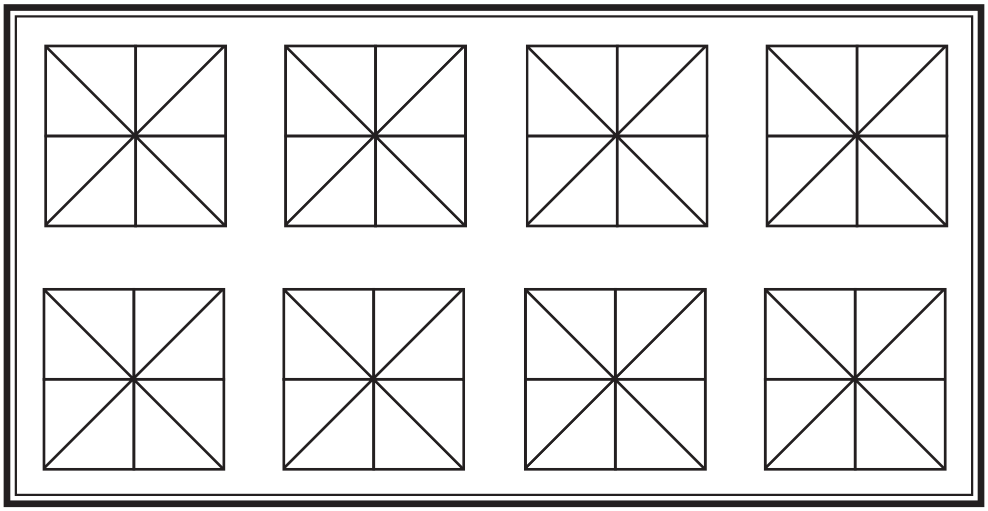


BLM 10: Fraction Strips

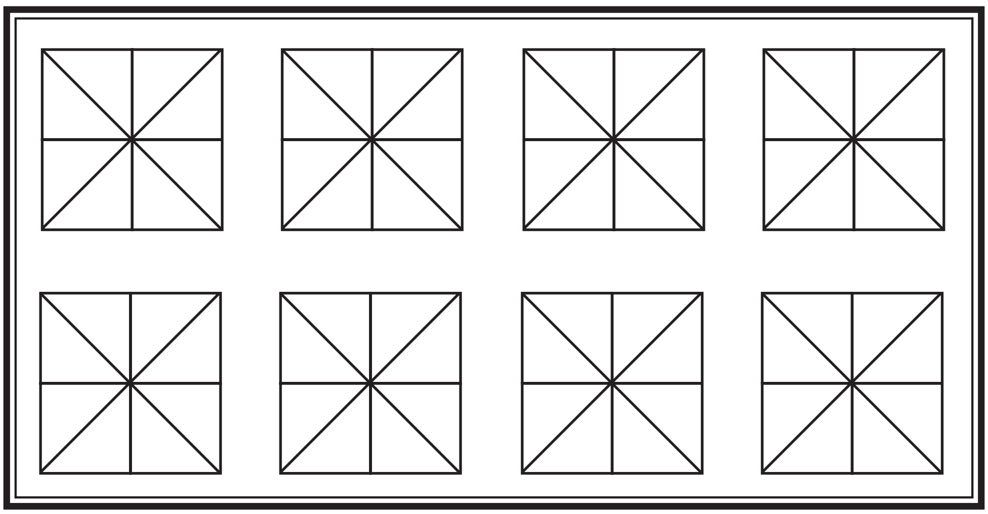


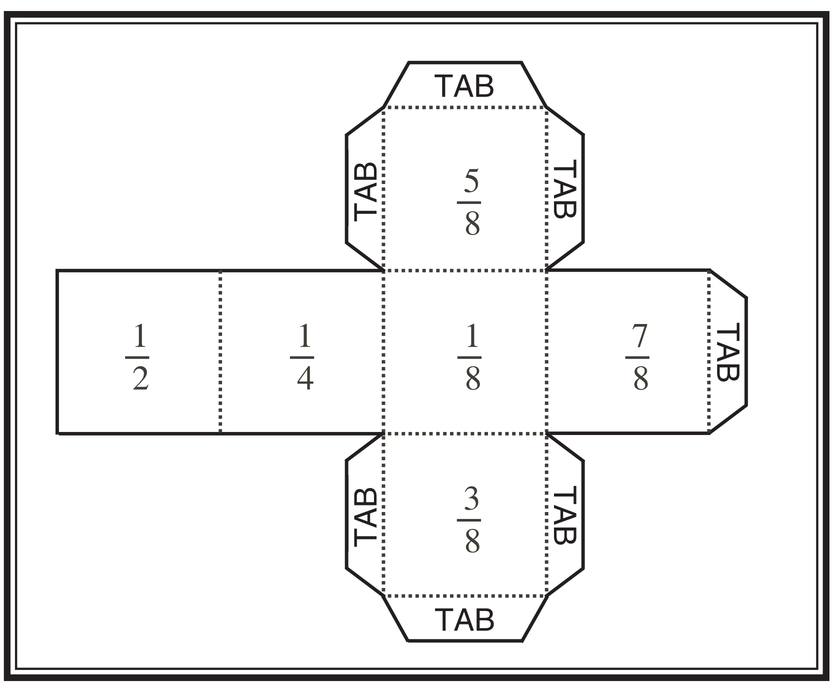
BLM 11: Counting to One

Gameboard 1:



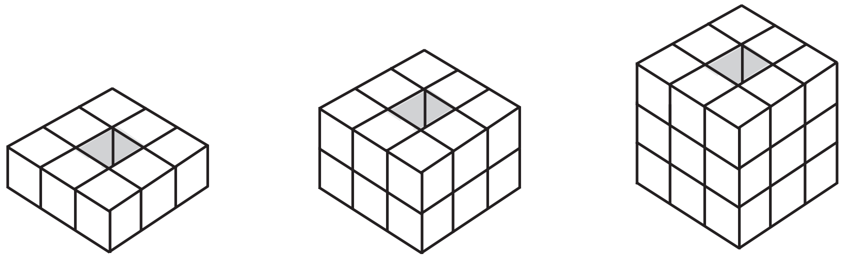
Gameboard 2:





BLM 12: Cubes, Squares, and Triangles

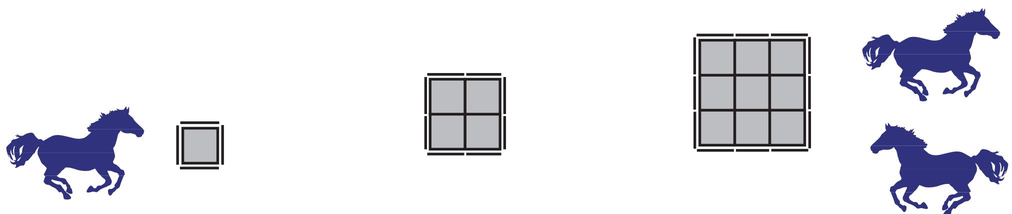
1.



This well uses 8 cubes. This well, with 2 layers, uses 16 cubes. This well, with 3 layers, uses 24 cubes.

How many layers will there be if the well is built using 64 cubes? How do you know?

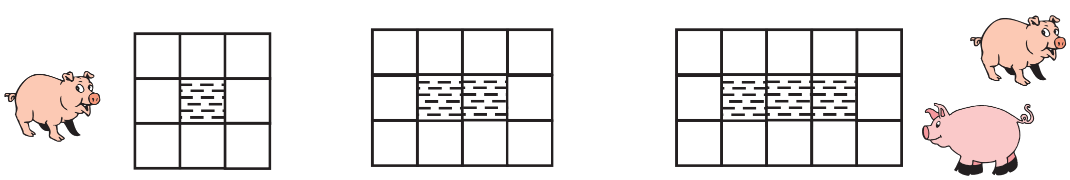
2. A farmer uses lengths of fencing to corral his ponies.



This pen holds 1 pony. This pen holds 2 ponies. This pen holds 3 ponies.

How many lengths of fencing will the farmer need for 8 ponies?

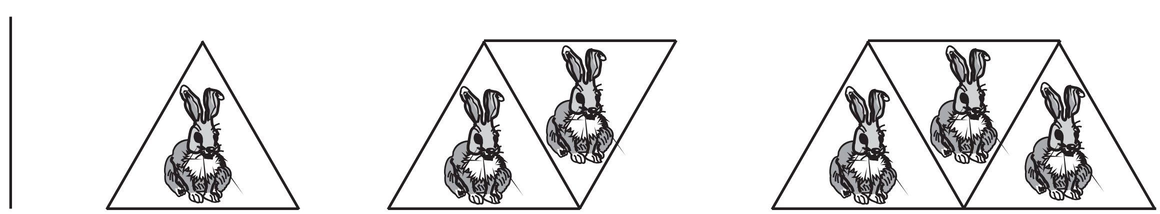
3. A farmer uses bales of hay to fence his pigs. Top views of his pens are shown below.



This pen holds 1 pig. This pen holds 2 pigs. This pen holds 3 pigs.

How many bales of hay will the farmer need for 10 pigs? How do you know?

4. Rail fences made of logs, all the same length, make individual pens for the rabbits.

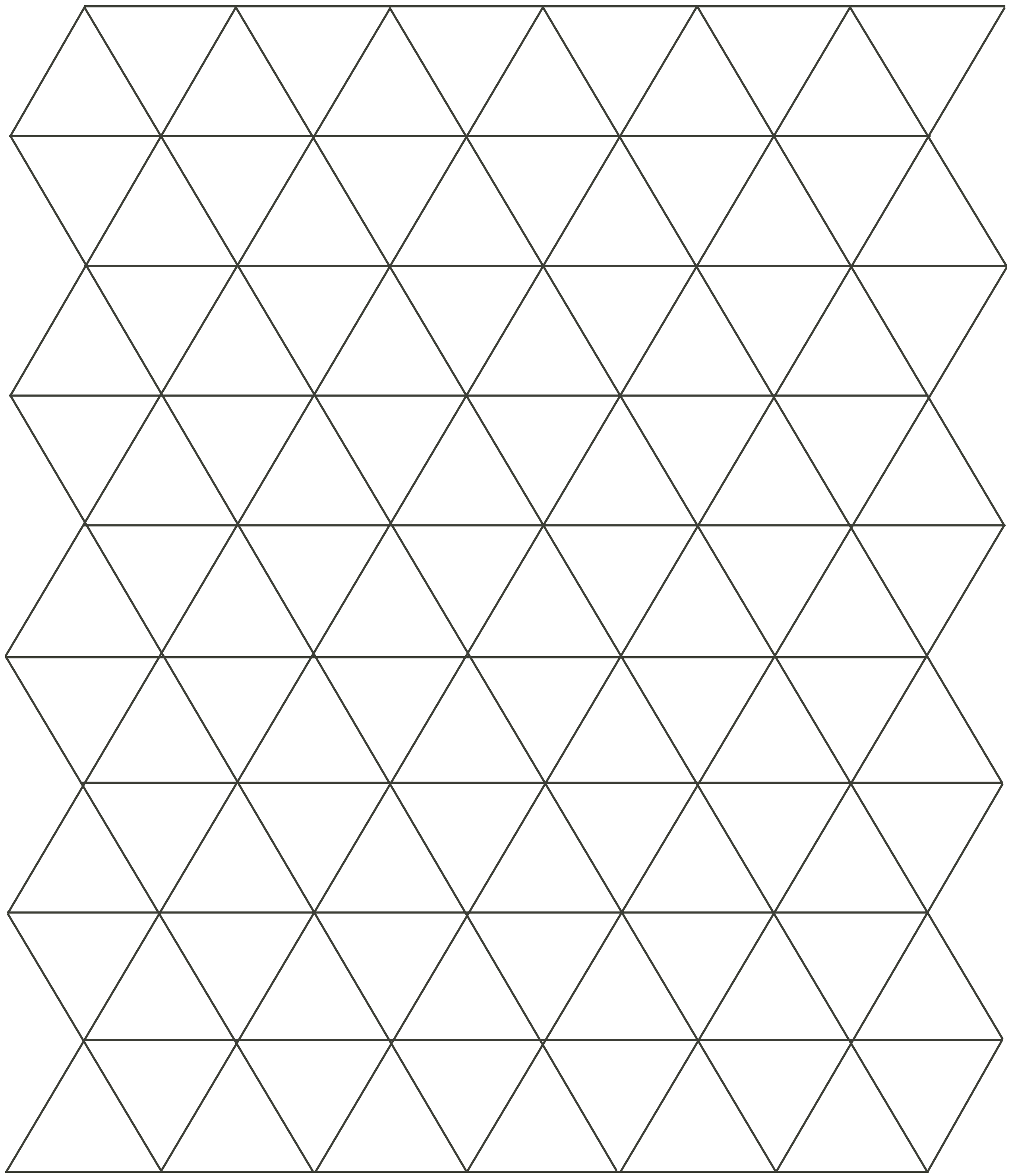


One log One pen for 1 rabbit Pens for 2 rabbits Pens for 3 rabbits

How many logs will be needed to pen 10 rabbits?

