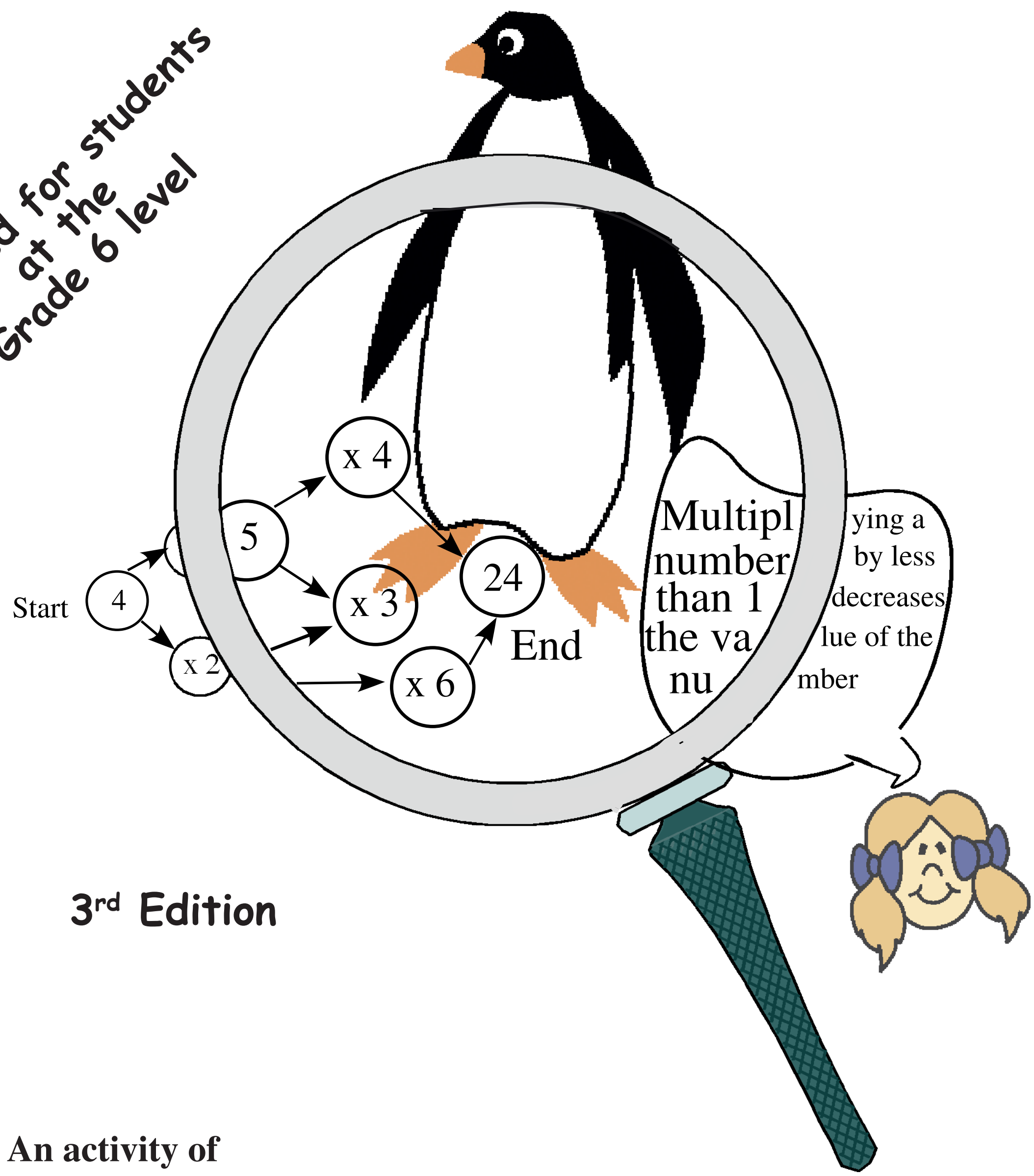
Invitations to Mathematics

Investigations in Number Sense and Estimation

"All 'Round Numbers"

Suggested for students at the Grade 6 level



3rd Edition

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Preface

The Centre for Education in Mathematics and Computing at the University of Waterloo is dedicated to the development of materials and workshops that promote effective learning and teaching of mathematics. This unit is part of a project designed to assist teachers of Grades 4, 5, and 6 in stimulating interest, competence, and pleasure in mathematics among their students. While the activities are appropriate for either individual or group work, the latter is a particular focus of this effort. Students will be engaged in collaborative activities which will allow them to construct their own meanings and understanding. This emphasis, plus the "Extensions" and related activities included with individual activities/projects, provide ample scope for all students' interests and ability levels. Related "Family Activities" can be used to involve the students' parents/care givers.

Each unit consists of a sequence of activities intended to occupy about one week of daily classes; however, teachers may choose to take extra time to explore the activities and extensions in more depth. The units have been designed for specific grades, but need not be so restricted. Activities are related to the Ontario Curriculum but are easily adaptable to other locales.

"Investigations in Number Sense and Estimation" is comprised of activities which explore the properties, estimation, and uses of whole numbers and fractions in mathematical and everyday settings. A reasonably level of numeracy is essential to navigating the complexities of the highly technical world in which we live. The activities in this unit develop many facets of number sense and apply them to a wide variety of practical situations.

Acknowledgements

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Overview

Common Beliefs

These activities have been developed within the context of certain beliefs and values about mathematics generally, and number sense and estimation specifically. Some of these beliefs are described below.

Numeracy involves an intuitive sense of the meanings of numbers and their various uses and interpretations. It is acquired slowly over a long period of time, and is fundamental, both to mathematics, and to the sciences which provide a quantitative understanding of the world around us.

While facility with number facts and algorithms is clearly important, the focus here is on developing students' thinking and reasoning abilities. This is achieved through investigation and sharing of ideas during group activities involving properties of numbers (whole numbers, decimals, and fractions), comparing and ordering, how numbers are used, and determining reasonable estimates. Problems using a variety of mechanisms (number lines, geometric quantities, mental manipulation, stories, games, etc) encourage flexibility in mthods os solution. Similarly, a variety of estimation situations increases students' awareness of the pervasive need for estimates in real life, and their ability to devise estimates competently. Students are encouraged not only to calculate in different ways but also to assess the reasonableness of their answers. In addition, by eliminating the need for boring computations, calculators can be used to permit students to focus on the process of obtaining solutions, and on their interpretation.

Throughout these activities, as they attempt to justify their conclusions using mathematical language, students deepen their insight into and understanding of how numbers relate to each other and to the world around them.

Essential Content

The activities herein explore numbers both in the abstract and in their connection to measures of real quantities, with the goal of developing students' ability to think and work flexibly with different kinds of numbers in a variety of contexts. In addition, there are Marginal Problems, Extensions in Mathematics, Cross-Curricular Activities, and Family Activities, which can be used prior to, during the activity, or following the activity. They are intended to suggest topics for extending the activity, assist integration with other subjects, and involve the family in the learning process.

During this unit, the student will:

- compare sizes of whole numbers (to billions), fractions, and decimals (to thousandths);
- explore properties of number (e.g., properties of zero; result of multiplying by a proper fraction);
- estimate products, quotients, sums, and differences;
- identify over- and under-estimates;
- explore number usage, place value, divisibility rules, and the meanings of remainders in division;
- practice skills in game situations;
- use mathematical language to express their results;
- work together to achieve success.

"Curriculum Expectations" are based on current Ontario curricula.

Overview

Curriculum Expectations

ACTIVITY	DESCRIPTION OF THE ACTIVITY	CURRICULUM EXPECTATIONS
Activity 1 **Using Numbers**	• place real-world measures from 1 to 1 million on a number line • decide whether given measures of specific items are reasonable • use scale drawings of large and small objects to determine their actual size	• understand the significance of numbers in the greater world and evaluate the use of numbers in the media • recognize and read numbers from 0.001 to 1 000 000 • solve simple ratio problems
Activity 2 **Comparing and Ordering**	• determine whether decimal numbers from 1 000 to 100 000, and fractions from 0 to 1 lie within a specified range • add and multiply fractions and decimal numbers • order fractions with denominators 1, 2, ... 9 • relate sums and products of decimal numbers and fractions as greater than or less than a specified given number	• compare, order, and represent decimals, or fractions • explain processes and solutions with fractions and decimals using mathematical language • multiply decimal numbers from 0.01 to 10.00 using a calculator
Activity 3 **Number Properties**	• determine the consequences of multiplying by a number >1, <1, or an even or odd number • determine how to use the remainder in specific real-world situations • explore patterns resulting from arithmetic operations with 2, 3, 4, 5 and 9 and 'digit sums', using a calculator • solve puzzles using both mental math and calculators	• recall multiplication and division facts and use them to estimate and do mental computation • use mental computation strategies to solve number problems • solve multi-step problems involving whole numbers and decimals with and without a calculator
Activity 4 **Fractions**	• find fractions satisfying a specific set of conditions • explore properties of sums and products of mixed numbers, whole numbers, and fractions • order fractions using number strips • play games involving sums of fractions • explore a way to graph equivalent fractions	• explain processes and solutions with fractions using mathematical language • compare and order mixed numbers and fractions with unlike denominators using concrete materials • relate fractions to ratios using drawings
Activity 5 **Estimation**	• estimate products of 2-digit and 3-digit numbers, and of 2-decimal numbers times a whole number • determine whether estimates are under-estimates or over-estimates • estimate sums, differences, products and quotients of 1-, 2-, and 3-digit numbers • assess accuracy of posted advertisements • assess whether specified real-world situations require an estimate or an exact value	• use and verify estimation strategies to determine the reasonableness of solutions • identify, interpret, and evaluate the use of numbers in the media • understand the significance of numbers in the greater world

Overview

Prerequisites

Although students should be able to deal with the activities in this book with an understanding of the previous grade's curriculum, it would help if they are familiar with the following:

- place value from tenths to millions;
- rounding to the nearest ten or hundred;
- adding, subtracting, and multiplying simple fractions $\left(\text{e.g., } \frac{2}{3}+\frac{1}{2},\ \frac{4}{5}+\frac{3}{10}\right)$
- the nature and use of estimates (i.e., what an estimate is and when an estimate is appropriate).

Logos

The following logos, which are located in the margins, identify segments related to, respectively:

Problem Solving Communication Assessment Use of Technology

Marginal Problems

Throughout the booklet you will see problems in the margin (see example to the right). These 'Marginal Problems' may be used as warm-ups to a lesson, as quick 'tests' or reviews, as 'problems-of-the-day' or in any other way your experience tells you could be useful. Some 'Marginal Problems' deal with the same topic as the activity and some with other topics in "Number Sense and Estimation". Discussion of individual problems can be found at the beginning of "Solutions and Notes".

Notes

Write a 3-digit number, *abc*, so that $a \times b = c$ and all 3 digits are different. How many such numbers are there?

Overview

MATERIALS

ACTIVITY	MATERIALS
Activity 1 **Using Numbers**	• Copies of BLMs 1, 2, and 3 • Chart paper and markers • Copies of BLMs 4, 5, and 6 (optional) • Calculators for each pair/group (optional)
Activity 2 **Comparing and Ordering**	• Copies of BLMs 7, 8, and 9 • Scissors for each pair/group • A calculator for each pair/group • Copies of BLMs 10, 11, and 12 (optional) • Two game markers for each player (optional)
Activity 3 **Number Properties**	• Copies of BLMs 13, 14, 15, and 16 • A calculator for each pair/group • Copies of BLMs 17 and 18 (optional)
Activity 4 **Fractions**	• Copies of BLMs 19 and 20 • Copies of BLMs 21, 22, and 23 (optional) • 2 dice of different colours or a simple spinner for each pair/group (optional)
Activity 5 **Estimation**	• Copies of BLMs 24 and 25 • Copies of BLMs 26, 27, and 28 (optional) • A calculator for each pair/group (optional)

Overview

Letter to Parents

SCHOOL LETTERHEAD

DATE

Dear Parent(s)/Guardian(s):

For the next week or so, students in our classroom will be participating in a unit titled “All ’Round Numbers”. The classroom activities will focus on expanding students’ understanding of numbers and estimation while exploring how numbers are related and how they are used. The emphasis will be on developing skill with mental manipulation, estimation, and computation.

You can assist your child in understanding the relevant concepts and acquiring useful skills by working together to perform number related tasks (e.g., comparing prices when shopping, estimating the total cost, calculating mileage for the family vechicle), and by helping to explore everyday ways numbers are used.

Various family activities have been included for use throughout this unit. Helping your child with the completion of these will enhance his/her understanding of the concepts involved.

If you work with measurement in your daily work or hobbies, please encourage your child to learn about this so that he/she can describe these activities to his/her classmates. If you would be willing to visit our classroom and share your experience with the class, please contact me.

Sincerely,

Teacher’s Signature

A Note to the Teacher:

If you make use of the suggested Family Activities, it is important to schedule class time for sharing and discussion of results.

Activity 1: Using Numbers

Focus of Activity:

- Exploring relative sizes of very 'large' and very 'small' numbers

What to Assess:

- Ability to put 'large' numbers in order
- Accurate identification of uses of 'large' and 'small' numbers

Preparation:

- Make copies of BLMs 1, 2, and 3.
- Provide chart paper and markers.
- Make copies of BLMs 4, 5, and 6 (optional).
- Provide a calculator for each pair/group (optional).

Activity:

Note: The terms 'large' and 'small' are relative; whether a number is 'large' or 'small' often depends on the context. For example, when discussing distances between planets, the diameter of Earth is 'small', but when discussing distances between towns in Ontario, the diameter of Earth would be considered 'large'.

To be accurate, we should speak of numbers as being 'greater' or 'lessor' than one another, since 'large' and 'small' refer more to physical size than to value. For example, in the line below, the '2' could be said to be 'larger' than the '7'.

2 7

This activity presents students with examples of 'large' and 'small' numbers in use. Ask students if they know of any place/time when 'large' numbers might be used (e.g., distances between planets, salaries of professional sports figures) and to give a possible number for each situation. Have each group list three or four such uses along with possible numbers used in these ways. Have them put the numbers in order, least to greatest.

Distribute copies of BLM 1 (Putting Numbers in Their Places) and tell them that the BLM gives several numbers less than 1 million that they are to put in order, and then mark their relative positions on the number line given. For numbers 'K' and 'L', have students include two examples of their own that were mentioned in the earlier discussion. Alternatively, have students explore resources (libraries, the Web, etc.) to discover other examples.

Allow students time to complete the exercise and ask them how they decided where to locate the numbers. For example, 26 000 (H) is more than, but close to, 21 000 (G); 662 200 (D) is more than half-way between 0 and 1 million.

See "Solutions and Notes", for a completed solution that could be copied onto acetate and shown to the class for checking. Note that placing these numbers involves estimating and it is only their relative positions that should be considered 'right' or 'wrong'.

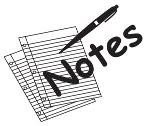
Notes

'BLM' refers to Black Line Masters, which follow the Activity Notes

See page 3 for a description of "Marginal Problems", an example of which is given here.

If you drove at the highway speed limit, it would take you a full year to circle the outer rings of Saturn. What is the circumference of these outer rings?

Have students decide if "highway speed" is 80 km/h or 100 km/h. Alternatively, have different groups calculate the distance using different speeds.



Activity 1: Using Numbers

Ask students why it is important to understand relative sizes of numbers (i.e., that 1 million is more than 100 000, or that 0.10 is less than 1.00). One reason is so that you know when a given number is reasonable. For example, if you did not understand relative sizes of numbers, you might think $500 000 was a good price for a small car, or that $0.10 was the same as $10 or $1.00.

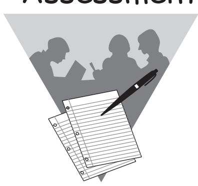
If students think that no one could possibly mistake numbers in this way, you might wish to use, at this point, BLM 6 (Errors, Errors, Errors!) which includes examples of such errors from various store signs and text books.

Problem Solving

BLM 2 (Numbers in Place) gives further examples of 'large' and 'small' numbers, ranging from 1 to 79 billion. Students are asked to draw a number line and locate these numbers on it. Since the range of numbers is so great, students will need to draw a fairly long line. Give each group a sheet of chart paper and markers to construct and label their line. These can then be posted for comparison.

Note: Some of the measures given are in the Imperial rather than the metric system. Since students are being asked only to locate the numbers, the units are irrelevant.

Assessment



BLM 3 (Being Reasonable) gives students the opportunity to decide whether or not some numbers are used in a reasonable manner. Read over the instructions with the students and have them start marking each statement as reasonable or unreasonable. While they are working, ask individual groups to justify their answers.

To complete the second part of the assignment, they should take their papers home and try to find reasonable numbers to replace unreasonable ones. This can be a whole family activity to be discussed the next day.

Extensions in Mathematics:

Use of Technology

1. Have students check library books, the Web, newspapers, magazines, etc., to find three examples of very large numbers (greater than 1 million) and three examples of very small numbers (less than 0.1) and tell how and by whom (e.g., astronauts, scientists) these numbers are used.

2. Distribute BLMs 4 (Scaling Large Objects) and 5 (Scaling Small Objects). These BLMs show two different types of scale drawings. For BLM 4 students should realize that the actual object is larger than the diagram and should judge their answers accordingly. BLM 5 consists of drawings larger than the actual objects. Students should be aware that the actual measures may use several places of decimal (e.g., #1; actual size is 0.005 mm).

You may wish to compare scale drawings with 'similar figures' in the Geometry curriculum.

Activity 1: Using Numbers

Note: Because of discrepancies created in formatting, photocopying and printing, the drawings may not match the given measurements exactly. That is, the arrow under the bicycle (#1, BLM 4) may not be exactly 5 cm. Calculations should be based on the measurements written adjacent to the arrows, not on the actual measurements of the lengths of arrows.

Cross–curricular Activities:

1. Many of the items on BLMs 1 and 2 can lead to interesting discussions or further problem solving. For example,

    If a bee flies 21 000 km to make 450 g of honey, how far must it fly to make enough honey to spread on a piece of toast?

    How would you count the number of hairs on a human head?
    Where did the biggest oil spill occur and what were some of the results?
    Is Hong Kong more or less heavily populated than a local city? (1 square mile is approximately 2.6 km^2)Can you see more stars in the city or in the country? Why?

2. Have students write about number sizes in response to some questions:

    Why would anyone want to make a drawing larger than the actual object?
    How is a photograph a scale drawing?
    Could a photograph be larger than the actual object? How?

3. A true story to share with students:
    A customer ordered a globe (a model of the earth) from a department store without specifying which of the available sizes was wanted. The store asked, "What size do you want?", and the customer replied, "Life size of course!"

    (If students don't laugh at this, they obviously need more experience with scale drawings and relative sizes.)

Family Activities:

1. See notes above regarding the latter part of BLM 3.

2. If BLM 6 (Errors, Errors, Errors!) was not used earlier, have students take it home to work with other family members.

Other Resources:

For additional ideas, see annotated "Other Resources" list on page 82, numbered as below.

2. "Developing Sense About Numbers".

Notes

At 100 km/h, how long would it take to drive around the world?

Assume you are driving a 'magic' car that actually prefers to drive on water. Where on the earth would your route take you?

Communication

Activity 2: Comparing & Ordering

'BLM' refers to Black Line Masters, which follow the Activity Notes

Focus of Activity:
- Identifying numbers (whole numbers, fractions, decimal numbers) that fall within a certain range
- Estimating sums of fractions that fall within a certain range
- Estimating products of decimal numbers that fall within a certain range.

What to Assess:
- Accuracy of ordering decimals and fractions
- Identification of numbers less than or greater than a given number (whole numbers, decimals and fractions)
- Reasonableness of estimates with fractions and decimals

Preparation:
- Make copies of BLMs 7, 8, and 9.
- Provide scissors and calculator for each pair/group.
- Make copies of BLMs 10, 11, and 12 (optional).
- Provide 2 game markers for each player (optional).

Activity:

For estimating and rounding, students need to be aware of the relative sizes of numbers. This activity gives several ways of helping students develop this skill. Although most of the activities are illustrated with only whole numbers or only decimals or only fractions, each can be easily adapted to include other types of numbers.

Shorter versions of some of the games can be used as "Bell Work" or "Problem-of-the-Day" throughout the year.

Write 2 multiplication questions whose answers have a zero in the tenths place and a 6 in the hundredths place.

THE RANGE GAME

Show the following on the blackboard or overhead projector:

10 + _____ = _________
(40 to 50)

Ask students to suggest a number that can be placed in the first blank so the second blank contains a number from 40 to 50. Ask students to suggest other numbers that satisfy the same condition.

Ask:

"What is the greatest/least number that can be used?"
"How many different whole numbers can be used?"
"If you use fractions or decimals, how many solutions do you think there are?"

Activity 2: Comparing & Ordering

The last question should lead students to (or remind them of) the idea that the number of fractions/decimals between any two (whole) numbers is infinite. Some students may realize that the number of fractions between <u>any two</u> numbers (whole or not) is infinite.

Increase the difficulty level by:

(i) narrowing the range: 131 + _____ = __________
(240 to 244)

[Possible whole number responses range from 109 to 113; students may suggest fractions as well.]

(ii) changing the operation: 23 x _____ = __________
(49 to 169)

[Possible whole number responses range from 3 to 7.]

(iii) using fractions or decimals: + _____ = __________ $3\frac{1}{2}$
(4 to 7)

5.23 – _____ = __________
(1 to 3)

A question

"Give the greatest possible whole number for which 17 *x* ______ = ______*"*
(380 to 400)

is analagous to the first step in the long division question 400 √ 17*. Problem 1 on BLM 7 (The Range Game) gives a few examples of this type.*

Narrow the range further with problems like the following:

Give whole number answers for the following:

(i) 6.9 is between 6 and 7 but closer to _____

(ii) 7.23 is between 7 and 8 but closer to _____

(iii) 0.035 is between 0 and 1 but closer to _____

Give decimal answers:

(i) 6.8 is between 6.___ and 6.___ but closer to 6.___

(ii) 7.23 is between 7.2___ and 7.2___ but closer to 7.2___

Notes

Write 2 subtraction questions whose answers have a 5 in the tens place and a 9 in the tenths place.

Notes

Write 3 different numbers between 3.4 and 3.5

Activity 2: Comparing & Ordering

Give fractional answers:

(i) $\frac{2}{3}$ is between 0 and ___ but closer to ___

(ii) $\frac{2}{5}$ is between ___ and ___ but closer to ___

Note: Answers equivalent to whole numbers such as $\frac{5}{5}$ *should be accepted here.*

BLM 7 (The Range Game) gives further examples of these types of problems, and includes some variations. Question 2 asks students to find the whole number limits for given decimal numbers. Question 3 asks for narrower limits.

Note that both of these allow for multiple correct answers. For example, 2(c) could be correctly completed in any of the following ways:

21.45 is between 21 and 22 but is closer to 21.

21.45 is between 20 and 30 but is closer to 20.

21.45 is between 0 and 25 but is closer to 25.

Such answers should be accepted and discussed. If only one answer is desired, ask students to write the narrowest possible range using whole numbers.

Problem Solving

Question 3 adds another difficulty. (See the example in the box below). For example, in 3(a), some possible choices for the first two blanks are 9.9 and 9.93, or 9.8 and 9.92, or 9.89 and 9.92. Note that, while the first blank may have one decimal place, but the second blank must have at least two decimal places. However, the student who completes the first two blanks with '9.9' and '9.92' will not be able to use one of these in the third blank, since 9.91 lies midway between 9.90 and 9.92 and is not 'closer' to one than to the other. Similarly in 3(d), it is necessary to go to three decimal places to solve the problem, such as "8.49 is between 8.48 and 8.495."

Excerpt from BLM 7

3 a) 9.91 is between 9.___ and 9.___ but is closer to 9.___

3 d) 8.49 is between 8.4___ and 8.4___ but is closer to ___

Activity 2: Comparing & Ordering

Identifying fractions as being close to $\frac{1}{2}$ or 1 (as in question 4) is a useful strategy when estimating with fractions and is dealt with further in Activity 4 (Fractions).

Range Points

Display several whole numbers:

2	5	24
32	6	19
8	7	15

Have students select two numbers and multiply them to reach a number in a specific range. (Possible answers are given in square brackets. Note that other answers may be possible.)

For example,

(i)	from 0 to 49	[2 × 5 or 5 × 6]
(ii)	from 50 to 99	[8 × 7 or 6 × 15]
(iii)	from 100 to 299	[7 × 15 or 5 × 24]
(iv)	300 or more	[15 × 24 or 32 × 15]

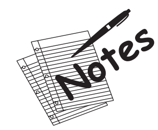
Have students find as many pairs of factors as possible (from the given set) for the given ranges.

Note: The rules do not specifically say "choose two different numbers", so students may choose to include a number paired with itself such as 2 x 2 for (i) or 15 x 15 for (iii).

When students are familiar with the idea, introduce a scoring system using narrower ranges:

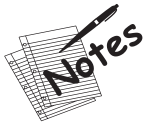
RANGE	SCORE
0 to 49	1 point
50 to 99	2 points
100 to 149	2 points
150 to 199	3 points
200 to 249	2 points
250 to 399	1 point
400 or more	1 point

Divide the class into groups of 6 or more. Have each child in the group select two numbers from the given set, multiply them, and record his/her score based on the



Write a 4-digit number, *abcd*, so that $a + b + c = d$, and the digits are all different. Do not use zero. What is the greatest such number? What is the least?

See "Solutions and Notes" for all possible combinations.



Communication

Write a 4-digit number, *abcd*, so that $a \times b \times c = d$, and all the digits are different. How many such numbers are there?

numbers from the given set, multiply them, and record his/her score based on the range in which the product falls. Students should be selecting the numbers based on their estimates, trying to choose numbers whose products will fall in the range 150 to 199 to earn 3 points. All members of the team can do this simultaneously. Allow calculator use to determine the actual products.

Ask each group for their total score. Ask if some ranges were easier to achieve than others and why. Ask if the scoring is fair -- i.e., “Do you have an equal chance for each range?” “Is this necessary?”

Since the aim of the activity is to provide practice in estimating answers, it does not really matter if the score for a range reflects the frequency of products for that range. However, students may find that this conflicts with their innate notion of “fairness”.

Students should find on analysing the game that there are nine different pairs of numbers that have a product from 0 to 49 but only four pairs with products from 150 to 199. However there are only two products from 200 to 249. Students may suggest that this range should give more points because it is harder to get.

You may wish to explore the probabilities of each range and have students devise a new scoring system based on the difficulty of reaching each range.

Ask, “What differences in the scoring would you make if you were adding the numbers instead of multiplying them?” “How would you change the game if you were multiplying <u>*three*</u> *of the numbers together?”*

Obviously, for addition the total range will be much smaller -- only from 7 (2 + 5) to 56 (24 + 32). For multiplication with 3 numbers the range will be extended from 60 (2 x 5 x 6) to 14 592 (32 x 24 x 19). It is not suggested that students should actually play these games unless they show some willingness to do so. The analysis of the game is the important thing.

BLM 8 (Range Points) gives numbers and rules for one game using fractions and another using decimals. The fraction game uses addition and the decimal game uses multiplication. To extend the activity further, have students devise games of their own.

Activity 2: Comparing & Ordering

Encourage students to use estimation techniques to determine the sums in Game 1.

For example, $\frac{1}{2}+\frac{2}{3}$ will be greater than 1 but not as great as 2.

$1\frac{1}{4}+1\frac{1}{3}$ will be between 2 and 3 since $1+1=2$ and $\frac{1}{4}+\frac{1}{3}<1$.

$\frac{4}{3}+\frac{5}{2}=1\frac{1}{3}+2\frac{1}{2}$ will be between 3 and 4 since $1+2=3$ and $\frac{1}{3}+\frac{1}{2}<1$.

Actual values are given in "Solutions and Notes". You may wish to have these available for students to check any disputed estimates.

The ranges given in the scoring chart are intended to include the whole number at the upper level of the range. That is "1 to 2" includes "2", "4 to 5" includes "5", and so on. This should be explained to the students.

A common mathematical notation for intervals uses a round bracket to indicate a number not included in the range, and a square bracket to indicate a number included in the range. Thus, (1 to 5] includes the '5' but not the '3' whereas [3 to 5) includes the '3' but not the '5'.

Similarly students should be encouraged to use estimation for Game 2. Calculators should be available for students to check their answers. *Once again you will find the actual products in "Solutions and Notes".*

Fraction tiles

BLM 9 (Fraction Tiles) presents two fraction problems for students to solve. Both involve inserting the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9 in the boxes to create fractions according to their values as less or greater than other fractions. Using the number tiles at the bottom of the BLM allows students to test their ideas and change them as they wish without the necessity of erasing wrong answers. Students should write the digits in place only after they are satisfied that all ten number tiles are in the correct place. *Two sets of answers are given in "Solutions and Notes", but others are possible.*

Notes

Problem Solving

Write a 4-digit number, *abcd*, so that $a \times b = c \times d$, and all the digits are different. What is the greatest such number? What is the least?

Notes

What fraction could you add to $\frac{1}{2}$ to give an answer less than 1? Give three different answers.

Extensions in Mathematics:

1. BLM 10 (Truths and Half-truths) asks students to consider the nature of numbers, given a sum or product, and to decide which statemennts about the numbers are true.

Introduce the idea with the following example.
If the sum of two different numbers is between 10 and 20, which of the following statements is true? Can you always tell?

(i) Both numbers are less than 10.
(ii) Both numbers are between 10 and 20.
(iii) At least one number is less than 5.
(iv) At least one number is less than 10.

Note: "Between 10 and 20" means neither 10 nor 20 is included.

(i) If both numbers are less than 10 the sum is obviously less than 20 but it may not be more than 10. Thus, we cannot say if this statement is true or false. It will be true sometimes and false sometimes.

(ii) It should be obvious that if both numbers are between 10 and 20 their sum will be greater than 20. Thus statement (ii) is false.

(iii) It is not essential that either number be less than 5 (e.g., $6+7<20$) but one may be (e.g., $4+15<20$). We can't tell whether the statement is true or not. In other words, it is sometimes true and sometimes false.

(iv) Yes, at least one number will be less than ten. We have already explored in part (ii) what will happen if neither one is less than 10. No matter what the two numbers are, at least one will be less than 10. So this statement is always true.

Assessment

The problems on BLM 10 deal with both whole numbers and fractions in a similar way, as well as some pure logic questions in problem 4. Students are asked to decide if statements are always true, never true (i.e., false), or sometimes true and sometimes false. Students should provide examples to illustrate their answers. The discussion within a group provides a good opportunity for assessment.

Activity 2: Comparing & Ordering

2. BLM 11 (Relative Sizes) provides several other questions for practice in estimation with whole numbers, fractions, and decimal numbers, and with all four operations. Statements in #2 should be identified as true or false. Students should be encouraged to use their answers for #1 to help them decide whether each statement is true or false. For example, 1e) shows that the first statement in #2 is true. Of the five statements, one is false.

Family Activities:

1. BLM 12 (Greater or Less?) is a game that involves identifying relative sizes of decimal numbers. Materials needed are two markers for each student. These can be buttons, bingo chips, bits of coloured paper, or any other available material that will fit on one of the numbered hexagons. Students also need a coin or pseudo-coin to flip. (Two-colour plastic counters are useful here and quieter than coins). Play the game a few times with students; then have them take copies home to play with family members.

   It is possible for 6 players to play on one board (one player starting from each side) but it does get crowded. Two or three players is better.

After students have played the game a few times, suggest one of the following variations.

- (i) Use only one marker. If you cannot move ahead on your turn, you miss that turn.
- (ii) Each player has several markers. Instead of moving one marker, place a new marker on the space to which you would normally move, to form a pathway across the board. This variation adds the factor of possibly blocking your opponent's path.

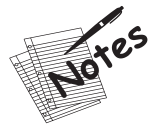
Other Resources:

For additional ideas, see annotated "Other Resources" list on page 82, numbered as below.

4. "How Much is a Million?"

5. "How Big is Bill Gates's Fortune?"

6. "A Game Involving Fraction Squares"



Activity 3: Number Properties

Focus of Activity:

- Identifying and using various number properties

What to Assess:

- Accuracy of numerical solutions
- Reasonableness of choices
- Ability to justify responses

'BLM' refers to Black Line Masters, which follow the Activity Notes

Preparation:

- Make copies of BLMs 13, 14, 15 and 16.
- Make copies of BLMs 17 and 18 (optional).
- Provide a calculator for each group/pair (optional).

Activity:

Knowing some of the many properties of number can help students with computation and estimation. These properties include such things as:

All even numbers are divisible by 2.

Multiplying a number by 1 does not change the value of a number.

When adding a string or column of numbers, the order in which we add doesn't matter. This means we can look for groups of ten to make the adding easier.

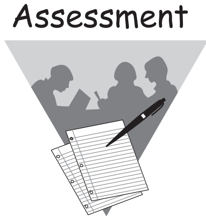
e.g., $3+5+9+7+1=(3+7)+(9+1)+5$

BLMs 13, 14, 15, and 16 deal with different types of number properties. These may be done in any order you wish, since they are not dependent on each other.

Communication

BLM 13 (Name That Number) explores some of these properties. Have students work in small groups with each student choosing a different number (a favourite number? a 'lucky' number?) to work with. Then students can compare results as they answer each question (See questions at the bottom of BLM 13). In some cases, the answers will be identical (See a) below); but depending on how students respond to questions such as e) below, answers may differ. For example, if students respond with a particular fraction, such as $\frac{1}{9}$ or $\frac{2}{3}$, their answers will differ. However, if they respond with a general answer as given below, their answers will be the same.

Assessment



> Excerpt from BLM 13:
>
> a) What number can you multiply your number by without changing its value? [Answer:1]
>
> e) What kind of number can you multiply your number by to get a lower value? [Answer: any fraction less than 1]

Listening to students as they discuss such answers can indicate how well they understand numbers and how well they generalize from examples.

Activity 3: Number Properties

See "Solutions and Notes" for possible responses and discussion of both problems and responses.

BLM 14 (Should Remainders Remain?) explores the practical meanings that remainders can have and how they should be treated. We are all familiar with students whose answer to problem #1 (see below) would be "4 buses" or even "$4\frac{17}{35}$ buses", completely losing sight of the reality of the situation.

> Excerpt from BLM 14:
> 1. A school with 150 students and 7 teachers is going to a concert. A school bus holds 35 people besides the driver. How many buses will be needed?

What is the greatest 4-digit number that can be multiplied by 9 to give a product that is a 4-digit number?

For this problem, the 'remainder' should be rounded up to the nearest whole number -- choice (ii) at the top of BLM 14.

> Excerpt from BLM 14:
> For each of the answers to the following problems, tell in which of the following ways the remainder should be used:
> (i) You should ignore the remainder.
> (ii) You should round the answer up to the next whole number.
> (iii) You should leave the remainder as it is.

See "Solutions and Notes" for possible answers.

As a challenge, have students work in groups to create a problem illustrating each of these "remainder treatments".

BLM 15 ("Four Patterns") goes beyond the better-known divisibility rules for 2, 5, and 10, using a little-known function of most four-function (i.e., non-scientific) calculators. Have students check their calculators to be sure they have the function described in #3 before proceeding with the problems on this BLM. A way of testing for divisibility by 4 is introduced: "Any number whose last two digits are divisible by 4 is itself divisible by 4". Because 100 is divisible by 4, we can ignore any digits other than the tens and units. That is, we test a number such as 2354 by recognizing that 2300 is divisible by 4, and checking only 54 for divisibility by 4.

What if the least whole number that is divisible by 2, 3, 4, and 5?

A basic property of number is embedded in this:

> If two numbers, a and b, are both divisible by n, then $a+b$ is divisible by n.
> For example, 2300 and 52 are both divisible by 4, so 2352 is divisible by 4.