BLM 13: Squares

BLM 14: Triangles



BLM 15: Building Patterns - 1

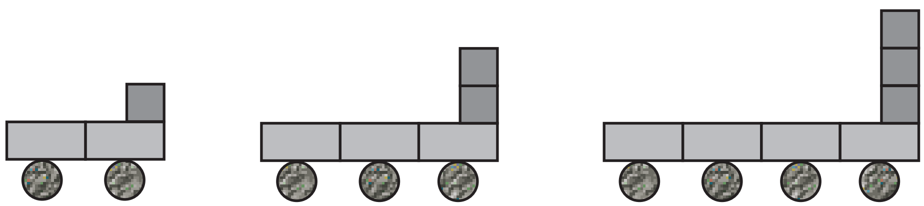
1. Chris was making patterns one day with rubber stamps. The pictures show the first three steps. Look for a pattern to help you complete the table.

Step 1 Step 2 Step 3

Step	1	2	3	4	5	10	100
Stars	1	2					
Squares	2	3					

Describe your pattern.

2. Chris's little sister started building trains with bricks, trying to copy Chris's pattern. Look for a pattern to help you complete the table.

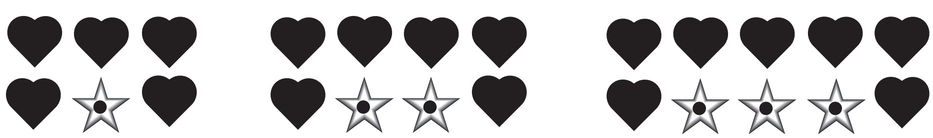


Train #1 Train #2 Train #3

Train	1	2	3	4	5	10	100
Squares	1	2					
Rectangles	2	3					
Circles	2						

Describe your pattern.

3. Count the stars and hearts used in making these designs. Look for a pattern to help you complete the table.



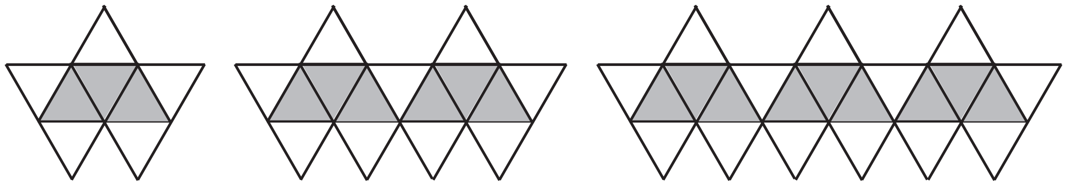
Step 1 Step 2 Step 3

Step	1	2	3	4	5	10	100
Stars	1	2					
Hearts	5	6					

Describe your pattern.

BLM 16: Building Patterns - 2

4. The stars below are made up of white triangles and shaded triangles.
Look for a pattern to help you complete the table.

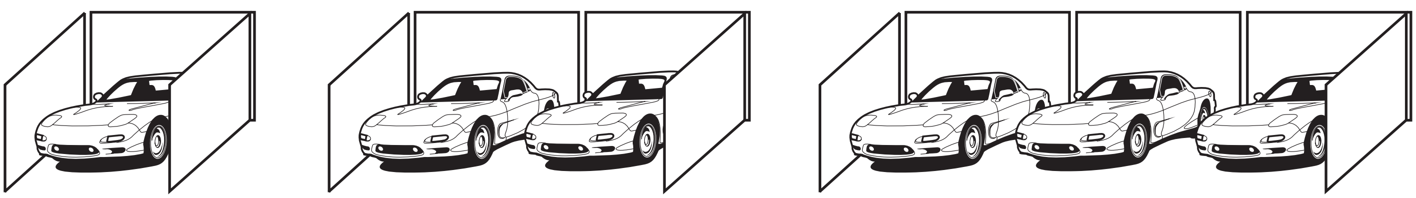


Star #1 Star #2 Star #3

Star	1	2	3	4	5	10	100
Shaded Triangles	3	6					
White Triangles	5	9					

Describe your pattern.

5. Chris's brother began building garages for his toy cars, using pieces of cardboard for the walls. For 1 car he used 3 wall pieces and had 1 door. For 2 cars he used 4 wall pieces and had 2 doors. His garage for 3 cars is shown below.

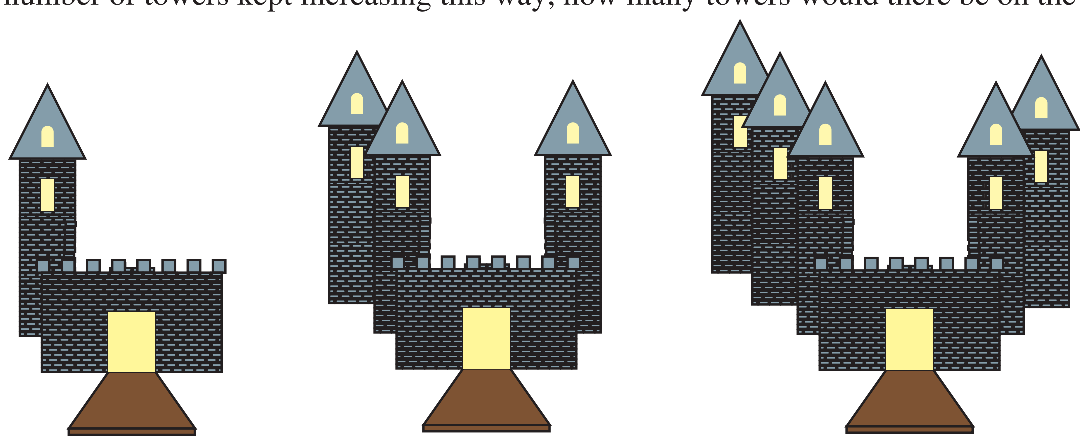


1 car 2 cars 3 cars

If he continues with this pattern, how many wall pieces would he need for 10 cars? for 100 cars (if he had 100 cars)? How many doors would he need for 100 cars?

CHALLENGE:
If each wall piece was 20 cm wide and 10 cm high, how much room would Chris's brother need for the garage for 100 cars?

6. In the Mythical land of Tara, a new wizard was testing his magic by making castles. His first three castles are shown below. If the number of towers kept increasing this way, how many towers would there be on the 100th castle?



First castle Second castle Third castle

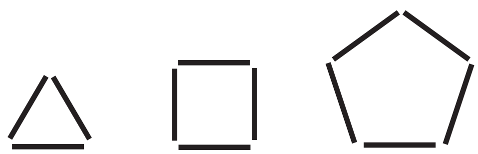
BLM 17: Toothpick Patterns

1. Use toothpicks to make boxes. Comtinue the pattern and complete the table.

Box Number	1	2	3	4	5	6	7	10	100
Number of Toothpicks	4	6							

Describe your pattern in words.

2. Use toothpicks to make polygons. Continue the pattern and complete the table.



Polygon 1 Polygon 2 Polygon 3

Polygon Number	1	2	3	4	5	6	7	10	100
Number of Toothpicks	3	4							

Describe your pattern in words.

3. Use toothpicks to build towers. Continue the pattern and complete the table.

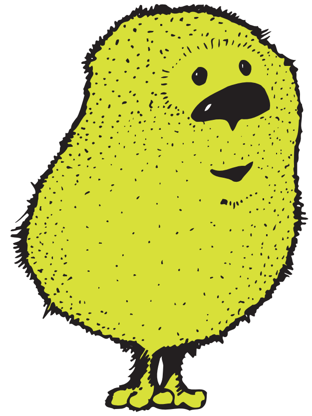
Tower 1 Tower 2 Tower 3 Tower 4

Tower Number	1	2	3	4	5	6	7	10	100
Number of Toothpicks	1	3	5						

Describe your pattern in words.

BLM 18: Wurffles and Garnobs

1. These are wurffles: 2 24 62 46 98
These are not-wurffles: 101 51 63 99 37
Which of these are wurffles? 3 27 34 50 82
What is a wurffle?
Write two more wurffles.



2. These are podlaps: 3 + 4 2 + 5 8 – 1 12 – 5 96 – 89
These are not-podlaps: 3 + 3 2 + 7 8 – 3 5 + 9 1 + 17
Which of these are podlaps? 6 + 1 12 – 4 20 – 13 80 + 7 99 – 2
What is a podlap?
Write two more podlaps.

3. These are garnobs: 123 678 579 356 189
These are not-garnobs: 251 768 404 521 769
Which of these are garnobs? 145 273 876 568 259
What are garnobs?
Write two more not-garnobs.

4. These are lundies: 33 55 99 88 11
These are not-lundies: 444 67 808 5 999
Which of these are lundies? 77 505 222 44 660
What is a lundy?
Write two more not-lundies.

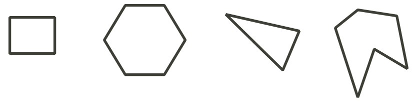
5. These are dawnlies: TWELVE TWENTY EIGHTY NINETY
These are not-dawnlies: TWO SIXTY FORTY FIFTY
Which of these are dawnlies? SEVEN ELEVEN THIRTY ONE FIVE
What is a dawnly?
Write two more not-dawnlies.

6. These are hapters: A M T H
These are not-hapters: C P G D
Which of these are hapters? V F W X O
What is a hapter?
Write one more hapter and one not-hapter.

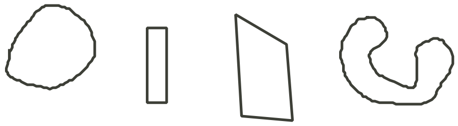
BLM 19: Find the Gurbs

1. These are gurbs:

These are not-gurbs:

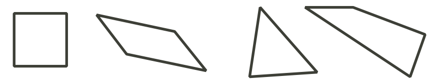


Which of these are gurbs? How do you know?

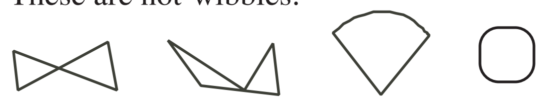


What is a gurb? Draw two more gurbs.