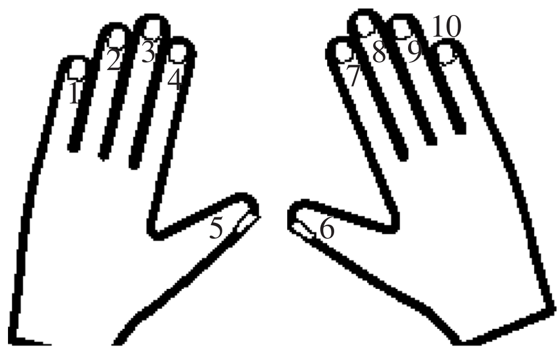
BLM 16 (Division Patterns) uses the 'counting by ....' function of the calculator to develop divisibility rules for 3 and 9.

Activity 3: Number Properties

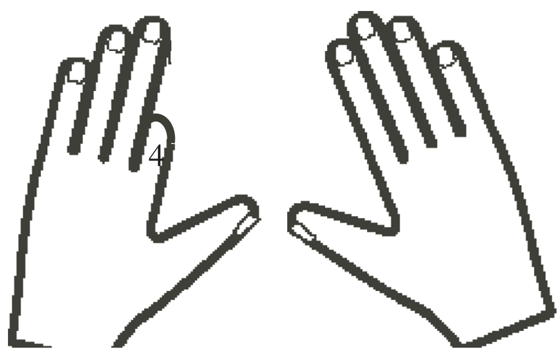
Most four-function calculators (and some scientific calculators) have built in automatic constants. Thus when you press $0+4$ *===..., the calculator adds 4 each time the equals sign is pressed.*

Similarly, the calculator will show -4, -8, -12, -16, etc. for the following sequence: $0-4$ ===... .

A diversion:
Students may be familiar with the following technique for multiplying single digits by 9. Start by placing both hands flat on the desk in front of you and imagining that the fingers are numbered as shown:

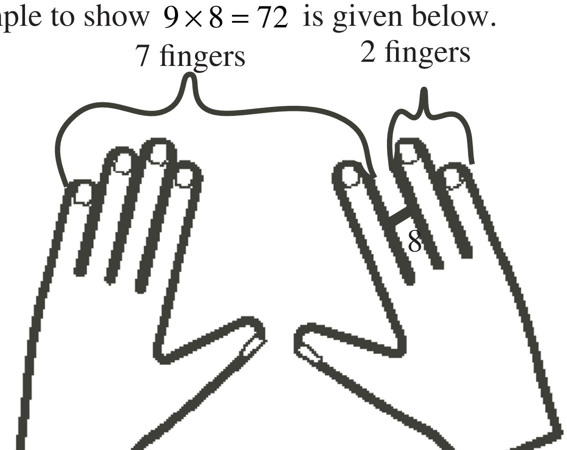


To multiply 9×4, for example, bend down the '4' finger:



Count the number of fingers to the left of the bent finger -- 3.
Count the number of fingers to the right of the bent finger -- 6.
Thus the answer to 9×4 is 36.

A second example to show $9\times8=72$ is given below.



Activity 3: Number Properties

This works because we are using base 10 numbers and 9 is one less than 10. The pattern of digits in the 9 times table is related to the number of fingers on either side of the bent finger.

For example, if we look at the tens digits in the 9 times table, we see that these increase by 1, while the units digits decrease by 1 each time.

Tens digits increase by 1. ↓		↓ Units digits decrease by 1.
	09	
	18	
	27	
	36	
	45	
	54	
	63	
	72	
	81	
	90	

Students can see that if each finger is bent in turn, the number of non-bent fingers to the left increases by one each time, and the number of non-bent fingers to the right decreases by one each time.

What is the least whole number that is divisible by every whole number from 1 to 10 inclusive?

Extensions in Mathematics:

1. Ask students how they can tell if a number is divisible by 6. [Answer: Check to see if it is divisible by both 2 and 3.]

2. You may wish to introduce students to a test for divisibility by 7: 957 343

   Remove the units digit, double it, and then subtract that product from the remaining number.

   $$\begin{array}{r} 95 \\ -14 \\ \hline 81 \end{array} \qquad \begin{array}{r} 34 \\ -6 \\ \hline 28 \end{array}$$

   In the first example, we delete '7', then subtract $2 \times 7 = 14$ from the remaining 95, leaving 81. Since 81 is not divisible by 7, we know that 957 is not divisible by 7.

   In the second example, 28 is divisible by 7, so 343 is divisible by 7.

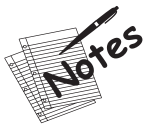
   Greater numbers will need doubling and subtracting more than once to reach a one – or two – digit number.

   For example, $25571 \to 2557 - 2 \to 2555 \to 255 - 10 \to 245 \to 24 - 10 \to 14$

3. A test that is easy to apply is a test for divisibility by 11. The following three numbers are all divisible by 11: 3916, 253, and 35112.

We test by alternately subtracting and adding the digits as follows:

$3 - 9 + 1 - 6 = -11 \qquad 2 - 5 + 3 = 0 \qquad 3 - 5 + 1 - 1 + 2 = 0$



If the decimal point on a calculator is broken, how could you use the calculator to determine $34.25 + 17 + 8.1$?

Activity 3: Number Properties

Since '–11' and '0' are both divisible by 11, the test shows that the original numbers are divisible by 11. Notice that no matter how many digits are in the number, we always start the left-most digit and *subtract* the next digit, then alternate addition and subtraction.

4. BLM 17 (Find the Path) gives students a chance to apply their knowledge of divisibiltiy rules. For example, in #1, the 'End' number is 1440. This means that one of the factors must be '5', and another must be an even number, since that is the only way to produce '0' as the final digit.

The digit sum of the 'End' number is $1+4+4+0=9$, which means the number is divisibile by 9. Since there is no '9' shown in the pathways, we must use 3×3 or 3×12 (since '12' has a factor of '3') or just $\times3$ or $\times12$, since the 'Start' number, 6, has a factor of '3'.

Testing a few combinations with a calculator, we find that $6\times4\times5\times12=1440$.

See "Solutions and Notes" for solutions and further discussion of techniques.

Family Activities:

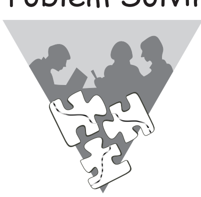
1. A simple calculator game, with variations, is given on BLM 18 (Reach A Number) can be introduced in class and then sent home to be played with other family members.

In Game 1 it is possible to reach exactly 400 since $16\times25=400$, but students may introduce other factors which make this result difficult to obtain – e.g., $16\times9=144$, and 144 is not a factor of 400. Because of this, students are instructed to reach a number 'as close as possible' to 400. This can be interpreted to mean a number like 399.1254 or simply as any number from, say 395 to 405. After playing the game once, students should decide on the degree of 'closeness'to the target number for which they will aim.

Technology

Notice that the Hints for Game 1 and the rules for Games 2 and 3 make use of the number properties described on BLM 13 (e.g., multiplying a number by a fraction – or in this case, a decimal – less than 1 will produce a lower value than the number you started with).

Problem Solving



Other Resources:

For additional ideas, see annotated "Other Resources" list on page 82, numbered as below.

7. "Multiplication Games: How We Made and Used Them"
8. "The Influence of Ancient Egypt on Greek and Other Numeration Systems"
9. "Understanding Aztec and Mayan Numeration Systems"
10. "Translating Number Words into the Language of Mathematics",

Activity 4: Fractions

Focus of Activity:

- Computation, comparing, and ordering of fractions

What to Assess:

- Accuracy of fraction computations and comparison
- Understanding properties of fractions
- Use of mathematical language

'BLM' refers to Black Line Masters, which follow the Activity Notes

Preparation:

- Make copies of BLMs 19, and 20.
- Make copies of BLM 21, 22, and 23 (optional).
- Provide two dice of different colours or simple spinners for each pair/group (optional).

Activity:

BLM 19 (The Problem with Fractions) is intended as a review (or pre-test) of fractional concepts and operations with which Grade 6 students are expected to be reasonably familiar. The BLM can be used in several ways:

(1) Use one each day as a "Marginal Problem" for the whole class.

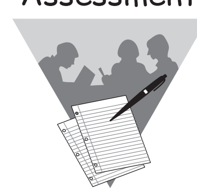
(2) Assign the BLM as a group activity.

(3) Use as a cooperative game: Cut the problems apart to make a deck of cards. Turn the cards face down in the centre of a group. Each student draws one problem and tries to solve it. If a student has difficulty, he/she is allowed to ask one other student in the group for help. When each student has an answer to the problem he/she drew, the answers are given to the whole group. If the group agrees that the answer is correct, that student scores one point. If the group disagrees, a correct answer must be found (and all members of the group must agree to this) before the game continues.

> How can you tell if a fraction is close to 1?

Communication

Assessment



The problems are designed to have students think about fractions in ways they probably don't often use, so as to develop 'fraction sense'. Discussion of students' solutions can bring out some techniques useful for estimating with fractions. For example, one way to estimate the sum of two proper fractions is to identify each fraction as greater than or less than $\frac{1}{2}$.

Example: $\frac{13}{24}+\frac{9}{20}$

$\frac{13}{24}$ is slightly more than $\frac{1}{2}$, since $\frac{1}{2}$ would be $\frac{12}{24}$

$\frac{9}{20}$ is slightly less than $\frac{1}{2}$, since $\frac{1}{2}$ would be $\frac{10}{20}$

Thus the sum is about 1.

Notes

Give a denominator for $\frac{5}{\square}$ so that the fraction is close to

i) 1   ii) 0   iii) $\frac{1}{2}$

Activity 4: Fractions

There are multiple answers to each problem. List several answers and have students look for similar characteristics. See the examples below.

> Excerpt from BLM 19:
>
> 6. Write a fraction that you can add to $\frac{1}{2}$ to give an answer greater than 1.
>
>   Possible answers: $\frac{2}{3}$, $\frac{3}{5}$, $\frac{7}{10}$, $\frac{8}{10}$, $\frac{9}{10}$
>
> 8. Write a fraction that you can add to $\frac{3}{4}$ to give a sum less than 1.
>
>   Possible answers: $\frac{1}{5}$, $\frac{1}{6}$, $\frac{1}{7}$, $\frac{2}{12}$, $\frac{3}{15}$

Discuss with students, for example, that each answer for #6 is greater than $\frac{1}{2}$ (because you have to add more than $\frac{1}{2}$ to $\frac{1}{2}$ to get a number greater than 1). Similarly, each answer for #8 is less than $\frac{1}{4}$ (because you have to add less than $\frac{1}{4}$ to $\frac{3}{4}$ to get a number less than 1).

Problem Solving

BLM 20 (Is It Ever True?) is an extension of BLM 19 and asks students to consider generalizaions about fractions. Students may find that their answers to some parts of BLM 19 will help with parts of BLM 20.

For example,

Excerpt from BLM 19:	Excerpt from BLM 20:
4. Write two fractions whose sum is greater than 1.	1. (d) The sum of two fractions is greater than 1.
12. Write two fractions whose difference is between $\frac{1}{2}$ and 1.	1. (l) The difference between two fractions is greater than 1.

Give a numerator for $\frac{\square}{7}$ so that the fraction is close to

i) 1   ii) 0   iii) $\frac{1}{2}$

Question 2 of BLM 20 asks students to write examples to justify their answers. Note that to justify a "Sometimes" response, students must write one true example and one false one.

Activity 4: Fractions

For example, for part (d) the example, $\frac{1}{2}+\frac{3}{4}=\frac{5}{4}$ *or* $1\frac{1}{4}$ *shows that, indeed, the statement "The sum of two fractions is greater than 1" is true at least sometimes. However students might argue (incorrectly) that this one example proves that the problem statement is* ***always*** *true. A second example such as* $\frac{1}{2}+\frac{1}{4}=\frac{3}{4}$ *shows that the problem statement is sometimes false and therefore not "Always True".*

Extensions in Mathematics:

1. BLM 21 (Number Strip Fractions) introduces a novel technique for identifying simple fractions that are greater or less than a given fraction.

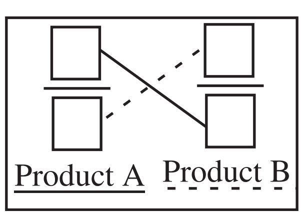
   Question 2 uses a series of fractions with which students will already be familiar:

   $$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \ldots$$

   Students may recognize that in this list the fractions are written in order of size from the least, $\frac{1}{2}$, to the greatest, $\frac{11}{12}$.

   To compare two fractions, such as $\frac{7}{8}$ and $\frac{8}{9}$, one can argue that $\frac{8}{9}$ is closer to 1 than $\frac{7}{8}$, since $\frac{7}{8}=1-\frac{1}{8}$ and $\frac{8}{9}=1-\frac{1}{9}$. That is, we subtract less from 1 to reach $\frac{8}{9}$ than to reach $\frac{7}{8}$. Therefore $\frac{8}{9}>\frac{7}{8}$.

2. Another technique for comparing fractions uses simple multiplication of numerators and denominators. The numerator of the first fraction is multiplied by the denominator of the second, and the denominator of the first is multiplied by the numerator of the second. For purposes of discussion, call these two products Product ***A*** and Product ***B***. It is important to record Products A and B in that order.



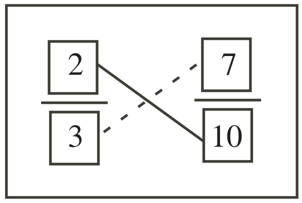
   If Product A is less than Product B, then the first fraction is less than the second. If Product A is greater than Product B, then the first fraction is greater than the second.

Activity 4: Fractions

Write three fractions between $\frac{1}{4}$ and $\frac{1}{3}$.

For example,

(i) Compare $\frac{2}{3}$ and $\frac{7}{10}$.



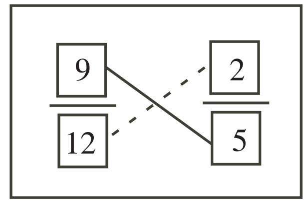
Product A $= 2 \times 10 = 20$

Product B $= 3 \times 7 = 21$

Product A < Product B so,

$$\frac{2}{3} < \frac{7}{10}$$

(ii) Compare $\frac{9}{12}$ and $\frac{2}{5}$.



Product A $= 9 \times 5 = 45$

Product B $= 12 \times 2 = 24$

Product A > Product B so,

$$\frac{9}{12} > \frac{2}{5}$$

This can be illustrated algebraically by examining the difference of the two fractions as follows:

$$\frac{a}{b} - \frac{c}{d} = \frac{a \times d - b \times c}{b \times d}$$

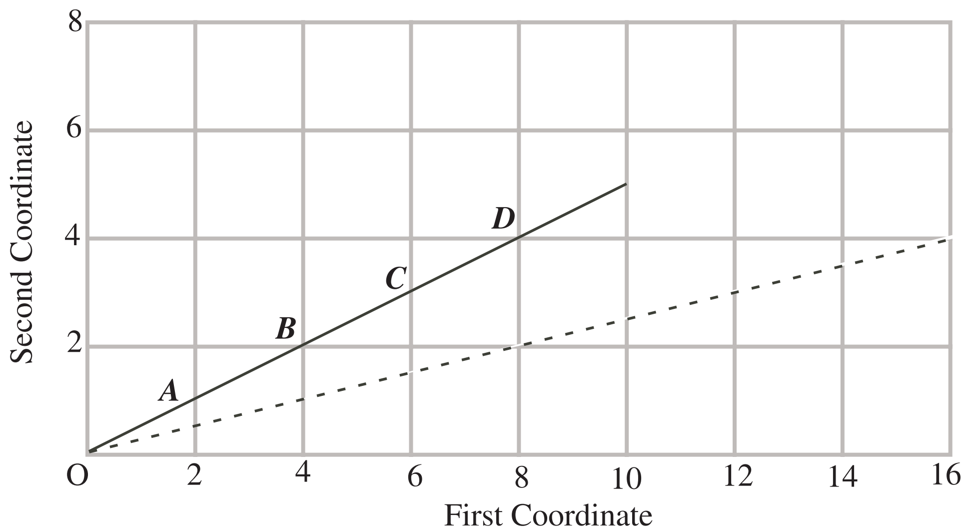
but $a \times d$ *is Product A and* $b \times c$ *is Product B. If* $a \times d > b \times c$, *then* $a \times d - b \times c$ *will be positive. That is,* $\frac{a \times d - b \times c}{b \times d}$ *is positive, which means* $\frac{a}{b} - \frac{c}{d}$ *is positive, i.e.* $\frac{a}{b} > \frac{c}{d}$. *If* $a \times d < b \times c$, *then* $a \times d - b \times c$ *will be negative. That is,* $\frac{a \times d - b \times c}{b \times d}$ *will be negative, which will happen only if* $\frac{a}{b} < \frac{c}{d}$.

3. A coordinate graph is a simple way of determining equivalent fractions. *In the graph below, point A has coordinate* (2,1). *Mathematicians say that the line from* (0,0) *to* (2,1) *has a slope of* $\frac{1}{2}$ *so the point* (2,1) *can be considered as indicating the fraction* $\frac{1}{2}$. *Point B has coordinates* (4,2) *so line OB has a slope of* $\frac{2}{4}$. *Point B can be considered as indicating the fraction* $\frac{2}{4}$. *Since the line from O through A to B is straight, slope* $\frac{1}{2}$ *equals* $\frac{2}{4}$, *or* $\frac{1}{2} = \frac{2}{4}$. *Similarly, Point C shows that* $\frac{3}{6} = \frac{1}{2} = \frac{2}{4}$ *and Point D shows that* $\frac{4}{8} = \frac{3}{6} = \frac{2}{4} = \frac{1}{2}$. *As long as the line starts at the origin and is straight, this idea can be used to identify equivalent fractions.*

Activity 4: Fractions

Have students mark axes on graph paper and plot the points (2,1), (4,2), (6,3), (8,4), joining them with a straight line. If (2,1) represents $\frac{1}{2}$, what does $\frac{4}{2}$ represent? $\left[\frac{2}{4}\right]$ What do (6,3) and (8,4) represent? $\left[\frac{3}{6} \text{ and } \frac{4}{8}\right]$ How are $\frac{1}{2}$ and $\frac{2}{4}$ related? How are $\frac{3}{6}$ and $\frac{4}{8}$ related? Which of the following points will be on the same line, and why: (10,6), (11,5), (14,7), (13, 6.5)? [(14,7), (13, 6.5)]

Ask students "What equivalent fractions are represented by the dotted line?" $\left[\frac{1}{4}, \frac{2}{8}, \frac{3}{12}, \frac{4}{16}, \ldots\right]$



Family Activities:

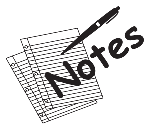
1. The game on BLM 22 can be sent home to play with other family members. If dice are not available, students can make simple spinners. (See BLM 23 for a sample spinner). Students should spin the spinner twice on each turn - once for the numerator and once for the denominator.

Other Resources:

For additional ideas, see annotated "Other Resources" list on page 82, numbered as below.

6. "A Game Involving Fraction Squares"

11. "Building Percent Dolls: Connecting Linear Mesurement to Learning Ratio and Proportion"



Activity 5: Estimation

Focus of Activity:

- Identifying 'over–estimate' and 'under–estimate'
- Identifying situations for which an estimate is appropriate
- Using estimation to identify errors

What to Assess:

- Reasonableness of answers
- Identification of an estimate as over or under
- Ability to justify answers; use of mathematical language

'BLM' refers to Black Line Masters, which follow the Activity Notes

Preparation:

- Make copies of BLMs 24 and 25.
- Make copies of BLM 26, 27, and 28 (optional).
- Provide a calculator for each pair/group (optional).

Activity:

Discuss with students when an under–estimate or over–estimate is acceptable or even preferred. For example, when keeping a running estimated total of the value of items you are buying, an under–estimate might lead you to conclude that you have enough money when in fact you haven't. Thus, an over–estimate is preferred in this case.

Sandpipers travel about 1920 km one-way during migration. If they travel 480 km per day, about how long does the trip take?

Ask students which would be acceptable/preferred (over - or under-estimate) in each of the following situations and why:

- determining a tip in a restaurant
- putting detergent in your washing machine (while considering the problem of pollution)
- predicting your grade on a test
- estimating your ability to carry heavy weights

Have students suggest other situations.

Problem Solving

BLM 24 (Over and Under) Ask students to make estimates and then determine if their estimates are over or under the actual value. You may wish to assign this BLM before discussing any technique for determining whether the estimate is over or under. In that case, students will be forced (encouraged?) to come up with their own methods in order to answer question 2.

In rounding, we do not always specify the range to be used. For example, 'to the nearest 10', or 'to the nearest 100'. Students may therefore vary in their responses. For example, in (a) 107 could become 100 or 110. It is not the rounding itself that is important here, it is the ability to determine whether an estimate is over or under the actual value and why.

Activity 5: Estimation

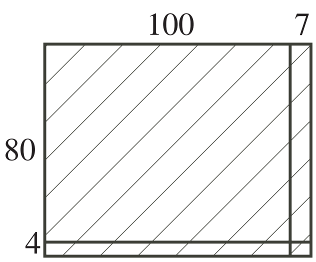
Some techniques are described here:

(1) For, say, 204×73, rounded to 200×70 to give an estimate of 14 000, it is clear that this is an under–estimate since both numbers were 'rounded down'.

(2) Similarly, if both numbers are 'rounded up' (e.g., 398×89 rounded to 400×90) this will give an over–estimate.

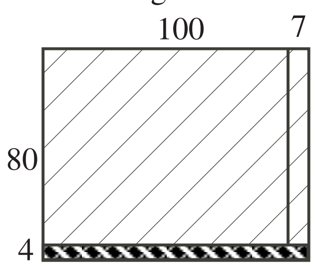
(3) However, if one factor is 'rounded up' and the other is 'rounded down', determining whether the result is an 'over–estimate' or 'under–estimate' is not quite this simple. For example, if the numbers in 107×84 are rounded to 110×80, then we have added 4×80 to the actual product while subtracting 4×107. Since we have subtracted more than we added, the result is an under–estimate. This can be shown in a diagram. (Note that the diagram is not to scale.)

Figure 1:



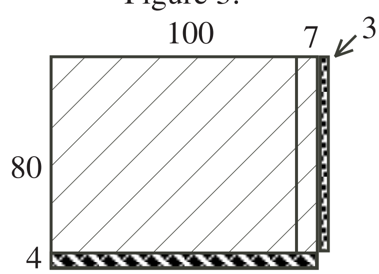
107 x 84

Figure 2:



107 x 80

Figure 3:



110 x 80

The original product, 107×84 is indicated in Figure 1 by the shading [hatched]. When 84 is rounded to 80, as in Figure 2, the cross-hatched area [cross-hatched] is being subtracted from the diagram, i.e., it is a loss. When 107 is rounded to 110, we are adding the section that is stippled [stippled], i.e., a gain. Thus, 110×80 is represented by [hatched] and [stippled].

We can see that the area added ([stippled]) is less than the area subtracted ([cross-hatched]). That is, 3×80 or 240 square units is less than 4×107 or 428 square units. Thus, the result is an under–estimate.

Notes

See "Solutions and Notes" for examples of this technique used on BLM 24.

Activity 5: Estimation

How many digits will there be in each answer?
i) $10298 - 2938$
ii) $1244 + 9873$

However, to simplify the calculations for students, the following 'gain' and 'loss' calculations will be accurate enough in the vast majority of cases.

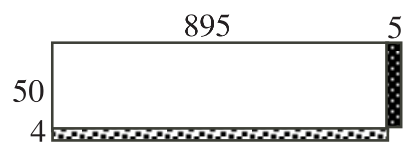
original numbers:	107	x	84
rounded numbers:	110	x	80
	rounded up		rounded down

Gain: $3 \times 80 = 240$ Loss: $4 \times 110 = 440$

Gain < Loss
This is an under–estimate.

Using the rounded numbers instead of the original numbers to calculate 'gain' and 'loss' means the arithmetic is mental, and much easier for students to calculate.

A second example is given below, showing that 900×50 is an under–estimate for part (c), 895×54, because the added area is 5×50 (250), but the subtracted area is 4×895 (4380). Students should be encouraged to explain how they know 4×895 is greater than 5×50 without actually calculating either value. For example, 5×50 or 250 is less than 895 and therefore much less than 4×895.



Using the simplified method, calculate $\begin{cases} \text{Gain: } 5 \times 50 = 250 \\ \text{Loss: } 4 \times 900 = 3600 \end{cases}$.

BLM 25 (Decoding with Estimates) gives further practice in identifying under–estimates or over–estimates. This is a self-checking exercise since the riddle answers will not make sense if students have identified estimates wrongly. Work through one example with students to be sure they understand the problem. For (a) the estimate of 450 is obviously 5×90. That is, the '89' has been rounded up to 90 while the '5' remains the same. Thus this must give an over-estimate. Students should circle the 'S' in the 'Over' column and then write the 'S' on any blank above the letter 'a'. The completed answer to #1 is "BASEBALL PLAYER", and to #2 "I'M DRESSING".

Activity 5: Estimation

Students will probably find it more difficult to determine over–estimates and under–estimates with division (question #2). They need to understand that if the dividend is rounded up, this will lead to an over–estimate, but if the divisor is rounded up, this will lead to an under–estimate. Specifically, dividing by a greater number gives a lesser quotient. For example,

$$24 \div 3 = 8 \quad \text{but} \quad 24 \div 4 = 6$$

Similarly, dividing by a lesser number gives a greater quotient.

Calculating gains and losses for division might look like this:

original numbers:	97	x	21
rounded numbers:	100	x	20
	rounded up		rounded down
	gain		gain <--rounding <u>down</u> the divisor gives a <u>gain</u>
	This is an over–estimate.		

In most cases it is not necessary to calculte the actual gain and loss.

Have students try to decide how the numbers were rounded to produce the given estimate. For example (a) was estimated as 630 √ 7, (b) as 810 √ 9, and (c) as 480 √ 6. In each case compatible numbers were chosen to make the division an easy mental question.

Communication

Extensions in Mathematics:

1. BLM 26 (Comparing Estimates) gives practice in estimating, in identifying over– estimates and under–estimates, and in justifying answers. Students should work in groups. Note that the estimates involve mixed operations.

2. BLM 27 (That Can't Be Right!) is intended for group discussion. Students should prepare their arguments for presentation to the class. This BLM presents some situations students may be familiar with, in which various characters are estimating, sometimes reasonably, sometimes doubtfully. Students should ignore taxes for these problems although as an extension you might wish to have them estimate 14% of the final totals (PST + GST). The need to calculate 14¢ for every dollar spent is a good reason for learning the 14 times table.

How many digits will there be in each answer?
i) $32456 \div 81$ (if the answer is a whole number)
ii) 6347 x 287

Cross-curricular Activities:

1. Have students examine newspapers to identify examples of estimates.

Family Activities:

1. BLM 28 (When Is an Estimate not an Estimate?) provides some 'real life' situations that students can discuss with family members. Students can ask family members for their opinions or help in identifying situations for which an

Notes

Activity 5: Estimation

estimate is appropriate and situations for which an estimate is not appropriate. For example, students may not recognize that a mechanic's 'estimate' (#6) is probably not an estimate.

Other Resources:

For additional ideas, see annotated "Other Resources" list on page 82, numbered as below.

13. "Mental Computation in the Middle Grades: The Importance of Thinking Strategies"

BLM 1: Placing Numbers

The number line below shows from 0 to 1 million.

A represents the length of the Great Wall of China in kilometres, 6000.
B represents the height of Mount Everest in centimetres, 900 000.

There are four other positions marked. Each one represents one of the following.
Mark C, D, E, and F in the proper places on the number line.

C: The mass of a blue whale, in kilograms is 190 000.

D: The area of Alberta, in square kilometres, is 662 000.

E: The speed of a jet-engined car, in km/h, is 10 000.

F: The world's worst oil spill, in tonnes, was 236 000.

The following are not marked. Estimate their positions and mark them on the number line.

G: The distance bees travel, in kilometres, to make a pound of honey, is 21 000.

H: The number of parts in a dragonfly's eyes is 26 000.

I: The average number of hairs on a human's head is 120 000.

Include two of your own examples:

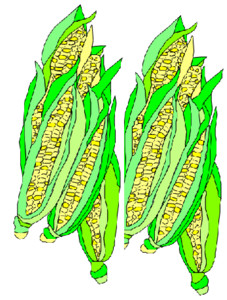
K:

L:

BLM 2: Numbers in Place

Draw another section of a number line to include the following numbers. Estimate the location of each number and mark it on the number line.

1. In 1977, Iowa's harvest of corn, in bushels, was 1 billion.



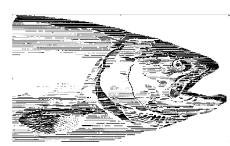
2. In the same year, the harvest in Kansas, in bushels, was 161 281 000.

3. In 1976 the total harvest of sea food from the world's oceans, in kilograms, was 79 billion.

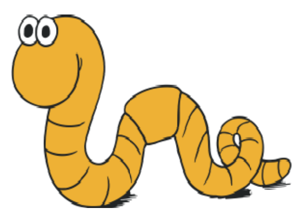
4. Hong Kong island has an area, in square kilometres, of about 80.

5. The population of Hong Kong, including tourists, is $7\frac{1}{2}$ million.

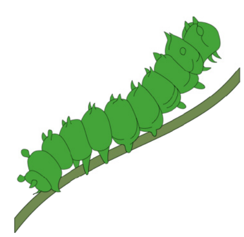
6. When travelling upstream to spawn, chinook salmon swim, in kilometres, up to 4635.



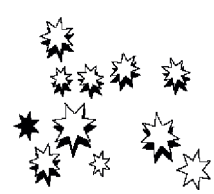
7. The number of earthworms in a field the size of a football field could be 2.5 million.



8. The number of muscles a caterpillar has is 4000. (You have 639.)



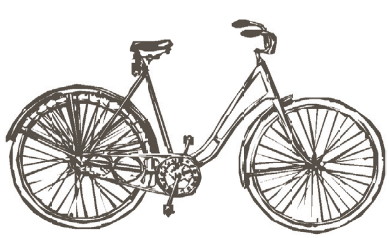
9. On a clear night, the number of stars you can see is about 3000.



BLM 3: Being Reasonable

Decide whether or not each of the following statements is reasonable and circle either "R" (for "Reasonable") or "U" (for "Unreasonable"). Tell why you think so.

1. A single potato chip weighs about 10 g. R U
2. A watermelon is about 93% water. R U
3. An average sized carrot weighs about 1 kg. R U
4. A CD usually has between 40 and 80 minutes of music. R U
5. Corn plants grow to about 2.2 m tall. R U
6. The seat of a dining room chair is about 75 cm above the floor. R U
7. Gasoline costs about $5.00 per litre. R U
8. A rock concert had an attendance of more than 20 000 people. R U
9. Alfonso's new computer had a hard drive of 10 megabytes. R U
10. Melinda's new bicycle cost $20. R U
11. Gerry's new digital camera holds 200 exposures. R U
12. A new minivan has seating for 12. R U
13. Mikhail lives in a castle that is at least 1200 years old. R U
14. Ten-year-old Zack claims that he can lift 2540 g. R U

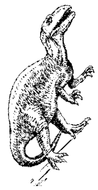


If you decided a statement was unreasonable, try to find a number that will make the statement reasonable. You may use any of the following resources: library books, text books, the Web, newspapers, magazines, members of your family, or your own knowledge.

BLM 4: Scaling Large Objects

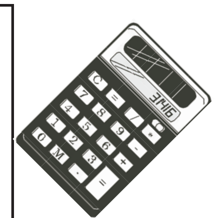
Each of the diagrams below represents an object that is larger than the diagram itself. The ratio of the size of the diagram to the size of the object is called the ‘scale’ of the diagram.

2 cm



Scale: 1:550

For example, the picture of the dinosur is 2 cm tall. The scale of the drawing is 1 to 550 (usually written as 1:550). This means that every 1 cm on the drawing represents 550 cm of the actual height of the animal. Since the drawing is 2 cm tall, this means the dinosaur was 2 x 550 cm or 1100 cm (or 11 m) tall.



Use the scale given to determine the actual size of the object.

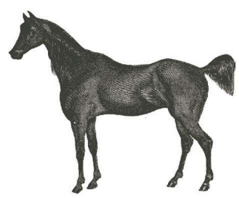
1. bicycle

5 cm

Scale: 1:40

Actual size: _______________

2. horse



3 cm

Scale: 1:70

Actual size: _______________

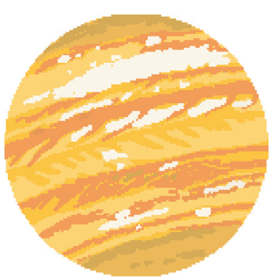
3. mountain

4 cm

Scale: 1:100 000

Actual size: _______________

4. Jupiter



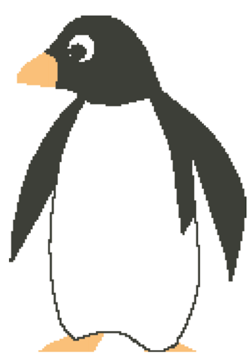
3.5 cm

Scale: 1:4 000 000 000

Actual size: _______________

5. penguin

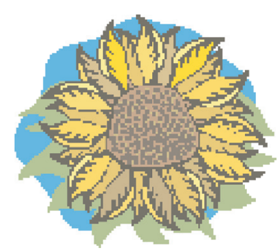
4 cm



Scale: 1:15

Actual size: _______________

6. sunflower



3 cm

Scale: 1:6

Actual size: _______________

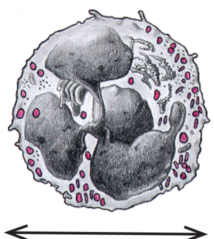
BLM 5: Scaling Small Objects

Each of the diagrams below represents an object that is smaller than the diagram itself. The ratio of the size of the diagram to the size of the object is called the 'scale' of the diagram.

For example, the picture of the red blood cell is 25 mm across. The scale of the drawing is 5000 to 1 (usually written as 5000:1). This means that the drawing is 5000 times the size of a real red blood cell. What is the actual measure of the red blood cell?

Use the scale to determine the actual sizer of each object.

1. a blood cell

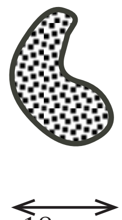


25 mm

Scale: 5000:1

Actual size: ________________

2. bacterium

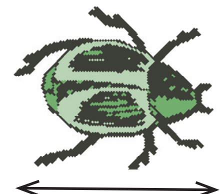


18 mm

Scale: 3000:1

Actual size: ________________

3. hairy-winged beetle

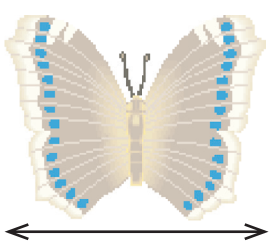


25 mm

Scale: 50:1

Actual size: ________________

4. dwarf blue butterfly

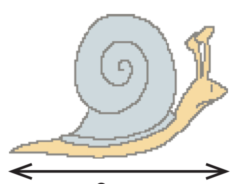


3 cm

Scale: 2:1

Actual size: ________________

5. smallest known snail

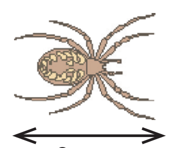


3 cm

Scale: 10:1

Actual size: ________________

6. smallest known spider



2 cm

Scale: 4:1

Actual size: ________________

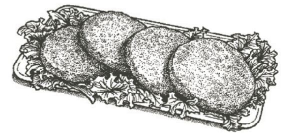
BLM 6: Errors, Errors, Errors!