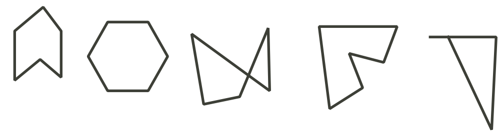
2. These are wibbles:



These are not-wibbles:

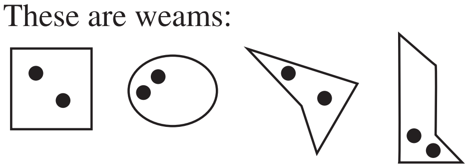


Which of these are wibbles? How do you know?

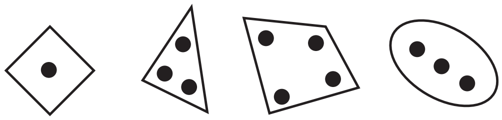


What is a wibble? Draw two more not-wibbles.

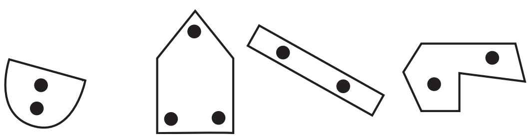
3. These are weams:



These are not-weams:

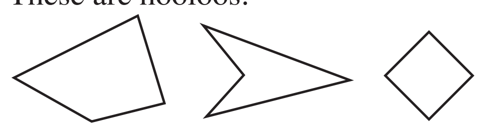


Which of these are weams? How do you know?



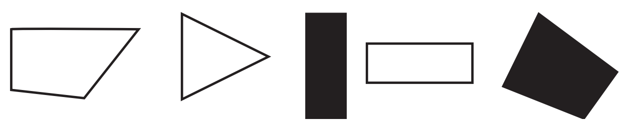
What is a weam? Draw two more weams.

4. These are hooloos:



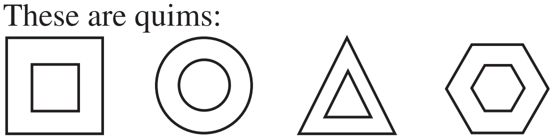
These are not-hooloos:

Which of these are hooloos? How do you know?

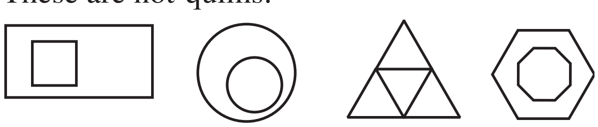


What is a hooloo? Draw two more not-hooloos.

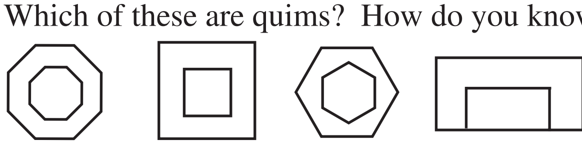
5. These are quims:



These are not-quims:

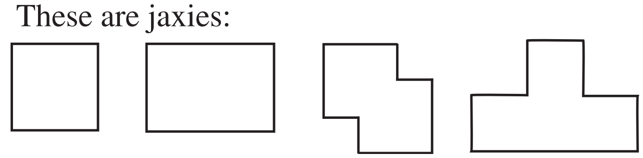


Which of these are quims? How do you know?

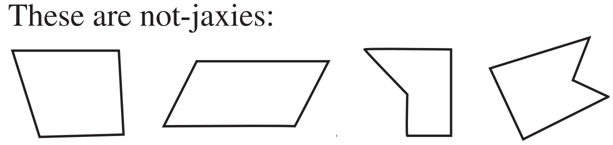


What is a quim?
Draw one more quim, and one more not-quim.

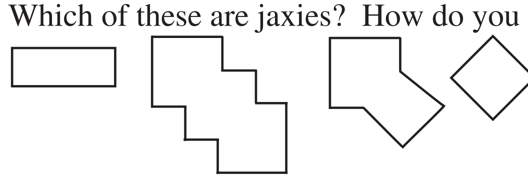
6. These are jaxies:



These are not-jaxies:



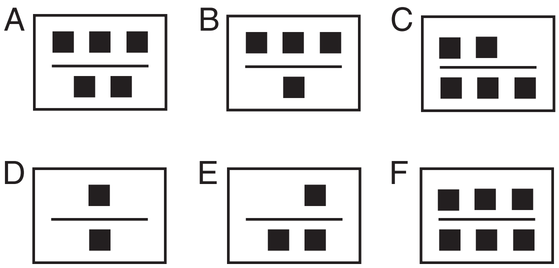
Which of these are jaxies? How do you know?



What is a jaxy?
Draw one more jaxy, and one more not-jaxy.

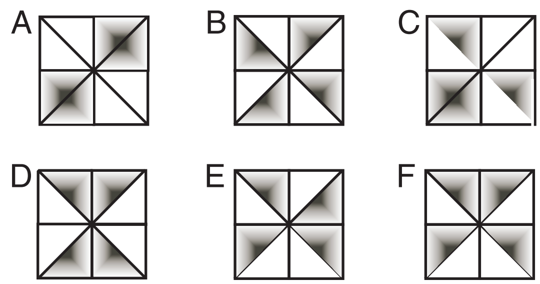
BLM 20: Which Two Are Different?

1. Two pictures are different from the rest.
   Which two?



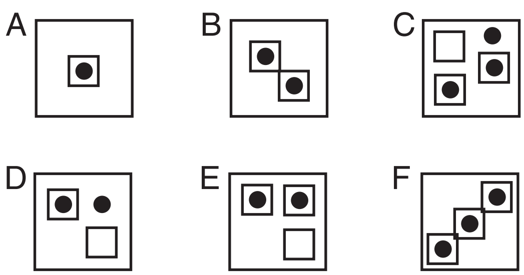
How are they different?

2. Two pictures are different from the rest.
   Which two?



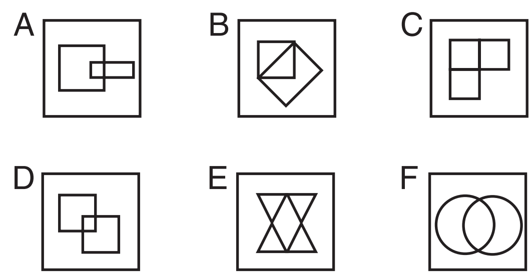
How are they different?

3. Two pictures are different from the rest.
   Which two?



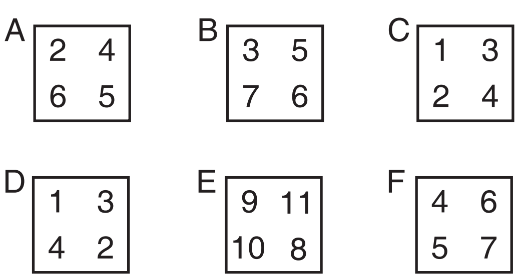
How are they different?

4. Two pictures are different from the rest.
   Which two?



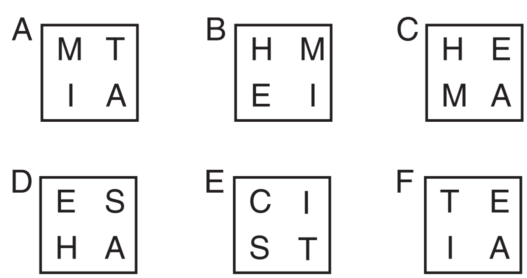
How are they different?

5. Two pictures are different from the rest.
   Which two?



How are they different?

6. Two pictures are different from the rest.
   Which two?



How are they different?

Solutions & Notes

Activity 1: Calendar Patterns

BLM 1

Students may be interested in doing some research about the well-known mathematicians whose birth dates are given on the calendars. Each one contributed a great deal to the development of mathematics. Students should be made aware of the fact that all of mathematics is not a "fait accompli", but that new mathematics is being developed all the time.

At one period, long division (done by a very complicated method not used today) was part of the mathematics taught at university, whereas now it is part of an elementary school curriculum.

Some information about each of the mathematicians mentioned on BLM 1 is given below.

1. Rene Descartes; March 31, 1596: developed the system of co-ordinate graphing that we use today.

2. Blaise Pascal; June 19, 1623: explored patterns in and applications of a particular number array known as "Pascal's Triangle". (For more on Pascal's Triangle see the grade 5 book in this series, Investigations in Patterns and Algebra.)

3. Maria Gaetana Agnesi; May 16, 1718: was so immersed in her studies that at one point she discovered that she had written out a solution for a particularly difficult problem in her sleep.

4. Marie Sophie Germain; April 1, 1776: studied symmetry as related to music, as well as modular arithmetic, which is similar to arithmetic in different bases. The digit sums used for BLM 8 are related to modular arithmetic.

5. Carl Frederick Gauss; April 30, 1777: checked the payroll for his father's business while still a young child; explored patterns with no end (e.g., 1 + 2 + 4 + 8 + 16 + ...). One type of infinite pattern can be seen on BLM 16 (Number Loops) in the grade 6 book of this series.

6. Evariste Galois; Oct. 25, 1811: was exceptionally good at complex mental mathematics; died in a duel at the age of 21, after spending the previous night at work trying to write as many of his discoveries in mathematics as he could in case he should be killed.

BLM 2: Hundred Charts

Students may identify a variety of patterns on the 100 charts. For example:

All the numbers in the first (or second or third ...) column (in either chart) end with '1'(or '2' or '3 '...).

All the numbers on one (oblique) line (in Chart 2) have two digits the same, except for 0." (i.e., 11, 22, 33, 44, ...)

When you count down a column (in either chart) you count by tens, starting with the top number.

All the numbers in the first row (in Chart 2) are single digits.

Whenever you move to the right (in either chart) you add one, and when you move to the left you subtract one.

All the numbers under the '2' (in either chart) can be divided by 2 exactly.

Solutions & Notes

To design a 100 chart that will not have the even numbers in straight columns (see the problem given just preceding Extensions in Mathematics) students need to be aware that in most of the charts given on BLMs 2 and 3, the even numbers occur in columns (or rows in Chart 3 on BLM 3). Students should realize that designing a 100 chart in which the even numbers are not in straight columns or rows means constructing a chart with an odd number of columns or rows. For example,

1	2	3	4	5	or	1	2	3	
6	7	8	9	10		4	5	6	
11	12	13	14	15		7	8	9	
16	17	18	19	20		10	11	12	
			etc.						etc.

Extensions in Mathematics

BLM 4: Addition and Multiplication Patterns

1. Students should realize that the numbers in, say, the first column of either chart are the same as the numbers in the first row.

2. In the addition chart the sixes are all on one oblique line, from 0 + 6 to 6 + 0 (and similarly for each number from 1 to 17). This will not be so in the multiplication chart.

3. One diagonal of the addition chart contains the even numbers 0, 2, 4, 6, 8, 10, 12, 14, 16. Students should recognize all of these as 'doubles' (i.e., 0 + 0, 1 + 1, 2 + 2, etc.)

4. Other patterns identified may include
   "In the addition chart you count by ones across each row and down each column."
   "For the addition chart, the numbers on one side of the 'doubles' (see above) are like a mirror image of the numbers on the other side." In other words there is a symmetry to the chart. Every number in the upper right half of the chart has a mirror image in the lower left portion. This is also true for the multiplication chart. Students should understand that this means if they know a fact, say 3 + 4 or 7×8, they also know the mirror image fact (4 + 3 or 8×7).

5. Comparing the rows and columns of the zeroes on the two charts can be used to reinforce the difference between adding zero and multiplying by zero.

6. In the multiplication chart all cells containing a particular number do not occur in a straight line as in the addition chart. In fact they are often separated from each other. Students should use the symmetry of the chart to help them locate all the sixes or all the eights or all instances of any other product.

7. The numbers in one diagonal line of the multiplication chart are all square numbers, formed by the multiplication of a number by itself.

Solutions & Notes

BLM 5: Lattice Multiplication

1. Students should be able to find similarities among the three examples to show them how to use lattice multiplication. The steps are described below:

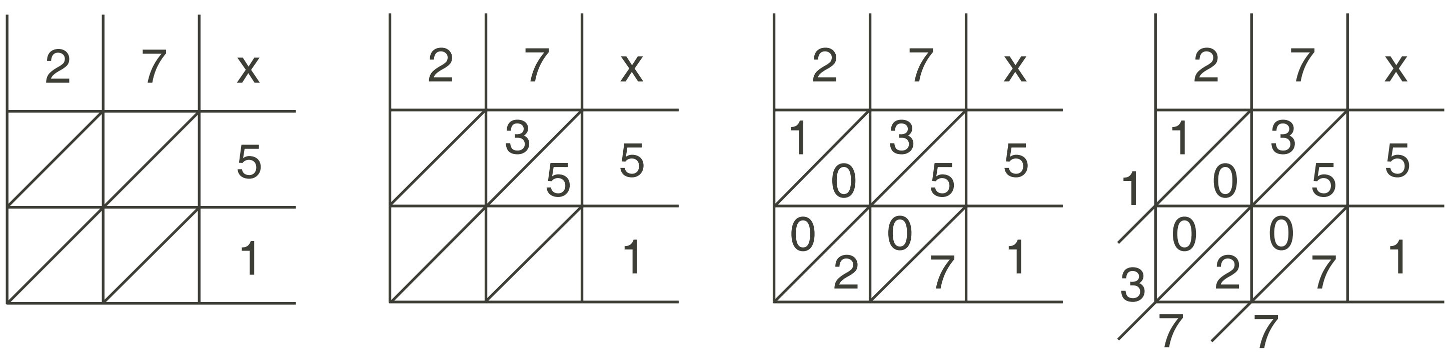
Step 1: Write the numbers being multiplied at the top and on the right of the lattice.

Step 2: Multiply one of the digits from one factor by one of the digits from the other factor (e.g., 7 x 5) and place the answer in the appropriate place. Note that the four multiplications in this Example 1 (i.e., , 7 x 5, 7 x 1, 2 x 5, and 2 x 1) may be done in any order, as long as the answers are placed correctly. The tens and units digits of each of these answers are separated by the diagonal of the square into which that answer is placed, with the tens digit in the upper left, and the units digit in the lower right.

Step 3: Complete all the single digit multiplications necessary and place the answers appropriately.

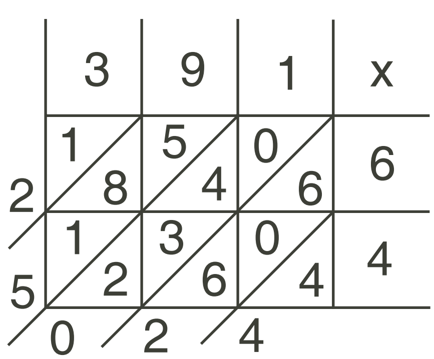
Step 4: Add the numbers along the diagonals, beginning at the right. If there is a need for regrouping during the addition, as there in is Example 2, BLM 5, do it. Notice that there is no regrouping necessary during the multiplication phase for any problem.

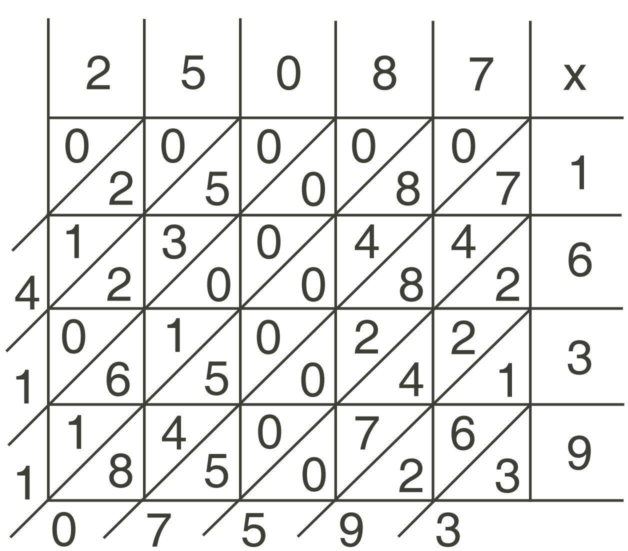
Example 1: 27 x 51 = 1377



Step 1 Step 2 Step 3 Step 4

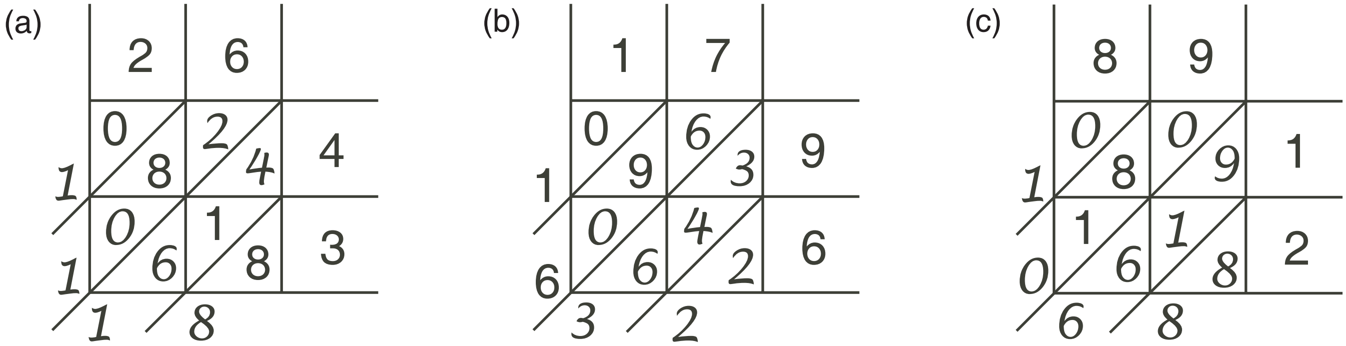
As can be seen by the examples below lattice multiplication can be extended to multipliers and multiplicands of any number of digits.





Solutions & Notes

Solutions for BLM 5, #5



Lattice multiplication is a technique that has been found helpful for some students who have learning difficulties and for students who remember their multiplication facts but forget to "carry" or "regroup" as in the traditional algorithm. Another advantage to this method is that it shows all the steps the students has taken to reach an answer. This can help the teacher identify any difficulties the student is having with multiplication.

BLM 6: Russian Peasant Multiplication

The rules for Russian Peasant multiplication are easy to learn, although the explanation as to why it works is somewhat complex.

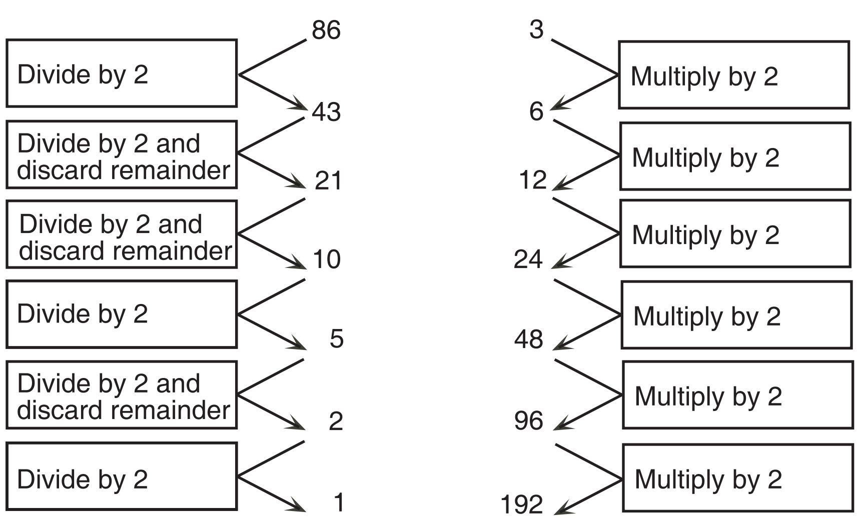
METHOD:

Start with the numbers you are multiplying, and place each of them at the top of a column. The numbers in one of these columns will be repeatedly divided by 2, while the other will be repeatedly multiplied by 2. Generally the greater number is the one divided by 2, though this is not essential.

One unusual aspect of this method is that, if there is a remainder of '1' after a division by 2, this remainder is simply discarded.

The examples given here and on BLM 6 have the "division by 2" column on the left, and the "doubling" column on the right, but this arrangement is not critical.

Example: 86 x 3



Solutions & Notes

To determine the product of 86 and 3, cross out any number pairs for which the number in the left column (or whichever column in which the numbers have been repeatedly divided by 2) is even. In the diagram below, these number pairs are in the shaded boxes.

86	3
43	6
21	12
10	24
5	48
2	96
1	192

Calculate the total of the numbers remaining in the column on the right (the column in which the numbers have been repeatedly doubled). This is the product of 86 and 3.

Thus, $86 \times 3 = 6 + 12 + 48 + 192 = 258$

Solutions for #4, BLM 6:

(a)

~~26~~	~~15~~
13	30
~~6~~	~~60~~
3	120
1	240
	390

(b)

~~54~~	~~35~~
27	70
13	140
~~6~~	~~280~~
3	560
1	1120
	1890

(c)

325	31
~~162~~	~~62~~
81	124
~~40~~	~~248~~
~~20~~	~~496~~
~~10~~	~~992~~
5	1984
~~2~~	~~3968~~
1	7936
	10075

Solutions & Notes

An explanation of the Russian Peasant Multiplication follows(not recommended for grade 4):

An understanding of base two is needed. Base two is compared with base ten (our standard decimal system) below.