The following are based on actual promotional flyers or advertisements, or found in books. Each has a math error. Find the error and explain how you would correct it.

1. PARTH COUNTY FAIR
Sept. 3-5
Save 25% by buying your ticket early!!
Price before Sept. 3: $7.00
Price after Sept. 3: $8.00

2. From a Science text book:
Your heart beats about 72 times a day, 37 843 200 times a year, and 28 382 400 000 times during a 75-year lifetime.

3. FRIED CHICKEN NUGGETS
5 for .99¢

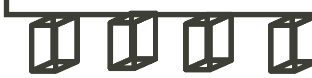


4. Save 31¢ on our $2.89 Bag of Snacks!!
Cost with coupon: $1.58

5. 

Was $3.50
Now $1.95
Save $2.55

6. GOOEY SNAK CUBES
NOW ONLY 3/$.00
REGULAR 99¢ VALUE



7. 25% OFF!
Back packs in all colours!!
Reg. $17.96
Sale: $23.95

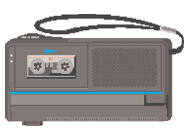
8. From a Math text-book:
Sam worked $8\frac{1}{2}$ h one week. He worked $10\frac{2}{3}$ h the next week. How many more hours did he work the first week than the second week?

9. 

Blaise Pascal
1623-1662

Pascal was an important mathematician in the 18th century

10. SAVE $100!!!
Compact Cassette Recorder less than 8 oz. 2-speed recording needs 2 AA batteries.
Now $39.95
Suggested list price $49.95
Adapter $15.95



11. A newspaper described a 12-foot snowman as follows:
The base is 22 ft. around;
the body is 13 ft. around;
the head is 8 ft. across.

12. From a math text-book:

OCTOBER

S	M	T	W	T	F	S
		1	2	3	4	5
6	7	8	9	10	11	12
13	14	15	16	17	18	19
20	21	22	23	24	25	26
27	28	29	30			

BLM 7: The Range Game

1. Give the greatest possible whole number factors for the following.

a) 47 x _____ = ____________ (350 to 390)

b) 94 x _____ = ____________ (650 to 700)

c) 29 x _____ = ____________ (630 to 680)

d) 53 x _____ = ____________ (240 to 280)

2. Give whole number answers for the following:

a) 3.09 is between 3 and 4 but closer to __________

b) 7.171 is between 7 and __________ but closer to __________

c) 21.45 is between __________ and __________ but closer to __________

d) 0.001 is between __________ and __________ but closer to __________

e) 53.019 is between __________ and __________ but closer to __________

3. Complete the decimal answers for the following:

a) 9.91 is between 9.__________ and 9.__________ but is closer to 9.__________

b) 5.454 is between 5.__________ and 5.__________ but is closer to 5.__________

c) 3.011 is between 3.0__________ and 3.0__________ but is closer to __________

d) 8.49__________ is between 8.4__________ and 8.4__________ but is closer to __________

e) 7.01 is between 7.0__________ and 7.0__________ but is closer to __________

4. Write either '0', "$\frac{1}{2}$" or '1' in each of the following blanks to make true statements.

a) $\frac{3}{5}$ is between $\frac{1}{2}$ and __________ but is closer to __________

b) $\frac{1}{5}$ is between __________ and $\frac{1}{2}$ but is closer to __________

c) $\frac{3}{8}$ is between __________ and __________ but is closer to __________

d) $\frac{3}{7}$ is between __________ and __________ but is closer to __________

e) $\frac{6}{11}$ is between __________ and __________ but is closer to __________

BLM 8: Range Points

Game 1: Choose two numbers from the set below. Add them and determine your score. The winner is the one with the most points after 10 turns.

$\frac{1}{2}$	$\frac{2}{3}$	$1\frac{1}{4}$	$\frac{1}{4}$
$2\frac{3}{4}$	$\frac{4}{2}$	$\frac{5}{2}$	$1\frac{2}{3}$
$\frac{4}{3}$	$1\frac{1}{5}$	$\frac{4}{5}$	$1\frac{1}{2}$

Scoring System: If the sum is a whole number, score 1 point.
If the sum is a fraction, use the following ranges to determine your score.

0 to 1	1 to 2	2 to 3	3 to 4	4 to 5
1 point	2 points	3 points	4 points	5 points

Game 2: Choose two numbers from the set below. Multiply the numbers to determine your score.

1.6	4.7	0.33
5.4	0.4	
2.03	8.1	3.0
9.1	6.01	

0 to 1	1 to 5	5 to 10	10 to 50	50 to 100
3 points	1 point	3 points	1 point	5 points

BLM 9: Fraction Tiles

1. Use all ten tiles to complete the following so that the mathematical statements are true.

$$\frac{1}{\square} < \frac{1}{\square} \qquad \frac{2}{\square} < \frac{2}{\square}$$

$$\frac{2}{\square} < \frac{\square}{8} \qquad \frac{3}{\square} < \frac{\square}{5}$$

$$\frac{\square}{1} < \square$$

2. Use all ten tiles to complete this set of five fractions in order from least to greatest.

$$\frac{\square}{\square} \quad \frac{\square}{\square} \quad \frac{\square}{\square} \quad \frac{\square}{\square} \quad \frac{\square}{\square}$$

Cut these tiles apart to use in the problems above.

0	1	2	3	4	5	6	7	8	9

BLM 10: Truths and Half - truths

For each of the following problems, tell whether each of the statements, a), b), c), or d), is always true (A), never true (N), or sometimes true and sometimes false (S). Circle the correct letter to show your answer. Provide examples to illustrate your answer.

1. If the sum of four two-digit numbers is less than 80, then

a) all four numbers are less than 100	A	N	S
b) at least one number is less than 50	A	N	S
c) if two numbers are less than 40, the other two must be greater than 40	A	N	S
d) the sum of the four numbers is an even number	A	N	S

2. If the sum of four fractions is less than one, then

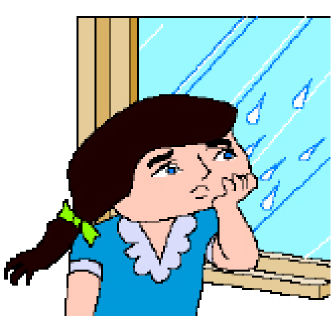
a) all four fractions are less than $\frac{1}{2}$	A	N	S
b) at least one fraction is less than $\frac{1}{2}$	A	N	S
c) at least one fraction is greater than $\frac{1}{2}$	A	N	S
d) at least one of the fractions is an improper fraction	A	N	S

3. If the product of a fraction and a whole number is a whole number, then

a) the product is greater than the whole number you started with	A	N	S
b) the whole number you started with is a multiple of the fraction's denominator	A	N	S
c) the whole number you started with is a multiple of the fraction's numerator	A	N	S
d) the fraction is less than 1	A	N	S

4.

a) If the grass is wet, it must be raining.	A	N	S
b) If the plows are out, it must have snowed.	A	N	S
c) If Jim got 100% on his math test, he must have studied.	A	N	S
d) If there are presents under the tree, then Santa must have come.	A	N	S




BLM 11: Relative Sizes

1. By estimating only, without calculating, decide whether the symbol > (is greater than) or < (is less than) or = (is equal to) should be in each box to make the statement true. Justify your answer.

a) $334 + 976 + 533$ ☐ $408 + 988 + 656$

b) $56 + 57 + 58$ ☐ 3×57

c) $760 \div 8$ ☐ $760 \div 9$

d) 512×1.1 ☐ 512

e) 674×0.9 ☐ 674

f) $5.01 + 5.2 + 5.03 + 5.04$ ☐ 4×5.02

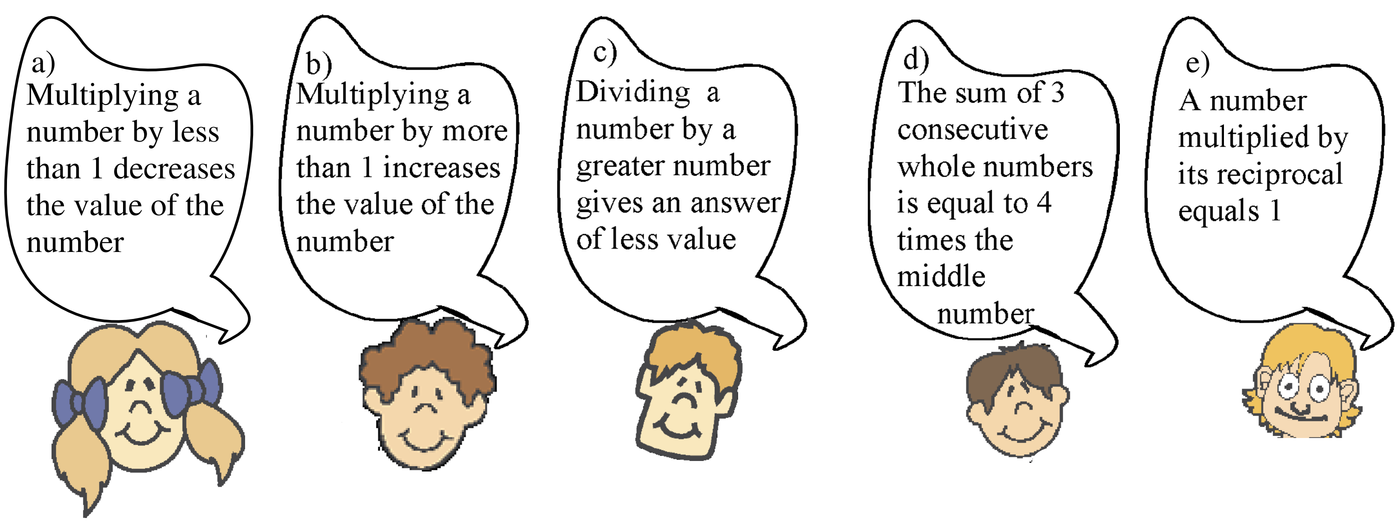
g) 88×1.001 ☐ 88

h) $78 \times 1\frac{1}{2}$ ☐ 100

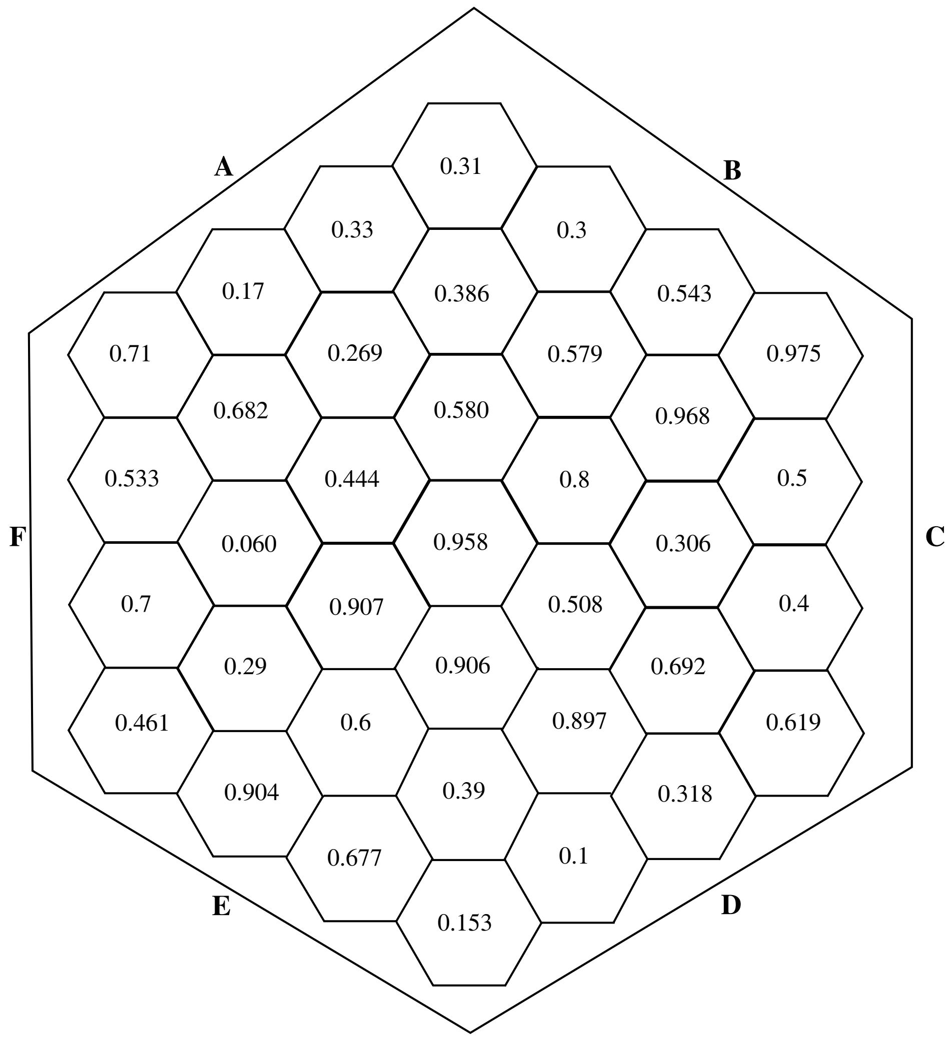
i) $6\frac{1}{4} \times \frac{4}{25}$ ☐ 1

j) 83×0.56 ☐ $83 \times \frac{1}{2}$

2. Tell whether each of the following is true or false. Give examples.



BLM 12: Greater or Less?



To play the game you will need two markers of the same colour for each player and a two-colour chip or a coin. Each player chooses a starting side on the board --A, B, C, D, E, or F. The aim of the game is to move one of your markers from one side of the board to the opposite side. For example, the player who starts on side A must move his/her markers toward the opposite side, D; the player who starts on B must move toward E, and so on.

Begin by placing your two markers on any of the numbers along your chosen side.

In turn, each player flips the two-colour counter or coin. If it comes up red (or heads) move one of your markers onto an adjacent space with a number that is greater than the number you are on. If the chip/coin turns up yellow (or tails) move to a number that is less than the one you are on.

You may move either of your markers on your turn. Do not move both on the same turn.

Every move must be in a forward direction.

If you cannot move either marker you lose your turn. Only one marker is allowed on a number at any one time.

The first person to get one of his/her markers to the opposite side of the board is the winner.

BLM 13: Name That Number

Choose a number — any whole number. It might be your favourite number or your lucky number. Do not choose 1 or 0.

1. Find a number for each of the following problems.

a) What number can you multiply your number by without changing its value? $\frac{3}{4}$

b) What number can you add to your number without changing its value? 0

c) What number can you subtract from your number without changing its value? 7.5

d) What number can you divide your number by without changing its value? $\frac{6}{12}$

e) What kind of number can you multiply your number by to get a lower value?

f) What kind of number can you multiply your number by to get a greater value? 1.0

g) What kind of number can you divide your number by to get a lower value?

h) What kind of number can you divide your number by to get a greater value? $\frac{3}{18}$

i) What kind of number do you get if you multiply your number by an even number?

j) What kind of number do you get if you multiply your number by an odd number?

k) What number can you multiply your number by to get nothing? **2**

l) What kind of number do you get if you divide your number by an even number?

2. Compare your answers with your classmates. For which of the problems do you have the same answers? Why?

3. For which of the problems do you have different answers? Why?

BLM 14: Should Remainders Remain?

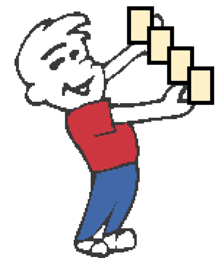
For each of the answers to the following problems, tell in which of the following ways the remainder should be used:

(i) You should ignore the remainder.
(ii) You should round the answer to the next whole number.
(iii) You should leave the remainder as it is.

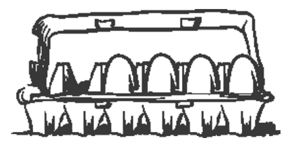
1. A school with 150 students and 7 teachers is going to a concert. A school bus holds 35 people besides the driver. How many buses will be needed?

2. a) David is gluing his baseball cards into an album. Each page will hold 6 cards. If David has 39 cards, how many pages will he fill?

   b) If the question had been, "How many pages will David need?", would your answer be different? Explain.



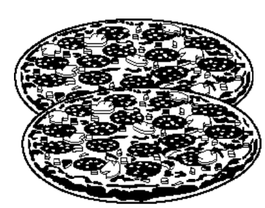
3. Ms. Sheff is baking cookies. She has 7 eggs. Each batch of cookies needs 2 eggs. How many batches can she bake?



4. A class of 36 students is learning to square dance. A 'square' consists of 8 people. How many squares can be formed?

5. Jessica is sharing her candies with her 2 friends. If Jessica has 17 candies, how many will each person have?

6. Sue ordered 2 large pizzas for her party. Each pizza was cut into 8 slices. If there were 6 people at the party, how many slices could each person have?



7. Stefan was mailing copies of his report card to his grandparents and aunts and uncles in Yugoslavia. Each package took three 50¢ stamps. If Stefan had $8.00, how many copies of his report card could he send?

BLM 15: '‘Four’ Patterns

1. a) List some numbers that are divisible by 2.

   b) Are these odd numbers or even numbers?

   c) Which of the numbers below are divisible by 2? Loop them.

   12 26 324 423 555 678 1092 2435 3000 5049 9450

   d) Check using your calculator.

2. a) Which of the following numbers are divisible by 5? Loop them.

   105 501 510 647 558 590 595 2345 5432 8905 8950

   b) Check using your calculator.

   c) Describe how you can tell, by looking at a number, whether or not it is divisible by 5.

3. a) Most four–function calculators will count by 4s if you enter

   (0) (+) (4) (=) (=) and continue to press (=).

   What arithmetic does your calculator do every time you press (=)?

   b) Count by 4s using your calculator. Loop the numbers from the list below that appear in the display, as you count by 4s..