Base ten	Base two
Numbers are grouped in tens Place value columns are units or ones (1) tens (10) tens of tens (10×10 or 100 or 10^2) tens of tens of tens($10\times10\times10$ or 1000 or 10^3) tens of tens of tens of tens ($10\times10\times10\times10$ or 10000 or 10^4) and so on	Numbers are grouped in twos Place value columns are units or ones (1) twos (2) twos of twos (2×2 or 4 or 2^2) twos of twos of twos ($2\times2\times2$ or 8 or 2^3) twos of twos of twos of twos ($2\times2\times2\times2$ or 16 or 2^4) and so on
Available digits are 0, 1, 2, 3, ..., 9 The greatest available digit is one less than the base number	Available digits are 0, 1 The greatest available digit is one less than the base number

Thus, a number written in base two will be composed of zeros and ones.
For example, 10110.
On a place value chart this is

2^8 or 256	2^7 or 128	2^6 or 64	2^5 or 32	2^4 or 16	2^3 or 8	2^2 or 4	2	1
				1	0	1	1	0

To translate this into base ten numbers, we calculate as follows:

$$1\times16+1\times4+1\times2=22$$

Now we return to the Russian Peasant multiplication example at the top of BLM 6, and label the numbers in the "dividing by 2" column, labelling the odd numbers with a '1' and the even numbers with a '0'. From the bottom up this gives us 1010110. On the place value chart this would be

86	0
43	1
21	1
10	0
5	1
2	0
1	1

2^8 or 256	2^7 or 128	2^6 or 64	2^5 or 32	2^4 or 16	2^3 or 8	2^2 or 4	2	1
		1	0	1	0	1	1	0

or $64 + 0 + 16 + 0 + 4 + 2 + 0 = 86$

Solutions & Notes

This is applied to the multiplication as follows, using the base two digits from right to left in the chart above:

86	0		3
43	1 x **2**	two groups of '3'	**6**
21	1 x **4**	four groups of '3'	**12**
10	0		24
5	1 x **16**	sixteen groups of '3'	**48**
2	0		96
1	1 x **64**	sixty-four groups of '3'	**192**

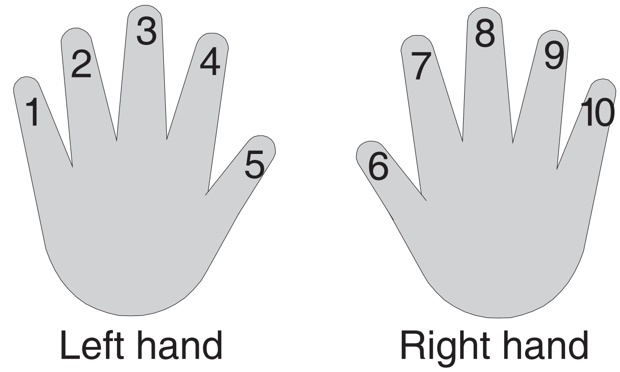
The numbers in the second column come from the place value table. These numbers are then used to 'count' groups of '3', the multiplier at the top of the right hand column. If all the groups of '3' are added together, we find that we have 2 + 4 + 16 + 64 = 86 groups of '3', which is our original multiplication question. Hence the sums of the products for these groups (6 + 12 + 48 + 192) gives the correct answer to 86×3.

Solutions & Notes

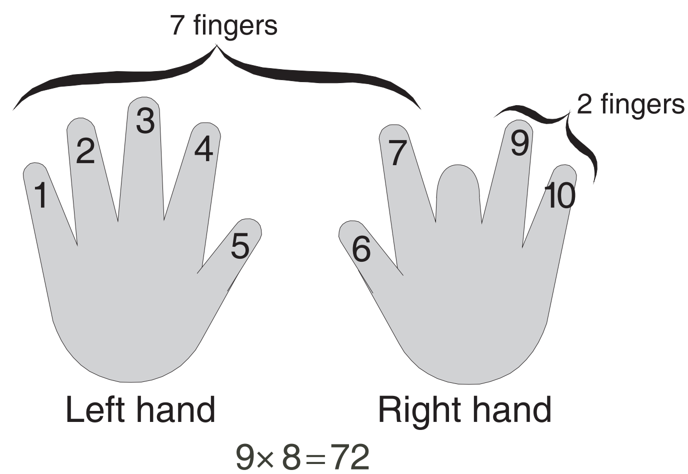
Activity 2: Circular Patterns

Multiplying by 9 with one's fingers.

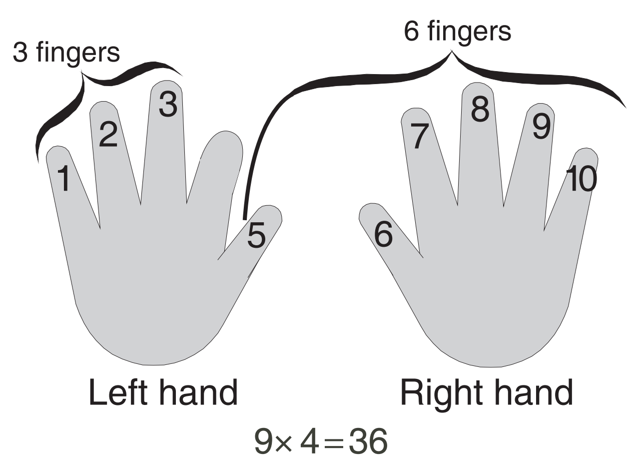
Number each finger as shown. If students are unable to remember which number belongs to which finger, have them write the numbers on bits of masking tape and stick these to their fingernails.



Example 1: 9×8
Bend 'finger 8'.
Count the number of fingers on each side of finger 8.
The number of fingers to the left of finger 8 gives the tens digit of the answer, 7.
The number of fingers to the right of finger 8 gives the units digit of the answer, 2.



Example 2: 9×4
Bend 'finger 4'.
Count the number of fingers on each side of finger 4.
The number of fingers to the left of finger 4 gives the tens digit of the answer, 3.
The number of fingers to the right of finger 4 gives the units digit of the answer, 6.



Ask students if they think this method (or a similar one) could be used to multiply by other numbers.
The answer is that this works only with muliplication by 9 since

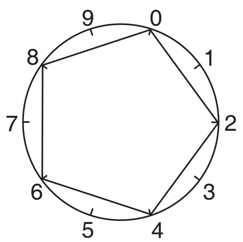
a) we have 10 fingers, and
b) answers in the 9 times table have digits that total 9 (the number of fingers left after we have folded one down.)

Solutions & Notes

Activity 2: Circular Patterns

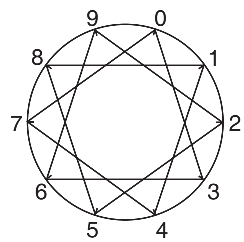
BLM 7: Skip Counting Patterns

(a) **x 2**



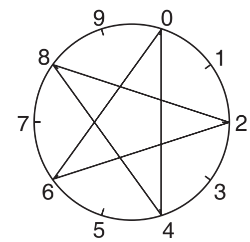
0, 2, 4, 6, 8, 0

(b) **x 3**



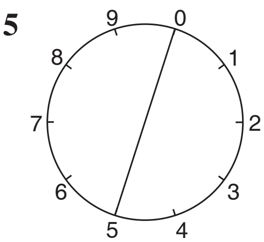
0, 3, 6, 9, 2, 5, 8, 1, 4, 7, 0

(c) **x 4**



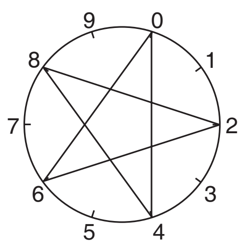
0, 4, 8, 2, 6, 0

(d) **x 5**



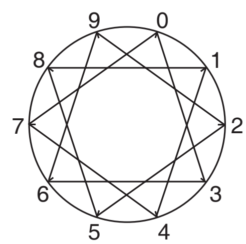
0, 5, 0

(e) **x 6**



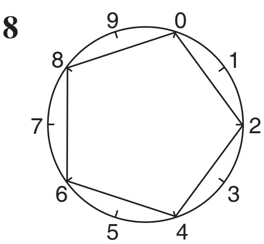
0, 6, 2, 8, 4, 0

(f) **x 7**



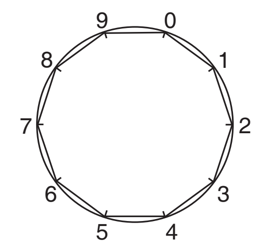
0, 7, 4, 1, 8, 5, 2, 9, 6, 3, 0

(g) **x 8**



0, 8, 6, 4, 2, 0

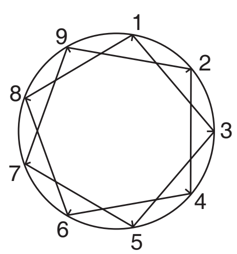
(h) **x 9**



0, 9, 8, 7, 6, 5, 4, 3, 2, 1, 0

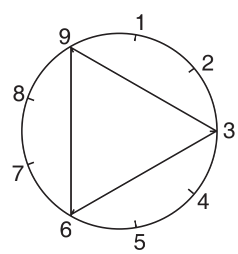
BLM 8: Digit Sums Patterns

(a) **x 2**



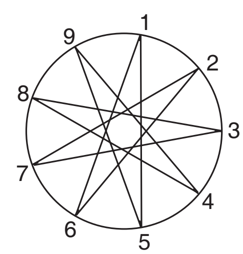
2, 4, 6, 8, 1, 3, 5, 7, 9, 2

(b) **x 3**



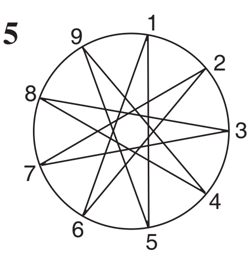
3, 6, 9, 3

(c) **x 4**



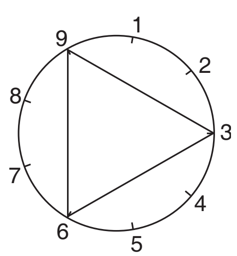
4, 8, 3, 7, 2, 6, 1, 5, 9, 4

(d) **x 5**



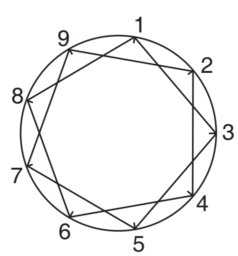
5, 1, 6, 2, 7, 3, 8, 4, 9, 5

(e) **x 6**



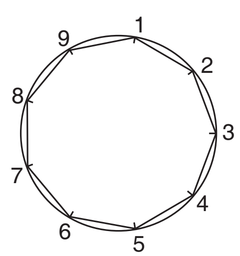
6, 3, 9, 6

(f) **x 7**



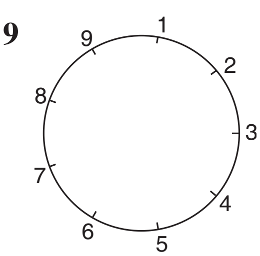
7, 5, 3, 1, 8, 6, 4, 2, 9, 7

(g) **x 8**



8, 7, 6, 5, 4, 3, 2, 1, 9, 8

(h) **x 9**



9, 9, 9, 9, 9

Solutions & Notes

BLM 9: Curve Stitching

Figure 1

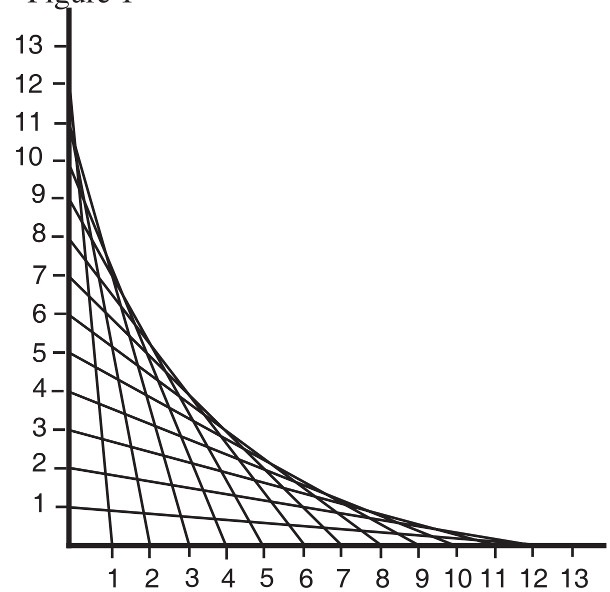
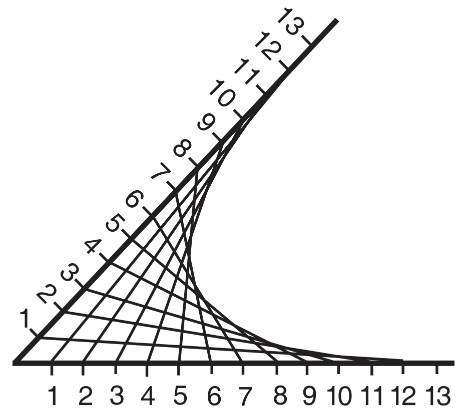


Figure 2



Repeat using Figure 2. How is this design the same as the first? How is it different?

Figure 3

Each section of the diagram has been completed. Students might complete only alternate sections or they may do something entirely different.

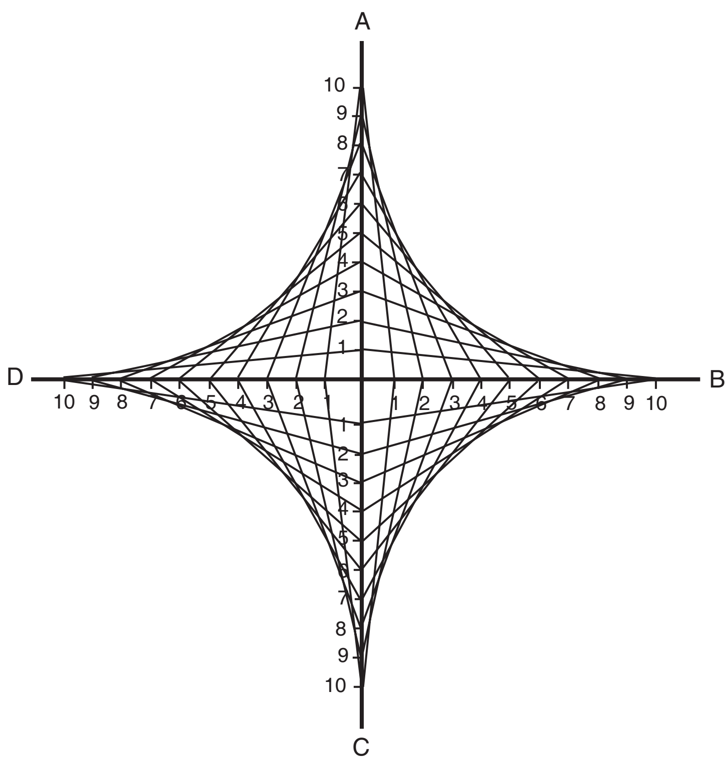
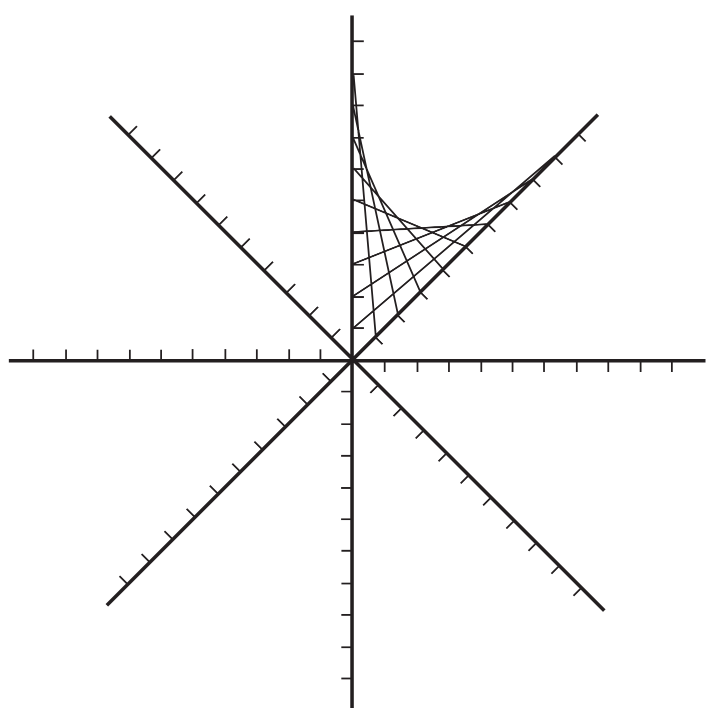


Figure 4

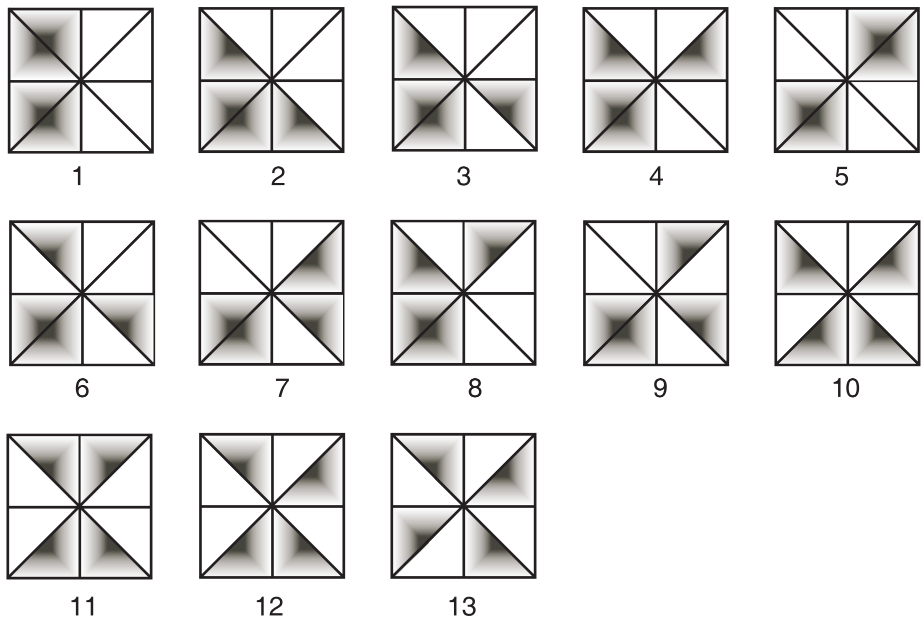
Students may complete each section of the diagram like the one shown, or they might complete only alternate sections or they may do something entirely different.



Activity 3: Fraction Patterns

BLM 11: Counting to One

The thirteen arrangements for colouring half the square are given below.



Have students identify designs with

- a) line symmetry [1, 2, 5, 6, 10, 11]
- b) rotation symmetry [5, 11, 13]
- c) no symmetry [3, 4, 7, 8, 9, 12]

Activity 4: Building Patterns

Number of bridges	1	2	3	4	5	10	100
Number of cubes	5	10	15	20	25	50	500

Square number	1	2	3	4	5	10	100
Number of squares	1	4	9	16	25	100	10000

Solutions & Notes

BLM 12: Cubes, Squares, and Triangles

If students record numbers in a table and develop a pattern, it should be relatively simple for them to solve problems 1-4. If they are familiar with spread sheets, they may wish to record the numbers using a computer.

1.

Number of layers in the well	1	2	3	4	5	6	7	8
Number of cubes needed	8	16	24	32	40	48	56	64

From the table we can see that a well of 8 layers will use 64 cubes.
Some students will need to continue the table from 1 layer to 8 layers to solve the problem posed. Others will see that the number of cubes used is 8 times the number of layers, and, therefore, 64 cubes will make a well of 8 layers.

2.

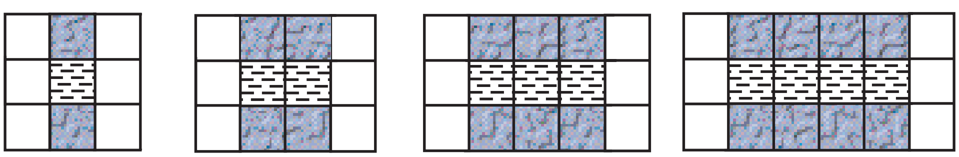
Number of ponies	1	2	3	4	5	6	7	8
Lengths of fencing	4	8	12	16	20	24	28	32

Some students will need to continue the table to 8 ponies to determine the number of lengths of fencing needed. Others will realize that the number of fencing lengths is 4 times the number of ponies.

3.

Number of pigs	1	2	3	4	5	6	7	8	9	10
Number of bales of hay	8	10	12	14	16	18	20	22	24	26

As before, some students will need to extend the table to 10 pigs to determine the number of bales of hay needed. Others may need to sketch (or build with squares) the various pens. Others may analyse the problem in different ways. One example is given here.



There are always 3 bales on each end (shown as). This makes 6.
For each pig (shown as) there are two other bales (shown as).
So, for 10 pigs, there would be $2\times10+6$, which is 26.

4.

Number of rabbits	1	2	3	4	5	6	7	8	9	10
Number of logs	3	5	7	9	11	13	15	17	19	21

As before, some students will continue the table and some may need to sketch the pens (or use toothpicks to construct the pens). Others will see that each additional rabbit needs 2 more logs.
A possible description of the pattern:
For 10 rabbits, the first will need 3 logs, and the other 9 will need 2 logs each. So 10 rabbits will need or 21 $2\times9+3$ logs.
Another possible description:
Every rabbit needs 2 logs, and the first rabbit needs one extra log. So 10 rabbits need $10\times2+1$ or 21 logs.

Solutions & Notes

BLM 15: Building Patterns-1

1.

Step	1	2	3	4	5	10	100
Stars	1	2	3	4	5	10	100
Squares	2	3	4	5	6	11	101

Some students will see that the number of squares is one more than the number of stars.
This pattern is similar to the rabbit-penning problem on BLM 12. In that one, each rabbit needed 2 logs plus 1 extra log at the beginning. Here, each star needs 1 square, plus 1 extra square at the beginning.

2.

Train	1	2	3	4	5	10	100
Squares	1	2	3	4	5	10	100
Rectangles	2	3	4	5	6	11	101
Circles	2	3	4	5	6	11	101

Students should see that the 'train number' (first row in the table) gives them the number of squares needed. The number of rectangles is one more than the number of squares and the number of circles equals the number of rectangles.

3.

Step	1	2	3	4	5	10	100
Stars	1	2	3	4	5	10	100
Hearts	5	6	7	8	9	14	104

The 'Step number' gives the number of stars. The number of hearts is 4 more than the number of stars.

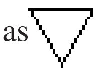
Some students may see a pattern similar to the pig-penning problem on BLM 12. Each star needs a heart, and there are 4 extra hearts (2 on each end), so the number of hearts needed is the number of stars plus 4.
Such a description will help students see why the number of hearts is always 4 more than the number of stars.

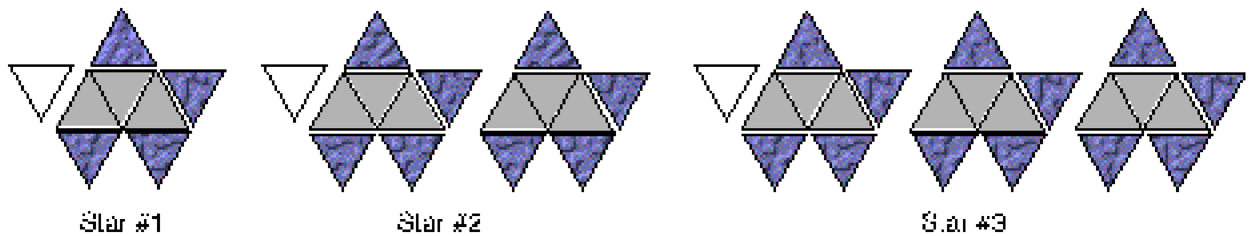
Solutions & Notes

BLM 16: Building Patterns-2

4.