   4 7 13 20 28 30 36 44 60 62

   64 70 72 88 100 112 124 150 152 200

   c) Based on your results from b), tell whether each of the following statements about multiples of 4 is true or false. (If a number is divisible by 4 it is a multiple of 4).

i)	Multiples of 4 are even numbers.	T	F
ii)	The last digit of any multiple of 4 is a multiple of 4.	T	F
iii)	The last two digits of any multiple of 4 form a two-digit number that is a multiple of 4.	T	F
iv)	Numbers ending in 00 are multiples of 4.	T	F

BLM 16: Division Patterns

1. Most four–function calculators will count by 3s if you enter

   (0) (+) (3) (=) (=) and continue to press (=).

   Predict which of the following numbers will appear in the display if you count by 3s using this method. Loop them.

   5 8 12 20 27 30 40 42 56

   60 63 69 72 75 77 78 90 111

   Now count by 3s using your calculator to see if you were correct.

2. You can count by 9s by entering

   (0) (+) (9) (=) (=) and so on.

   Predict which of the following numbers will appear in the display if you count 9s using this method. Loop them.

   12 18 27 34 38 45 56 63

   67 76 84 99 100 126 134 144

   Now use your calculator to see if you were correct.

3. The 'digit sum' of a number is found by adding all the digits of a number.
   For example, the digit sum of 18 is $1+8$ or 9. We write $18 \rightarrow 9$.
   Sometimes we have to add more than once to reach a single digit.
   For example, the digit sum of 75 is 3, as shown by $75 \rightarrow 12 \rightarrow 3$.

   a) Calculate the digit sum of each of the numbers in question 1. Examine the numbers you found when counting by 3s. How are their digit sums alike? How are they different from the digit sums of the other numbers?

   b) Use your conclusions to identify the numbers below that are divisible by 3. Then check using your calculator.

   68 94 99 101 120 243 415 514 678 789

4. a) Calculate the digit sum of each of the numbers in question 2. Examine the numbers you found when counting by 9s. How are their digit sums alike? How are they different from the digit sums of the other numbers?

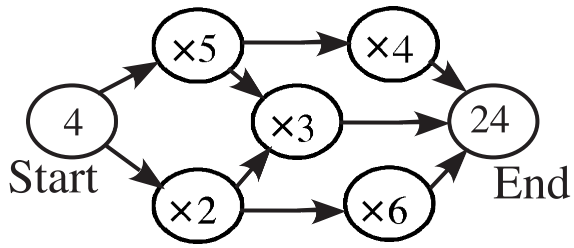
   b) Use your conclusions to identify the numbers below that are divisible by 9. Then check using your calculator.

   541 567 657 699 702 720 742 801 810 935 991

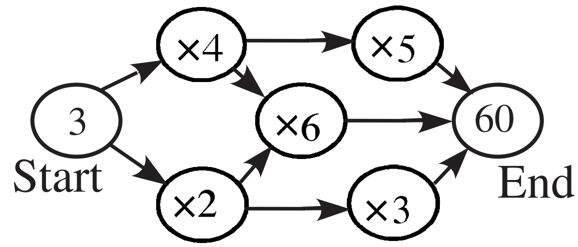
BLM 17: 'Find the Path

Begin with the "Start" number and find the correct path to reach the "End" number.
Check with your calculator.

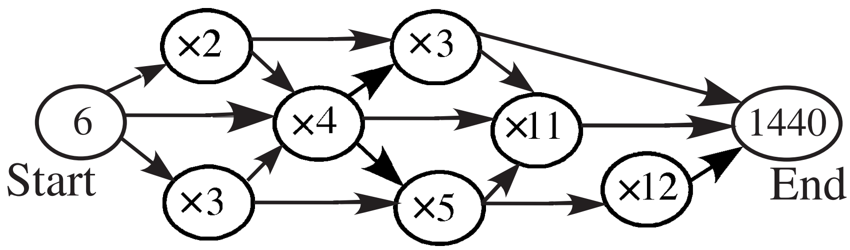
1.



2.



3.



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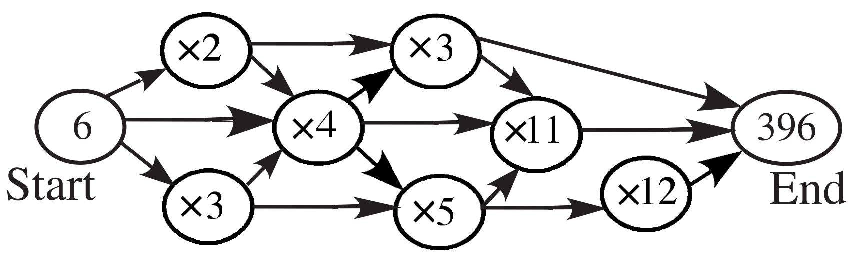

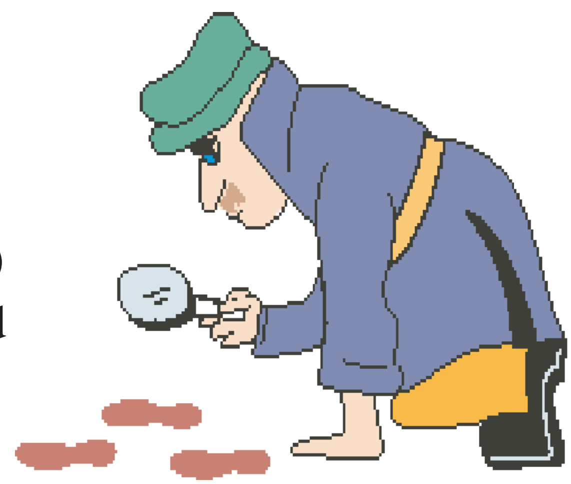

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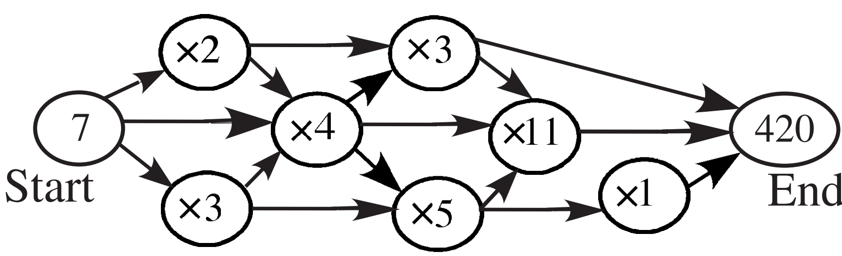

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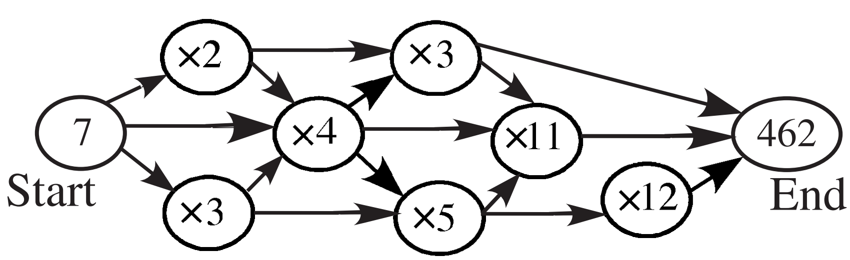


BLM 18: Reach A Number

These are games for two players and one calculator.

The only keys you may use are

(1) (2) (3) (4) (5) (6) (7) (8) (9) (0) (•) (×) (=)

Game 1:

The first player enters 16 on the calculator.

The second player presses (×), any number, and (=).

The first player then presses (×), any number, and (=).

Do NOT clear the calculator, but continue in this way until the display shows 400 or a number as close as possible to 400. The player who reaches this number is the winner.

Hints:

Suppose the display shows 320. What could you multiply this by to get a number less than 400?

Suppose the display goes over 400. What could you multiply this by to get a number less than 400?

Game 2:

Start with 9 and try to reach the target of 300 or a number as close to 300 as you can.

Is it possible to get exactly 300? Explain.

Game 3:

Start with 200 and try to reach the target of 8. What kinds of numbers should you multiply by?

Game 4:

Start with 200 and try to reach the target of 8, but by dividing instead of multiplying.

Is this easier or harder than Game 3? Why?

BLM 19: The Problem with Fractions

1. Write two fractions between $\frac{1}{2}$ and 1.	2. Write a fraction between $\frac{1}{4}$ and $\frac{1}{2}$ with a denominator of 10.	3. Write a fraction between 0 and $\frac{1}{2}$ whose numerator is not 1.
4. Write two fractions whose sum is greater than 1.	5. Write two fractions whose sum is less than 1.	6. Write a fraction that you can add to $\frac{1}{2}$ to give an answer greater than 1.
7. Write two fractions, with numerators greater than 1, whose sum is less than 1.	8. Write a fraction that you can add to $\frac{3}{4}$ to give a sum less than 1.	9. Write two fractions with different denominators whose sum is 1.
10. Write two fractions between $\frac{1}{4}$ and $\frac{1}{3}$.	11. Write two fractions whose difference is 1.	12. Write two fractions whose difference is between $\frac{1}{2}$ and 1.
13. Write two fractions whose difference is less than 1.	14. Write two fractions whose sum is between 0 and $\frac{1}{2}$.	15. Write two fractions whose product is less than $\frac{1}{2}$.
16. Write two fractions whose product is greater than 1, and whose difference is less than 1.	17. Write two fractions with different denominators whose product is 1.	18. Write two fractions whose product is greater than 1 and whose sum is greater than their product.
19. Write two fractions whose difference is less than $\frac{1}{2}$, and whose sum is greater than 1.	20. Write a fraction that you can add to $\frac{1}{2}$ to give an answer less than 1.	21. Write two fractions whose product is between $\frac{1}{2}$ and 1.

BLM 20: Is It Ever True?

1. Tell whether each of the following statements is Always True (Write "A"), Sometimes True (Write "S") or Never True (Write "N"). Be prepared to justify your answer.

A / S / N

a) A mixed number can be expressed as an improper fraction (a fraction in which the numerator is greater than the denominator). ___________

b) The value of a mixed number is less than 1. ___________

c) A mixed number is made up of two whole numbers. ___________

d) The sum of two fractions is greater than 1. ___________

e) The sum of two fractions is less than 1. ___________

f) The product of a whole number and a fraction is greater than the whole number you started with. ___________

g) The product of a whole number and a fraction is less than the whole number you started with. ___________

h) The product of two fractions is one. ___________

i) The product of two fractions is less than one. ___________

j) The product of two fractions is greater than one. ___________

k) The product of two fractions is less than either fraction. ___________

l) The difference between two fractions is greater than one. ___________

2. If your answer to any part of #1 is "Sometimes", give examples to justify this answer.

BLM 21: Number Strip Fractions

1. Make two number strips like the ones shown below, or cut out the two at the bottom of the page to use. If you are making your own, use graph paper to make sure each number has the same amount of space on the strip.

1	2	3	4	5	6	7	8	9	10	11	12
1	2	3	4	5	6	7	8	9	10	11	12

The top number strip will give the numerators of fractions. The bottom number line will give the denominators.

2. Slide one number strip to the right so that '3' is over '4'. This gives us the fraction $\frac{3}{4}$. Look at the fractions to the left of $\frac{3}{4}$. One of these is $\frac{1}{2}$. What is the other one? _____

	1	2	3	4	5	6	7	8	9	10	11	12
1	2	3	4	5	6	7	8	9	10	11	12	

Is $\frac{1}{2}$ greater or less than $\frac{3}{4}$? _________ Is $\frac{2}{3}$ greater or less than $\frac{3}{4}$? ________

3. Write three of the fractions to the right of $\frac{3}{4}$. __________________ Are they greater or less than $\frac{3}{4}$? ______

4. Move the numerator number strip again to get the following fractions. Find $\frac{5}{8}$.

			1	2	3	4	5	6	7	8	9	10	11	12
1	2	3	4	5	6	7	8	9	10	11	12			

List three fractions to the left of $\frac{5}{8}$. ______________________ Are they greater or less than $\frac{5}{8}$? _________

List three fractions to the right of $\frac{5}{8}$. ______________________ Are they greater or less than $\frac{5}{8}$? ________

5. Move the numerator line strip to make improper fractions (with the numerator greater than the denominator).

1	2	3	4	5	6	7	8	9	10	11	12			
			1	2	3	4	5	6	7	8	9	10	11	12

Pick any fraction showing and examine the fractions to the left and right of that one to see if they are greater or less than the fraction you picked. Write your conclusions.

1	2	3	4	5	6	7	8	9	10	11	12
1	2	3	4	5	6	7	8	9	10	11	12

BLM 22: More or Less

FOR: 2-4 Players
SKILL: Estimating sums of fractions
MATERIALS: 2 regular dice of different colours
1 playing board
several markers for each player

RULES:
1. Decide which die will represent the numerator and which the denominator.
2. Roll the dice. Use the two numbers to make a fraction and put a marker on any one block that describes your fraction.
3. Take turns rolling the dice and placing one of your markers on the board. More than one marker may be on any one square.
4. The winner is the first one to get 4 of his/her markers in a row.

VARIATIONS:
1. Allow only one marker on any one square.
2. Allow either die to be the numerator or denominator.
3. Enlarge the Playing Board and add more fraction descriptions.

ANALYSIS:
1. Which game is fairer -- the original or Variation 1? Why?
2. How does Variation 2 change the game?

Playing Board

Numerator is odd	**Fraction is less than 1**	**Fraction is greater than $\frac{3}{4}$**	**Numerator is less than 4**
Fraction is greater than $\frac{1}{3}$	**Fraction is less than $\frac{1}{2}$ or more than $\frac{3}{4}$**	**Denominator is even**	**Fraction is greater than 1**
Fraction does not equal $\frac{1}{2}$ or 1	**Numerator is even**	**Fraction is less than $\frac{1}{2}$**	**Fraction is greater than $1\frac{1}{4}$**
Fraction is less than 1	**Fraction is equal to $\frac{1}{2}$ or 1**	**Denominator is greater than 3**	**Denominator is odd**

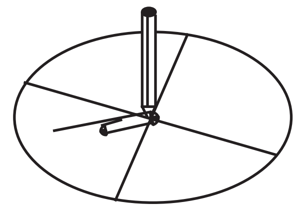
BLM 23: Constructing Spinners

To construct spinners, use the templates at the bottom of the page. Paste the spinners on to bristol/cardboard.

Method 1:

For the spinner, straighten a paper clip as shown.

Hold the spinner in place with a pen or pencil at the centre of the circle.
Flick the point of the paper clip with a finger.
This is the simplest way to construct an acetate spinner for use with an overhead projector.

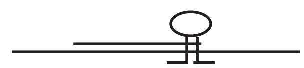


Method 2:

Cut arrows from the bristol board or cardboard and punch a hole in one end.

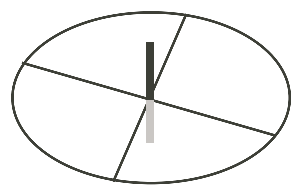
Punch a hole in the centre of each spinner.

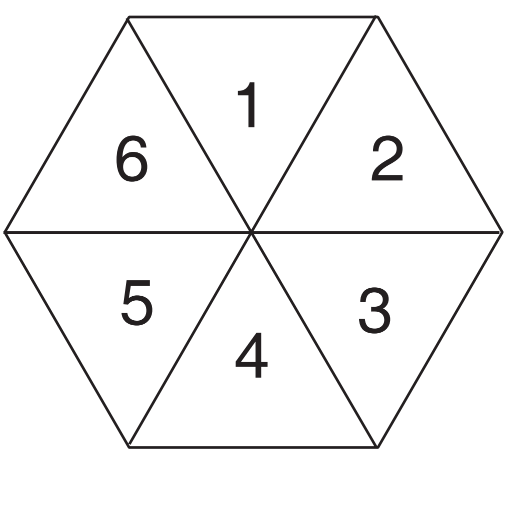
Use a paper fastener to fasten the two pieces together.
The connection should be tight enough so that the arrow doesn't wobble, but loose enough so that it spins freely.

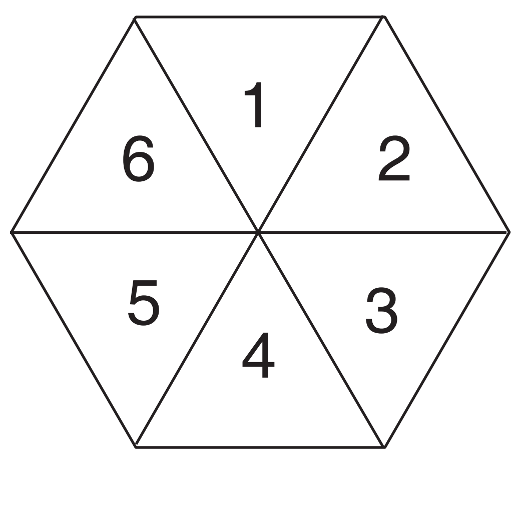


Method 3:

Punch a small hole in the centre of the spinner.
Put a round toothpick about one-third of the way through the hole.
Spin like a top, using the number that touches the desk/table.







BLM 24: Over and Under

1. Estimate an answer to each question by rounding each number to the nearest 10, 100, or 1000. Tell whether your estimate is over or under the actual answer without calculating the actual answer.

	Question	Rounded Numbers	Estimated Answer	Over or Under?
Example →	78 x 90	80 x 90	7200	over
a)	107 x 84			
b)	7 x 19			
c)	895 x 54			
d)	102 x 28			
e)	58 x 113			
f)	$98.15 x 29			
g)	18 x $3.99			

2. For each question above, explain how you can tell that your estimate is over or under the actual value. Use the back of this page if you need more room.

BLM 25: Decoding with Estimates

In the table below, an estimate is given for each question. If you think this is an under-estimate, circle the letter in the column titled "Under". If you think the estimate is an over-estimate, circle the letter in the "Over" column.

Put the circled letters in the proper blanks to find the answer to the riddle.

1.

	Question	Estimate	Under	Over
a)	5 x 89	450	T	S
b)	8 x 715	5600	L	F
c)	73 x 21	1400	B	C
d)	8 x 273	2400	I	P
e)	73 x 912	63 000	E	O
f)	178 x 38	8000	U	A
g)	52 x 595	30 000	Y	D
h)	625 x 18	12 000	Z	R

Riddle: What catches flies?

___ ___ ___ ___ ___ ___ ___ ___
c f a e c f b b

___ ___ ___ ___ ___ ___
d b f g e h

2.

	Question	Estimate	Under	Over
a)	623 ÷ 7	90	B	M
b)	821 ÷ 9	90	E	F
c)	456 ÷ 6	80	P	S
d)	243 ÷ 3	80	N	O
e)	710 ÷ 63	12	L	I
f)	397 ÷ 42	10	H	G
g)	1227 ÷ 29	40	R	Q
h)	3716 ÷ 38	90	D	A

Riddle: What did □ the mayonnaise say to the salad?

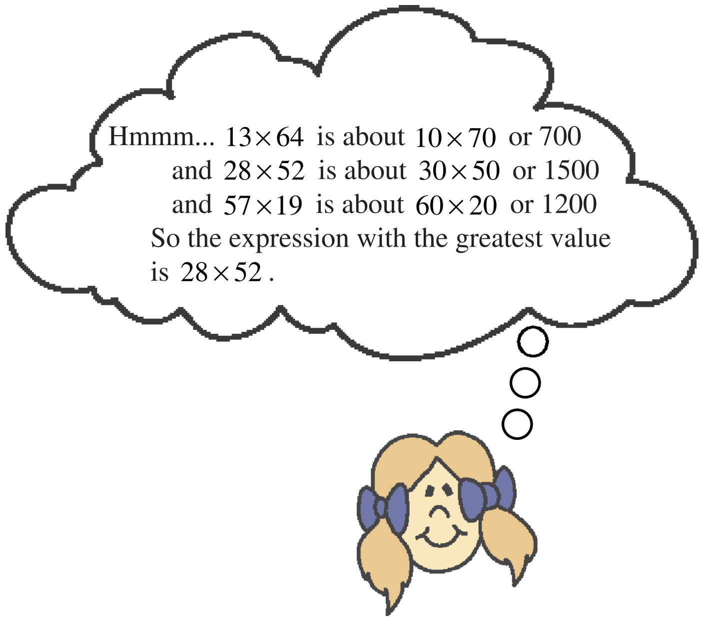
"Go away, ___□' ___
e a

___ ___ ___ ___ ___ ___ ___ ___ "
h g b c c e d f

BLM 26: Comparing Estimates

1. Estimate an answer for the three expressions in each line.
   Loop the one that you think has the greatest value, as in the example.
   Example: 13×64 (28×52) 57×19

a)	29×41	52×48	19×89
b)	26×19	400	21×21
c)	$489+612$	31×35	23×42
d)	$900-496$	$615-101$	$891-525$
e)	$243\div8$	$430\div6$	$323\div4$
f)	89×41	121×12	65×21
g)	27×48	672×8	136×15
h)	$12\times8\times5$	$15\times4\times7$	$9\times5\times6$



2. Explain how you made your choice for any three of the above. Use the back of this sheet if you need more space.

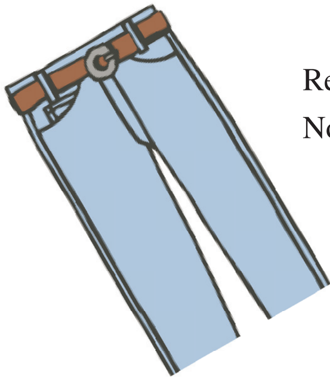
3. Check your answers using a calculator. Were you correct for most of the questions? If not, try to explain how you made an error.

BLM 27: That Can't Be Right!

There are peculiar answers given for some of the problems/situations below. Find the ones that "can't be right" and tell why they can't. Ignore taxes.

1.

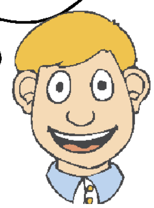
DENIM JEANS



Regular $29.99

Now 30% off!

That means I can get 2 pair for $50.00!



2.

DANI'S DINER

HAMBURGER

SHAKE

FRIES

TOTAL: $9.54

A 15% tip is about $2.00

3.

CAR RALLY!!!

Toronto to Barrie: 50 km

London: 157 km

London to Toronto: 102

Let's see ... that's about 300 km, and my car uses about 10 L per 100 km, so I'll need 3000 L!!

COMFORT CARPET

Soft on your toes

Warm on cold mornings

Special $12/m²

My bedroom is 3 m by 3.6 m, so I'll need 9.5 m² for wall-to wall carpeting.

5.

SPECIAL!

Blank CDs

Buy 2 boxes

Get 1 free!

$7.98 ea.

Reg. $13.95

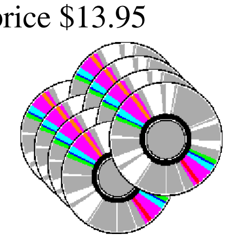
I need 1 box and Mom needs 2. I have only $25. That's not enough.

6.

CDs

SAVE 714%

List price $13.95



You only pay $3.99

BLM 28: When Is an Estimate Not an Estimate?

For each of the following, tell whether or not an estimate is appropriate. Explain your choices.

1. A waiter calculates a 5% tax on a meal.

2. A customer calculates a 15% tip on the same meal.

3. A clerk calculates 6% sales tax on a purchase.

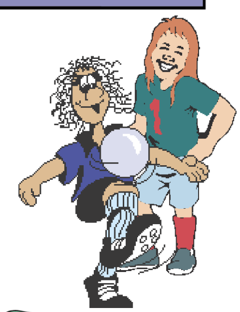
4. Mr. Mujeer figures he has enough money to pay for his purchases.

5. Maria figures that shovelling snow for 3 hours at $4.50 per hour will earn her enough money to buy a CD for $12.95.

6. The mechanic gave Chris an estimate of $354.75 for car repairs.

7. A newspaper headline tells how many people attended a Blue Jays game.

8. The management calculates how much money was paid for tickets at a soccer game.



9. An airline determines how many dinners will be needed for a particular flight.

10. A newpaper headline says "Will Grates is worth 4.5 billion dollars."

Solutions & Notes

Marginal Problems

Page	Problem	Discussion
3	Write a 3-digit number, *abc*, so that $a \times b = c$, and all 3 digits are different. How many such numbers are there?	Since the digits are all different, neither of the first two digits can be '1'. Since '*c*' is a single digit, the first two digits cannot be '34' or anything greater. There are only two 3-digit combinations meeting the criteria: 236 and 248. However, the first two digits can be interchanged to give 326 and 428 as well. Thus there are 4 such 3-digit numbers.
7	If you drove at the highway speed limit, it would take you a full year to circle the outer rings of Saturn. What is the circumference of these outer rings?	A year of 365 days has 8760 hours. At 80 km/h, you could travel 700 800 km in a year. At 100 km/h, you could travel 876 000 km. The circumference of the outer rings is estimated at between 600 000 km and 900 000 km.
9	At 100 km/h, how long would it take to drive around the world?	The earth's circumference at the equator is about 40 000 km. At 100 km/h, it would take 400 hours driving non-stop (about 17 days). Going "around the world" following, say, the Tropic of Capricorn, will take less time.
9	Assume you are driving a 'magic' car that actually prefers to drive on water. Where on the earth would your route take you?	Answers will vary.
10	Write 2 multiplication questions whose answers have a zero in the tenths place and a 6 in the hundredths place.	The answer will be of the form $\square.06$. Answers may vary from the easy 0.06×1 to, for example, 9.02×3, to more complicated examples such as 4.011×6.
11	Write two subtraction questions whose answers have a 5 in the tens place and a 9 in the tenths place.	Answers will be of the form $5\square.9$. Possible answers include $60 - 0.1$, $58.9 - 7.0$, $2753.64 - 2699.74$.
12	Write 5 different numbers between 3.4 and 3.5.	Possible answers are 3.41, 3.42, 3.43, ... 3.49.
13	Write a 4-digit number, *abcd*, so that $a + b + c = d$, and the digits are all different. What is the greatest such number? What is the least?	There are only 7 different combinations of digits meeting these criteri, but each combination can be arranged in different ways. For example, 1236, 1326, 3126, 3216, etc. The greatest such number is 4329 and the least is 1236.
14	Write a 4-digit number, *abcd*, so that $a \times b \times c = d$ and the digits are all different. How many such numbers are there?	There are only two combinations of digits meeting these criteria (1, 2, 3, and 6 and 1, 2, 4, and 8), but each set can be arranged in 6 different ways. Thus, there are 12 such 4-digit numbers.

Solutions & Notes

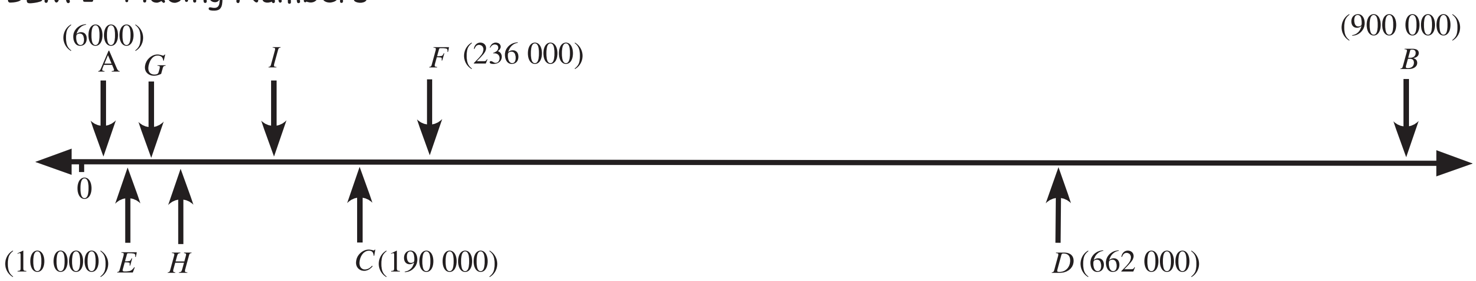
Page	Problem	Discussion
15	Write a 4-digit number, *abcd*, so that $a \times b = c \times d$, and the digits are all different. What is the greatest such number? What is the least?	The only possible combinations are $1 \times 6 = 2 \times 3$ and $1 \times 8 = 2 \times 4$. The greatest such number is 8142 and the least is 1623.
16	What fraction could you add to $\frac{1}{2}$ to give an answer that is less than 1? Give three different answers.	Any fraction between 0 and $\frac{1}{2}$ is suitable.
19	What is the greatest 4-digit number that can be multiplied by 9 to give a product that is a 4-digit number?	The greatest 4-digit number is 9999. This is equal to 1111×9. The answer to the problem, then, is 1111.
19	What is the least whole number that is divisible by 2, 3, 4, and 5?	If the number is divisible by 4 it is also divisible by 2. Thus the number is $3 \times 4 \times 5 = 60$.
21	What is the least whole number that is divisible by every whole number from 1 to 10 inclusive?	If the number is divisible by 6, it is also divisible by both 2 and 3. If it is divisible by 8, it is also divisible by 4. If it is divisible by both 8 and 9, it is also divisible by 6. Thus the number is $5 \times 7 \times 8 \times 9 = 2520$.
22	If the decimal point on a calculator is broken, how could you use the calculator to determine $34.25 + 17 + 8.1$?	Write each number with 2 decimal places $(34.25 + 17.00 + 8.10)$, add without the decimal points $(3425 + 1700 + 810 = 5935)$, and write the decimal point in the proper place (59.35).
23	How can you tell if a fraction is close to 1?	If a fraction is close to 1, its numerator is close in value to its denominator. For example, $\frac{24}{25}, \frac{26}{25}, \frac{12}{13}, \frac{14}{13}$.
24	Give a denominator for $\frac{5}{\square}$ so that the fraction is close to i) 1 ii) 0 iii) $\frac{1}{2}$	Possible answers: i) 4 or 5 or 6 ii) 100 or 200 or 99 iii) 9 or 10 or 11
24	Give a numerator for $\frac{\square}{7}$ so that the fraction is close to i) 1 ii) 0 iii) $\frac{1}{2}$	Possible answers: i) 6 or 7 or 8 ii) 0 or 1 iii) 3 or 4

Solutions & Notes

Page	Problem	Discussion
26	Write three fractions between $\frac{1}{4}$ and $\frac{1}{3}$.	The simplest way to determine these fractions is to write $\frac{1}{4}$ and $\frac{1}{3}$ with common denominators. However, $\frac{3}{12}$ and $\frac{4}{12}$ have no obvious fractions between them. With a different denominator, say 48, we have $\frac{1}{4}=\frac{12}{48}$ and $\frac{1}{3}=\frac{16}{48}$. Three fractions between them are $\frac{13}{48}$, $\frac{14}{48}$ and $\frac{15}{48}$.
28	Sandpipers travel about 1920 km one way during migration. If they travel 480 km per day, how long does the trip take?	Method 1: Round the numbers to 2000 ÷ 500 giving about 4 days. Method 2: 1920 ÷ 480 = 192 ÷ 48 = 48 ÷ 12 = 4.
30	How many digits will there be in each answer? i) 10 298 – 2938 ii) 1244 + 9873	i) This is about 10 000 – 3000 = 7000. Hence there will be 4 digits in this answer. ii) Either round as above or use leading digits. This latter gives 1000 + 9000 which will obviously give an answer of 5 digits.
31	How many digits will there be in each answer? i) 32 456 ÷ 81 (if the answer is a whole number) ii) 6347 x 287	i) This is about 32 000 ÷ 80 = 3200 ÷ 8 = 400, a 3-digit answer. ii) This is about 6000 x 300 = 1 800 000, a 7-digit answer.

Activity 1: Using Numbers

BLM 1: Placing Numbers



If students have difficulty placing the numbers, suggest that they first list all the numbers in order of size, beginning with A:

A (6000), E (10 000), G (21 000), H (26 000), I (120 000), C (190 000), F (236 000), D (662 000), B (900 000).

Solutions & Notes

BLM 2: Numbers In Place

The numbers, listed in order, are

(4) 80, (9) 3000, (8) 4000, (6) 4635, (7) 2.5 million, (5) $7\frac{1}{2}$ million, (2) 161 281 000, (1) 1 billion, (3) 79 billion.

BLM 3: Being Reasonable

Students may not agree with the answers given below. If they can justify their answers, with good reasoning, their answers should be accepted. For example, whether 12 is reasonable or not depends on one's definition of "minivan".

Reasonable statements are 2, 4, 5, 8, 13, and 14. Considering how rapidly technology is changing, answers to 9 and 11 will probably change every few months (if not weeks). Statement 7 may become reasonable eventually.

BLM 4: Scaling Large Objects

1. 200 cm or 2 m
2. 210 cm or 2.1 m
3. 400 000 cm or 4000 m or 4 km
4. 14 000 000 000 cm or 140 000 000 m or 140 000 km
5. 60 cm
6. 18 cm

BLM 5: Scaling Small Objects

1. 0.005 mm
2. 0.006 mm
3. 0.5 mm
4. 1.5 cm
5. 0.3 cm
6. 0.5 cm

BLM 6: Errors, Errors, Errors!

1. If you buy your ticket early, you save \$1.00, which is $\frac{1}{8}$ or $12\frac{1}{2}$ % of the ticket price of \$8.00. This can be corrected either by changing 25% to $12\frac{1}{2}$ % or by changing \$7.00 to \$6.00.
2. To correct this, change the rate of heart beat to "about 72 times a minute".
3. The price given, ".99¢", is $\frac{99}{100}$ of a cent, less than 1¢. Delete the decimal point and give the price as "99¢".
4. The savings here are \$2.89 – \$1.58 or \$1.31, not just 31¢. This can be corrected by changing any one of the numbers to make the arithmetic correct.
5. The difference between \$3.50 and \$1.95 is \$1.55, not \$2.55. This can be corrected by changing any one of the numbers to make the arithmetic correct.
6. The new price could be 3/\$1.00 or 3/\$2.00 to be reasonable.
7. The sales price given is greater than the regular price.
8. Sam obviously worked more hours in the second week. This can be corrected by switching the numbers or by changing the wording in the question.

Solutions & Notes

9. Pascal lived in the 17th. century. The dates 1623 and 1662 are correct.

10. Buying the recorder at $39.95 rather than $49.95 saves $10 not $100. The value of the adapter is extraneous data, not needed in the problem.

11. If the circumference of the base is 22 ft., the diameter is about 7 ft. The diameter of the body is about 4 ft. If the diameter of the head is 8 ft., the snowman would be top heavy. Also, the diameters of the three parts have a total of 19 ft., more than the 12 foot height given. This can be corrected by saying the circumference of the head is 8 ft. around.

12. October has 31 days, not 30.

Activity 2: Comparing and Ordering

RANGE POINTS
This table displays all possible products of the numbers given on page 13.

	2	5	6	7	8	15	19	24	32
2	4	/	/	/	/	/	/	/	/
5	10	25	/	/	/	/	/	/	/
6	12	30	36	/	/	/	/	/	/
7	14	35	42	49	/	/	/	/	/
8	16	40	48	56	64	/	/	/	/
15	30	75	90	105	120	225	/	/	/
19	38	95	114	133	152	285	361	/	/
24	48	120	144	168	192	360	456	576	/
32	64	160	192	224	256	480	608	768	1024

From the table above, it can be seen that there are 16 possible combinations giving a product in the 0 – 49 range, 6 possible combinations giving a product in the 50 – 99 range, 15 possible combinations giving a product in the 100 – 299 range, and 8 possible combinations giving a product in the 300 or more range.

Note that these numbers include products such as 2 × 2 or 24 × 24 in which the same number is selected twice in one turn.