Star	1	2	3	4	5	10	100
Shaded Triangles	3	6	9	12	15	30	300
White Triangles	5	9	13	17	21	41	401

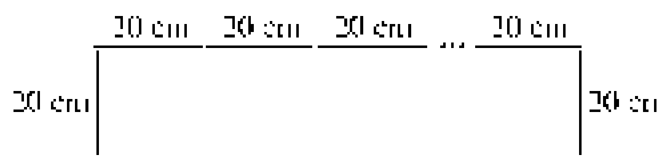
One possible description of the pattern:
For every star, the number of shaded triangles is 3 times the 'star number'. The number of white triangles (shown as  below) is 4 times the 'star number', plus 1 extra at the beginning (shown as  below).



5. If this problem is analysed as problem 4 above (see also problems 1 on BLM 15 and 4 on BLM 12) we can see that there is one wall piece behind each car plus 2 wall pieces on the ends. So, for 10 cars he will need 10 + 2 or 12 wall pieces. For 100 cars he will need 100 + 2 or 102 wall pieces.
The number of doors (not shown on the diagram but represented by the open front of each garage) is the same as the number of cars.

Challenge

Students may sketch a diagram showing the garage space.
For example:



The garage for 100 cars would be cm or 1000 cm or 10 m long and 20 cm wide.
Explore with students the possibility of a garage of this size. Could it be built in the classroom for example.
Note: The word "room" in the problem is deliberately vague. If you wish to have students compute either area or volume, this is possible with the information given.

6.

Number of castles	1	2	3	4	5	6
Number of towers	1	3	5	7	9	11

A possible description of the pattern:
After the first castle, the wizard managed to add a tower on each side. So, for 100 castles, he would have 200 towers minus the 1 he didn't make on the first castle. He would have 199 towers on the 100th castle.

Solutions & Notes

BLM 17: Toothpick Patterns

1.

Box Number	1	2	3	4	5	6	7	10	100
Number of Toothpicks	4	6	8	10	12	14	16	22	202

This can be analysed in a manner similar to #3 on BLM 12. There is always 1 toothpick on each end. This makes 2. For each 'box number' there are twice that many toothpicks.
thus, for Box 100 there are or 202 toothpicks.

2.

Polygon Number	1	2	3	4	5	6	7	10	100
Number of Toothpicks	3	4	5	6	7	8	9	12	102

The number of toothpicks is 2 more than the 'polygon number'.
You might wish to review the names of some of the polygons and reinforce how the names can indicate the number of toothpicks to build the polygon (i.e., the number of sides of the polygon).
For example, the triangle has 3 sides (needs 3 toothpicks)
the quadrilateral has 4 sides
the pentagon has 5 sides
the hexagon has 6 sides
Students could search dictionaries to discover other words that begin the same way and that refer to a number of something.
For example, triangle, triplets
quartet, quadruplets

3.

Tower Number	1	2	3	4	5	6	7	10	100
Number of Toothpicks	1	3	5	7	9	11	13	19	199

The number of horizontal toothpicks is the same as the tower number.
The number of vertical toothpicks is one less than the tower number.
Thus the number of toothpicks needed for the 100th tower is 100 + 99 or 199.
Some students may see this as "one less than twice the tower number" or "two times the tower number take away one."

Students may recognize the pattern in the table as identical with the pattern in #6 of BLM 16.
If they have not recognized the similarity, it would be worthwhile asking them to compare the two.

Solutions & Notes

Activity 5: What's The Pattern?

Students may identify different patterns. Discussion will allow them to defend their choices. The solutions given below are suggestions:

1. A wurffle is an even number.
2. A podlap has a sum or difference of 7. Students may suggest more elaborate possibilities for the "two more podlaps" (e.g., , ,). This is a good opportunity for assessment.
3. Garnobs have 3 digits in order of size. Thus 145 is a garnob because 1 is less than 4, and 4 is less than 5. Numbers such as (a) 143 or (b) 745 are not-garnobs because (a) 4 is not less than 3, and (b) 7 is not less than 4.
4. Lundies have two identical digits. The inclusion of 444 and 999 in the not-lundies shows clearly that lundies have two digits only.
5. Two possible solutions are:
   (a) dawnlies are number names that include an 'E';
   (b) dawnlies are number names having 6 letters.
6. Three possible solutions are:
   (a) a hapter is one of the letters of the word MATH;
   (b) a hapter is a letter with straight lines only;
   (c) a hapter is a letter with a vertical line of symmetry.

   Some students may identify what they think are the important characteristics of one of the sets of numbers, but fail to see that the negative examples do not agree with their conclusions. for example, a student may identify a lundy (#4) as having identical digits, but ignore the fact that it must have only 2 digits. Such a student will pick 222 as a lundy.

BLM 19: Find the Gurbs

As in BLM 18 problems, students may identify different solutions that given below. They should be encouraged to defend their opinions.

Suggested responses:
1. Gurbs are figures made entirely with curved lines.
2. Wibbles have straight sides and only one inside space, or they have straight sides that do not cross.
3. Weams may have straight or curved sides, but must have two dots.
4. Hooloos are white and have 4 sides.
5. Quims are pairs of similar figures with the inside one centred in the larger one.
   The word "similar", mathematically, means figures with the same shape, but not necessarily the same size.
6. Jaxies have only 90° angles (or "square corners").

Solutions & Notes

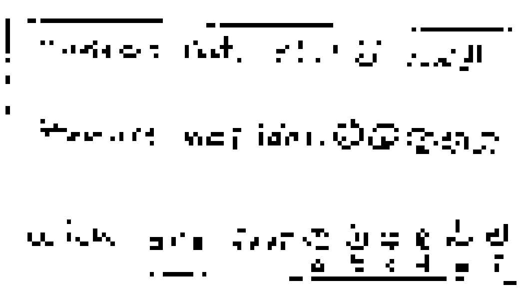
BLM 20: Which Two Are Different?

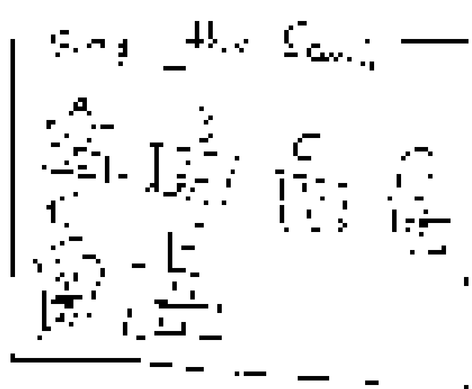
These problems are designed to be sure there are at least two possible solutions for each. Encourage students to identify as many different answers as they can. Some are given below.

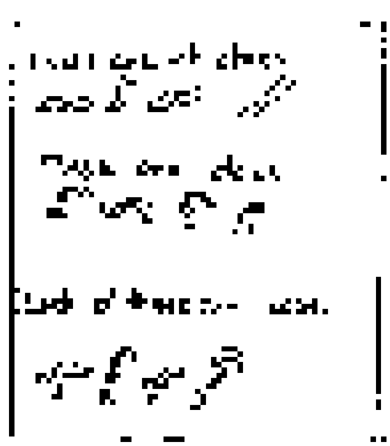
1. D and F are different, because they both have the same number of squares above the line as below the line.
   D and E are different, because all the others have at least one row of three squares.
2. A and C are different, because they are the only ones with a small shaded square.
   A and D are different, because they are the only ones with just two shaded areas.
   B and E are different because all the others have line symmetry.
3. B and D are the only ones two with two small squares.
   A and F are the only two with a small square right in the middle of the larger square.
   C and D are the only ones with a dot outside a small square.
4. E and F are the only two without squares.
   A and B are the only two that do not have identical figures.
5. A and B are the only two that do not contain four consecutive numbers.
   A and B are the only two that do not contain two even and two odd numbers.
   A and B are the only two whose total is not an even number.
   Note: The second and third rules given above are true only because the first one is. Challenge students to find a set of four consecutive numbers whose total is odd. Ask them why this is impossible.
6. B and C are the only two that contain letters from the first half of the alphabet only.
   D and E are the only two that contain curved letters.

Extensions in Mathematics

1. Samples of student work are given below. Notice that the differences are either very obvious or very confusing. The examples here were the students' first efforts.







Suggested Assessment Strategies

Investigations

Investigations involve explorations of mathematical questions that may be related to other subject areas. Investigations deal with problem posing as well as problem solving. Investigations give information about a student's ability to:

- identify and define a problem;
- make a plan;
- create and interpret strategies;
- collect and record needed information;
- organize information and look for patterns;
- persist, looking for more information if needed;
- discuss, review, revise, and explain results.

Journals

A journal is a personal, written expression of thoughts. Students express ideas and feelings, ask questions, draw diagrams and graphs, explain processes used in solving problems, report on investigations, and respond to open-ended questions. When students record their ideas in math journals, they often:

- formulate, organize, internalize, and evaluate concepts about mathematics;
- clarify their thinking about mathematical concepts, processes, or questions;
- identify their own strengths, weaknesses, and interests in mathematics;
- reflect on new learning about mathematics;
- use the language of mathematics to describe their learning.

Observations

Research has consistently shown that the most reliable method of evaluation is the ongoing, in-class observation of students by teachers. Students should be observed as they work individually and in groups. Systematic, ongoing observation gives information about students':

- attitudes towards mathematics;
- feelings about themselves as learners of mathematics;
- specific areas of strength and weakness;
- preferred learning styles;
- areas of interest;
- work habits — individual and collaborative;
- social development;
- development of mathematics language and concepts.

In order to ensure that the observations are focused and systematic, a teacher may use checklists, a set of questions, and/or a journal as a guide. Teachers should develop a realistic plan for observing students. Such a plan might include opportunities to:

- observe a small number of students each day;
- focus on one or two aspects of development at a time.

Suggested Assessment Strategies

Student Self-Assessment

Student self-assessment promotes the development of metacognitive ability (the ability to reflect critically on one's own reasoning). It also assists students to take ownership of their learning, and become independent thinkers. Self-assessment can be done following a co-operative activity or project using a questionnaire which asks how well the group worked together. Students can evaluate comments about their work samples or daily journal writing. Teachers can use student self-assessments to determine whether:

- there is change and growth in the student's attitudes, mathematics understanding, and achievement;
- a student's beliefs about his or her performance correspond to his/her actual performance;
- the student and the teacher have similar expectations and criteria for evaluation.

Resources for Assessment

"For additional ideas, see annotated Other Resources list on page 72, numbered as below."

1. The Ontario Curriculum, Grades 1-8: Mathematics.

2. *Linking Assessment and Instruction in Mathematics: Junior Years*, Ontario Association of Mathematics Educators/OMCA/OAJE, Moore et al., 1996.
   The document provides a selection of open-ended problems tested in grades 4, 5, and 6. Performance Rubrics are used to assess student responses (which are included) at four different levels. Problems could be adapted for use at the Junior Level. Order from OAME/AOEM, P.O. Box 96, Rosseau, Ont., P0C 1J0. Phone/Fax 705-732-1990.

3. *Mathematics Assessment: Myths, Models, Good Questions, and Practical Suggestions*, by Jean Kerr Stenmark (Ed.), NCTM, 1991.
   This book contains a variety of assessment techniques and gives samples of student work at different levels.
   Order from Frances Schatz, 56 Oxford Street, Kitchener, Ont., N2H 4R7. Phone 519-578-5948; Fax 519-578-5144. email: frances.schatz@sympatico.ca

4. *How to Evaluate Progress in Problem Solving*, by Randall Charles et al., NCTM, 1987.
   Suggestions for holistic scoring of problem solutions include examples of student work. Also given are ways to vary the wording of problems to increase/decrease the challenge. A section on the use of multiple choice test items shows how these, when carefully worded, can be used to assess student work.

Suggested Assessment Strategies

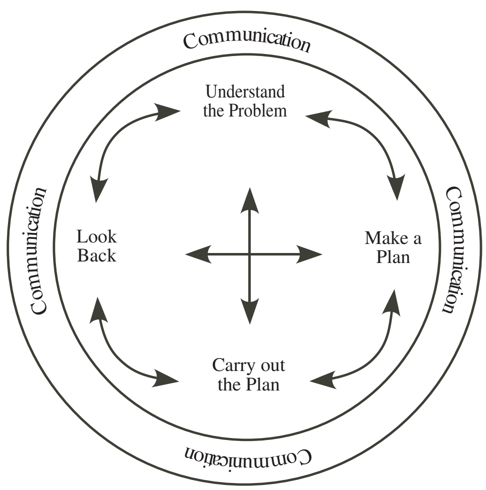
A GENERAL PROBLEM SOLVING RUBRIC

This problem solving rubric uses ideas taken from several sources. The relevant documents are listed at the end of this section.

"US and the 3 R's"

There are five criteria by which each response is judged:
Understanding of the problem,
Strategies chosen and used,
Reasoning during the process of solving the problem,
Reflection or looking back at both the solution and the solving, and
Relevance whereby the student shows how the problem may be applied to other problems, whether in mathematics, other subjects, or outside school.

Although these criteria can be described as if they were isolated from each other, in fact there are many overlaps. Just as communication skills of one sort or another occur during every step of problem solving, so also reflection does not occur only after the problem is solved, but at several points during the solution. Similarly, reasoning occurs from the selection and application of strategies through to the analysis of the final solution. We have tried to construct the chart to indicate some overlap of the various criteria (shaded areas), but, in fact, a great deal more overlap occurs than can be shown. The circular diagram that follows (from OAJE/OAME/OMCA "Linking Assessment and Instruction in Mathematics", page 4) should be kept in mind at all times.



There are four levels of response considered:
Level 1: Limited identifies students who are in need of much assistance;
Level 2: Acceptable identifies students who are beginning to understand what is meant by 'problem solving', and who are learning to think about their own thinking but frequently need reminders or hints during the process.
Level 3: Capable students may occasionally need assistance, but show more confidence and can work well alone or in a group.
Level 4: Proficient students exhibit or exceed all the positive attributes of the **Capable** student; these are the students who work independently and may pose other problems similar to the one given, and solve or attempt to solve these others.

Suggested Assessment Strategies

LEVEL OF RESPONSE →

CRITERIA FOR ASSESSMENT ↓

	Level 1: Limited	Level 2: Acceptable	Level 3: Capable	Level 4: Proficient
UNDERSTANDING	• requires teacher assistance to interpret the problem • fails to recognize all essential elements of the task	• shows partial understanding of the problem but may need assistance in clarifying	• shows a complete understanding of the problem	• shows a complete understanding of the problem
STRATEGIES	• needs assistance to choose an appropriate strategy	• identifies an appropriate strategy	• identifies an appropriate strategy	• identifies more than one appropriate strategy
	• applies strategies randomly or incorrectly • does not show clear understanding of a strategy[1] • shows no evidence of attempting other strategies	• attempts an appropriate strategy, but may not complete it correctly[2] • tries alternate strateges with prompting	• uses strategies effectively • may attempt an inappropriate strategy, but eventually discards it and tries another without prompting	• chooses and uses strategies effectively[3] • recognizes an inappropriate strategy quickly and attempts others without prompting
REASONING	• makes major mathematical errors • uses faulty reasoning and draws incorrect conclusions • may not complete a solution	• may present a solution that is partially incorrect	• produces a correct and complete solution, possibly with minor errors	• produces a correct and complete solution, and may offer alternative methods of solution
	• describes[4] reasoning in a disorganized fashion, even with assistance • has difficulty justifying[5] reasoning even with assisstance	• partially describes[4] a solution and/or reasoning or explains fully with assistance • justification[5] of solution may be inaccurate, incomplete or incorrect	• is able to describe[4] clearly the steps in reasoning; may need assistance with mathematical language • can justify[5] reasoning if asked; may need assistance with language	• explains reasoning in clear and coherent mathematical language • justifies[5] reasoning using appropriate mathematical language
REFLECTION	• shows no evidence of reflection or checking of work • can judge the reasonableness of a solution only with assistance	• shows little evidence of reflection or checking of work • is able to decide whether or not a result is reasonable when prompted to do so	• shows some evidence of reflection and checking of work • indicates whether the result is reasonable, but not necessarily why	• shows ample evidence of reflection and thorough checking of work • tells whether or not a result is reasonable, and why
RELEVANCE	• unable to identify similar[6] problems	• unable to identify similar[6] problems	• identifies similar[6] problems with prompting	• identifies similar[6] problems, and may even do so before solving the problem
	• unlikely to identify extensions[7] or applications of the mathematical ideas in the given problem, even with assistance	• recognizes extensions[7] or applications with prompting	• can suggest at least one extension[7], variation, or application of the given problem if asked	• suggests extensions[7], variation, or applications of the given problem independently

Suggested Assessment Strategies

Notes on the Rubric

1. For example, diagrams, if used, tend to be inaccurate and/or incorrectly used.

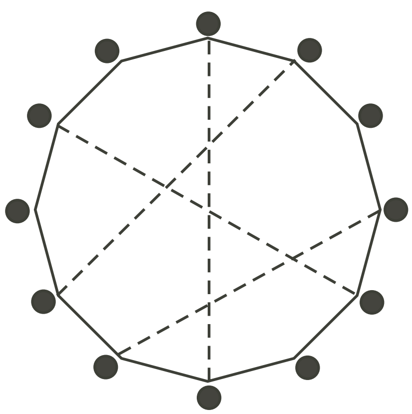
2. For example, diagrams or tables may be produced but not used in the solution.

3. For example, diagrams, if used, will be accurate models of the problem.

4. To *describe* a solution is to tell *what* was done.

5. To *justify* a solution is to tell *why* certain things were done.

6. *Similar* problems are those that have similar structures, mathematically, and hence could be solved using the same techniques.
   For example, of the three problems shown below right, the better problem solver will recognize the similarity in structure between Problems 1 and 3. One way to illustrate this is to show how both of these could be modelled with the same diagram:



Problem 1: There were 8 people at a party. If each person shook hands once with each other person, how many handshakes would there be? How many handshakes would there be with 12 people? With 50?

Problem 2: Luis invited 8 people to his party. He wanted to have 3 cookies for each person present. How many cookies did he need?

Problem 3: How many diagonals does a 12-sided polygon have?

   Each dot represents one of 12 people and each dotted line represents either a handshake between two people (Problem 1, second question) or a diagonal (Problem 3).

   The weaker problem solver is likely to suggest that Problems 1 and 2 are similar since both discuss parties and mention 8 people. In fact, these problems are alike only in the most superficial sense.

7. One type of extension or variation is a "what if...?" problem, such as "What if the question were reversed?", "What if we had other data?", "What if we were to show the data on a different type of graph?".

Suggested Assessment Strategies

Suggested Adapted Rubric for Activity 4, BLM 12

The rubric below has been adapted for the problem on BLM 12 for Activity 4: Building Patterns. This rubric considers the understanding of the problem, the selection and application of strategies, reasoning and reflection.

Level 1: Limited	Level 2: Acceptable	Level 3: Capable	Level 4: Proficient
• needs assistance to build correct models even with concrete materials	• prefers to use concrete materials	• may begin using concrete materials, but eventually uses drawings	• may use concrete materials to verify results from drawings
• needs assistance to organize results in tables or charts	• may not use tables or charts initially, but will do so if this is suggested	• organizes results in some fashion, recognizing the need for this	• organizes results in a way that makes them easy to use
• has difficulty identifying patterns in the results • does not build models beyond the ones pictured	• identifies a pattern for each problem, but may do so from only 3 or 4 examples, and such patterns may not be accurate beyond this level	• identifies a pattern for each problem • extends table or chart beyond the examples pictured	• recognizes the need to extend the patterns in order to validate them • identifies a valid pattern for each problem
• patterns identified may be true for the first 3 or 4 examples, but not for subsequent ones • may identify a different pattern if asked to construct more examples; may not be concerned if the identified patterns differ	• may be unaware that identified patterns may not be valid beyond the first few examples • will extend the number of models if this is pointed out	• may identify patterns based on a few examples, but will adjust these as necessary when more examples are examined • will adjust patterns if necessary so that they are true for all examples shown	• identifies patterns based on a number of examples • verifies that patterns are valid for several examples
• has difficulty describing patterns, even with assistance • is happy with a pattern that extends only as far as the examples • has difficulty generalizing to, for example, 10 pigs	• can describe, and will attempt to justify, patterns identified • patterns identified are likely to be "along the table" : that is, the student needs to build/draw the pens from 1 pig to 10 pigs, rather than identifying a pattern to give the number of bales for any given number of pigs	• can describe and justify patterns identified • may identify a pattern that will give, say, the number of bales of hay for a given number of pigs without building or drawing pens from 1 to 10	• can describe and justify patterns identified • will identify a pattern that will give, for example, the number of bales of hay needed for any given number of pigs

Other Resources

1. *Patterns, Addenda Series, K-6*, Miriam A Leiva (Ed.), 1993, NCTM
   Detailed notes for activities include patterns on a hundred chart, patterns built with manipulatives, patterns on the calculator, 'What's My Rule?' games, and tiling patterns. Three or more activities for each grade are given. Some Black Line Masters are included.

2. *Multicultural Math Posters and Activities*, Wayne Peterson (Ed.), 1984, NCTM
   This package contains eighteen different 28 cm by 42 cm posters along with a book of suggestions for use. Some of the patterns explored are patterns in early number systems, Japanese and Arabic geometric patterns , Pascal's Triangle, and Russian Peasant multiplication (See Gr. 4 *Investigations in Pattern and Algebra*)

3. *Multicultural Mathematics Materials*, Marina C. Krause, 1983, NCTM
   This booklet contains Black Line Masters and Teacher Notes concerning patterns from several different areas, including Egypt, Japan, China, the Philippines, Norway, Peru, and ancient Rome, as well as several from the American Indians of south-west U.S.A.

4. "The Wonderful World of Digital Sums" by Martin V. Bonsangue, Gerald E. Gannon, and Karen L. Watson, *Teaching Children Mathematics*, January 200, NCTM
   This article describes the results when an interesting pattern-problem was given to students. We have included the problem in the grade 4 *Investigations in Pattern and Algebra*.

5. "Algebraic Thinking", Focus Issue of *Teaching Children Mathematics*, February 1977, NCTM
   Articles describe the use of spreadsheets to study patterns, explore ways students of different ages deal with a problem using square tiles to develop a pattern, illustrate the use of Logo and function machines, and suggest a way to introduce to students the use of variables (e.g., the letter 'x' or 'n' to represent a number).

6. "Math Magic", by Hari P. Koirala and Phillip M. Goodwin, pp 562-566, *Mathematics Teaching in the Middle School* , May 2000, NCTM
   The article describes students' reasoning about problems such as "Think of a number; add 5; multiply by 3; subtract 3; divide by 3: subtract your original number" and how they used algebraic notation to explain the "magic" in such problems.

7. "Responses to the "What Are the Clues? Patterns in More Than One Direction" Problem, pp115-117, *Teaching Children Mathematics*, October 1999, NCTM
   The article presents responses to a computation problem using patterns that was presented in an earlier version of the journal. Teachers have described the variety of ways their students dealt with the problem and how they extended the problem further.