BLM 7: The Range Game

1. a) $47 \times \underline{8} = \underline{376}$ b) $94 \times \underline{7} = \underline{658}$ c) $29 \times \underline{23} = \underline{667}$ d) $53 \times \underline{5} = \underline{265}$

2. a) 3 b) 8, 7 c) 21, 22, 21 d) 0, 1, 0 e) 53, 54, 53

3. Many answers are possible. The answer to which is closer will depend on the range chosen.Samples are given here.
   a) 9.90, 9.93, 9.90 or 9.89, 9.92, 9.92 b) 5.450, 5.5, 5.450 c) 3.00, 3.02, 3.02
   d) 8.40, 8.491, 8.491 e) 7.00, 7.03, 7.00

Solutions & Notes

4. a) 1, $\frac{1}{2}$ b) 0, 0 c) 0, $\frac{1}{2}$, $\frac{1}{2}$ d) 0, $\frac{1}{2}$, $\frac{1}{2}$ e) $\frac{1}{2}$, 1, $\frac{1}{2}$

BLM 8: Range Points

Game 1:

	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{4}{5}$	$1\frac{1}{5}$	$1\frac{1}{4}$	$\frac{4}{3}$	$1\frac{1}{2}$	$1\frac{2}{3}$	$\frac{4}{2}$	$\frac{5}{2}$	$2\frac{3}{4}$
$\frac{1}{4}$	$\frac{1}{2}$											
$\frac{1}{2}$	$\frac{3}{4}$	1										
$\frac{2}{3}$	$\frac{11}{12}$	$1\frac{1}{6}$	$1\frac{1}{3}$									
$\frac{4}{5}$	$1\frac{1}{20}$	$1\frac{3}{10}$	$1\frac{7}{15}$	$1\frac{3}{5}$								
$1\frac{1}{5}$	$1\frac{9}{20}$	$1\frac{7}{10}$	$1\frac{13}{15}$	2	$2\frac{2}{5}$							
$1\frac{1}{4}$	$1\frac{1}{2}$	$1\frac{3}{4}$	$1\frac{11}{12}$	$2\frac{1}{20}$	$2\frac{9}{20}$	$2\frac{1}{2}$						
$\frac{4}{3}$	$1\frac{7}{12}$	$1\frac{10}{12}$	2	$2\frac{2}{15}$	$2\frac{8}{15}$	$2\frac{7}{12}$	$2\frac{2}{3}$					
$1\frac{1}{2}$	$1\frac{3}{4}$	2	$2\frac{1}{6}$	$2\frac{3}{10}$	$2\frac{7}{10}$	$2\frac{3}{4}$	$2\frac{5}{6}$	3				
$1\frac{2}{3}$	$1\frac{11}{12}$	$2\frac{1}{6}$	$2\frac{1}{3}$	$2\frac{7}{15}$	$2\frac{13}{15}$	$2\frac{11}{12}$	3	$3\frac{1}{6}$	$3\frac{1}{3}$			
$\frac{4}{2}$	$2\frac{1}{4}$	$2\frac{3}{4}$	$2\frac{2}{3}$	$2\frac{4}{5}$	$3\frac{1}{5}$	$3\frac{1}{4}$	$3\frac{1}{3}$	$3\frac{1}{2}$	$3\frac{2}{3}$	4		
$\frac{5}{2}$	$2\frac{3}{4}$	3	$3\frac{1}{6}$	$3\frac{3}{10}$	$3\frac{7}{10}$	$3\frac{3}{4}$	$3\frac{5}{6}$	4	$4\frac{1}{6}$	$4\frac{1}{2}$	5	
$2\frac{3}{4}$	3	$3\frac{1}{4}$	$3\frac{5}{12}$	$3\frac{11}{20}$	$3\frac{19}{20}$	4	$4\frac{1}{12}$	$4\frac{1}{4}$	$4\frac{5}{12}$	$4\frac{3}{4}$	$5\frac{1}{4}$	$5\frac{1}{2}$

There are 11 whole number answers.

There are 3 combinations giving a fraction sum in the 0 to 1 range, 16 combinations giving a fraction sum in the 1 to 2 range, 23 combinations giving a fraction sum in the 2 to 3 range, 16 combinations giving a fraction sum in the 3 to 4 range, 6 combinations giving a fraction sum in the 4 to 5 range, and 2 combinations giving a sum greater than 5. There are 12 sums that are whole numbers.

Solutions & Notes

Game 2:

	0.33	0.4	1.6	2.03	3.0	4.7	5.4	6.01	8.1	9.1
0.33	0.11									
0.4	0.13	0.16								
1.6	0.53	0.64	2.56							
2.03	0.67	0.81	3.25	4.12						
3.0	0.99	1.20	4.80	6.09	9.00					
4.7	1.55	1.96	7.52	9.54	14.1	22.09				
5.4	1.78	2.16	8.64	10.96	16.2	25.38	29.16			
6.01	1.98	2.40	9.62	12.20	18.03	28.25	32.94	36.12		
8.1	2.67	3.24	12.96	16.44	24.30	38.07	43.74	48.68	65.61	
9.1	3.00	3.64	14.56	18.47	27.30	42.77	49.14	54.69	73.71	82.81

Answers are given to the nearest hundredth.

There are 8 combinations giving a product in the 0 to 1 range, 15 combinations giving a product in the 1 to 5 range, 6 combinations giving a product in the 5 to 10 range, 22 combinations giving a product in the 10 to 50 range and 4 combinations giving a product over 50.

BLM 9: Fraction Tiles

1. Two solutions are given. There are others, but the zero can be in only one box.

$\frac{1}{\boxed{9}} < \frac{1}{\boxed{2}}$ $\quad \frac{2}{\boxed{7}} < \frac{2}{\boxed{3}}$

$\frac{2}{\boxed{4}} < \frac{\boxed{6}}{8}$ $\quad \frac{3}{\boxed{5}} < \frac{\boxed{8}}{5}$

$\frac{\boxed{0}}{1} < \boxed{1}$

$\frac{1}{\boxed{4}} < \frac{1}{\boxed{1}}$ $\quad \frac{2}{\boxed{6}} < \frac{2}{\boxed{5}}$

$\frac{2}{\boxed{2}} < \frac{\boxed{9}}{8}$ $\quad \frac{3}{\boxed{3}} < \frac{\boxed{8}}{5}$

$\frac{\boxed{0}}{1} < \boxed{7}$

Solutions & Notes

2. Two solutions are given. There are others.

$$\frac{\boxed{0}}{\boxed{1}} \quad \frac{\boxed{2}}{\boxed{3}} \quad \frac{\boxed{4}}{\boxed{5}} \quad \frac{\boxed{6}}{\boxed{7}} \quad \frac{\boxed{8}}{\boxed{9}} \qquad\Big|\qquad \frac{0}{\boxed{9}} \quad \frac{\boxed{1}}{\boxed{8}} \quad \frac{\boxed{2}}{\boxed{7}} \quad \frac{\boxed{3}}{\boxed{6}} \quad \frac{\boxed{4}}{\boxed{5}}$$

BLM 10: Truths and Half Truths

1. If the sum of four two-digit numbers is less than 80, then
   a) all four numbers are ALWAYS less than 100
   b) at least one number is ALWAYS less than 50
   c) if two numbers are less than 40, the other two must NEVER be greater than 40
   d) the sum of four numbers is SOMETIMES an even number

Rationale:
   a) If even one number is not less than 100, the total will be more than 80.
   b) If even two numbers are greater than or equal to 50, the total will be more than 80.
   c) If two of the numbers are greater than 40, the total will be more than 80.
   d) $5+9+11+15=40$ (even)
   $4+9+11+15=39$ (odd)

2. If the sum of four fractions is less than one, then
   a) all four fractions are SOMETIMES less than $\frac{1}{2}$

   e.g., $\frac{1}{2}+\frac{1}{10}+\frac{1}{10}+\frac{1}{10}<1$; three fractions are less than $\frac{1}{2}$

   $\frac{1}{3}+\frac{1}{10}+\frac{2}{10}+\frac{3}{10}<1$; four fractions are less than $\frac{1}{2}$

   b) at least one fraction is ALWAYS less than $\frac{1}{2}$

   If even two fractions are greater than or equal to $\frac{1}{2}$, the total will be more than or equal to 1. In fact, at least three of the fractions must be less than $\frac{1}{2}$.

   c) at least one fraction is SOMETIMES greater than $\frac{1}{2}$.

   e.g., $\frac{2}{3}+\frac{1}{9}+\frac{1}{10}+\frac{1}{10}<1$; one fraction is greater than $\frac{1}{2}$.

   $\frac{1}{3}+\frac{1}{9}+\frac{1}{10}+\frac{1}{10}<1$; all four fractions are less than $\frac{1}{2}$.

   d) at least one of the fractions is NEVER an improper fraction, since an improper fraction is greater than 1.

3. If the product of a fraction and a whole number is a whole number, then
   a) the product is SOMETIMES greater than the number you started with.

   e.g., $12\times\frac{1}{2}=6<12$; $12\times\frac{3}{2}=18>12$

Solutions & Notes

b) the whole number you started with is ALWAYS a multiple of the fraction's denominator

e.g., $8\times\frac{1}{4}=\frac{8}{4}=2$; $2\times\frac{1}{2}=1$

c) the whole number you started with is SOMETIMES a multiple of the fraction's numerator

e.g., $12\times\frac{3}{4}=9$; $12\times\frac{5}{6}=10$

d) the fraction is SOMETIMES less than 1

e.g., $12\times\frac{1}{4}=3$; $12\times\frac{5}{4}=15$

4. a) SOMETIMES (The grass could be wet with dew, or from a water sprinkler.)
   b) SOMETIMES (Some plows plough dirt or gravel. The plow may be simply returning to the garage.)
   c) SOMETIMES (Jim may have done his homework regularly, and not needed to study.)
   d) SOMETIMES (This could provoke interesting discussions.)

BLM 11: Relative Sizes

1. a) $334+976+533<408+908+656$
   b) $56+57+58=3\times 57$
   c) $760\div 8>760\div 9$
   d) $512\times 1.1>512$
   e) $674\times 0.9<674$
   f) $5.01+5.2+5.03+5.04>4\times 5.02$
   g) $88\times 1.001>88$
   h) $78\times 1\frac{1}{2}>100$
   i) $6\frac{1}{4}\times\frac{4}{25}=1$
   j) $83\times 0.56>83\times\frac{1}{2}$

2. a) true; see e) above
   b) true; see d) above
   c) true; see c) above
   d) false: the sum of three consecutive numbers is equal to 3 times the middle number. See b) above.
   e) true: see i) above.

Activity 3: Number Properties

BLM 13: Name That Number

a) $\times\ 1$ b) $+\ 0$ c) $-\ 0$ d) $\div\ 1$

e) any fraction less than 1

f) any number greater than 1

g) any number greater than 1

h) any fraction less than 1

i) an even number

j) even if your number is even; odd if your number is odd

k) $\times\ 0$

l) answers will vary

BLM 14: Should Remainders Remain?

1. $(150+7)\div 35=4R17$

(ii) round to the next whole number: 5 buses

Solutions & Notes

2. a) $39 \div 6 = 6R3$. David will fill 6 pages: (i) ignore the remainder
   b) (ii) round to the next whole number: David will need 7 pages.

3. $7 \div 2 = 3R1$ (i) Discard the remainder. Ms. Sheff can bake 3 batches of cookies.

4. $36 \div 8 = 4R4$ (i) Discard the remainder. 4 squares can be formed.

5. $17 \div 3 = 5R2$ (i) Discard the remainder. Each person will have 5 candies. OR
   (ii) Use the remainder. Two people will have 6 candies and 1 will have 5.

6. $2 \times 8 \div 6 = 2R4$ (i) Discard the remainder. Each person will have 2 slices of pizza. OR
   (ii) Use the remainder: four people will have 3 slices, and two will have 2.

7. 800¢ ÷ 150¢ = $5R50$ (i) Discard the remainder. Stefan could send 5 copies.

BLM 15: 'Four' Patterns

Students should draw the following conclusions:
- Even numbers (i.e., those ending in 0, 2, 4, 6, or 8) are divisible by 2.
- Numbers ending in 0 or 5 are divisible by 5.
- Numbers in which the last two digits are divisible by 4 are divisible by 4, e.g., 5<u>24</u>, 17<u>20</u>, 7<u>96,</u> are all divisible by 4. Numbers ending in 00 are also multiples of 4.

BLM 16: Division Patterns

Students should draw the following conclusions:
- If the digit sum is divisible by 3, the number is divisible by 3.
- If the digit sum is 9, the number is divisible by 9.

BLM 17: Find the Path

1. Since $4 \times 6 = 24$, the correct path is $4 \times 2 \times 3 \rightarrow 24$.

2. Since the end number ends in zero, both 5 and some even number must be used as factors. $3 \times 4 \times 5 \rightarrow 60$

3. This path also must have 5 and an even number as factors. The digit sum of 1440 is 9, so one of the factors is 9 or two of the factors are 3×3. $6 \times 4 \times 5 \times 12 \rightarrow 1440$

4. Since 396 is divisible by 11(see Extension 3), and $396 \div 11 = 36.$ the correct path is $6 \times 2 \times 3 \times 11 \rightarrow 396$.

5. $420 \div 7 = 60$ so the missing factors must have a product of 60. One of the factors is 5, so the others must have a product of 12. $7 \times 3 \times 4 \times 5 \times 1 \rightarrow 420$

6. Since 462 is divisible by 11, and $462 \div 11 = 42$ and $42 \div 7 = 6$, the missing factors must have a product of 6. $7 \times 2 \times 3 \times 11 \rightarrow 462$

Solutions & Notes

Activity 4: Fractions

BLM 19: The Problem with Fractions

Answers will vary. Samples are given.

1. $\frac{2}{3}$; $\frac{3}{4}$; $\frac{4}{7}$

2. $\frac{3}{10}$; $\frac{4}{10}$

3. $\frac{3}{11}$; $\frac{5}{12}$; $\frac{4}{13}$

4. $\frac{1}{2}+\frac{3}{4}$; $\frac{3}{4}+\frac{2}{3}$

5. $\frac{1}{4}+\frac{1}{4}$; $\frac{1}{3}+\frac{1}{4}$

6. any fraction greater than $\frac{1}{2}$

7. $\frac{2}{5}+\frac{2}{7}$; $\frac{2}{9}+\frac{3}{9}$

8. any fraction less than $\frac{1}{4}$

9. $\frac{1}{2}+\frac{2}{4}$; $\frac{1}{3}+\frac{4}{6}$

10. $\frac{10}{36}$ and $\frac{11}{36}$

11. $\frac{5}{2}-\frac{3}{2}$; $\frac{9}{8}-\frac{1}{8}$

12. $\frac{5}{4}-\frac{4}{4}$; $\frac{4}{3}-\frac{2}{3}$

13. $\frac{7}{8}-\frac{1}{8}$; $\frac{3}{2}-\frac{2}{2}$

14. $\frac{1}{9}+\frac{1}{3}$; $\frac{1}{4}+\frac{1}{8}$

15. $\frac{2}{3}\times\frac{3}{5}$; $\frac{5}{8}\times\frac{3}{5}$

16. $\frac{9}{7}\times\frac{6}{7}=\frac{54}{49}>1$
    $\frac{9}{7}-\frac{6}{7}=\frac{3}{7}<1$

17. $\frac{2}{3}\times\frac{3}{2}$; $\frac{4}{7}\times\frac{7}{4}$

18. $\frac{5}{4}\times\frac{6}{5}=1\frac{1}{2}$
    $\frac{5}{4}+\frac{6}{5}=2\frac{9}{20}$

19 $\frac{3}{4}-\frac{2}{4}=\frac{1}{4}$ and $\frac{3}{4}+\frac{2}{4}=\frac{5}{4}=1\frac{1}{4}$
   $\frac{7}{8}-\frac{6}{8}=\frac{1}{8}$ and $\frac{7}{8}+\frac{6}{8}=\frac{13}{8}=1\frac{5}{8}$

20. Any fraction less than $\frac{1}{2}$.

21. $\frac{1}{2}\times\frac{3}{2}=\frac{3}{4}$
    $\frac{3}{4}\times\frac{4}{5}=\frac{3}{5}$

BLM 20: Is It Ever True?

1. a) ALWAYS $2\frac{1}{3}=\frac{7}{3}$; $5\frac{2}{4}=\frac{22}{4}$

   b) NEVER

   c) NEVER

   d) SOMETIMES $\frac{3}{4}+\frac{1}{2}=1\frac{1}{4}$; $\frac{1}{4}+\frac{1}{2}=\frac{3}{4}$

   e) SOMETIMES see examples just above

   f) SOMETIMES If the fraction is greater than 1, the product will be greater than the starting number. If the fraction is less than 1, the product will be less than the starting number.

   g) See f)

   h) SOMETIMES If the fractions are reciprocals, this is true: $\frac{4}{5}\times\frac{5}{4}=1$

Solutions & Notes

i) SOMETIMES
j) SOMETIMES
k) SOMETIMES
l) SOMETIMES

} Students should provide an example showing the statement to be true, and a second example showing the statement to be false.

BLM 21: Number Strip Fractions

Answers will vary.

Fractions to the right of a given fraction are greater when the top strip is moved to the right (as in problems 2, 3, 4), and less when it is moved to the left (as in problem 5).

Activity 5: Estimation

BLM 24: Over and Under

1.

	Question	Rounded Numbers	Estimated Answer	Over or Under?
Example →	78×90	80×90	7200	over
a)	107×84	100×80	8000	under
b)	7×19	10×20	200	over
c)	895×54	900×50	45000	under
d)	102×28	100×30	3000	over
e)	58×113	60×100	6000	under
f)	\$98.15 × 29	\$100 × 30	\$3000	over
g)	18 × \$3.99	20 × \$4	\$80	over

2. Example: One factor is rounded up, and the other one not changed, producing an over-estimate.

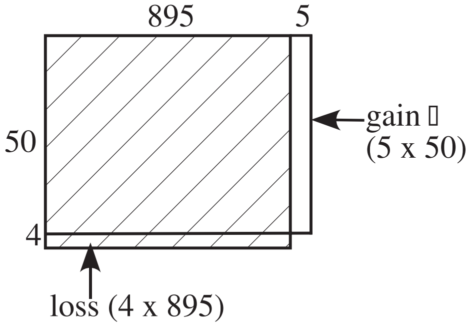
a) Both factors are rounded down, producing an under-estimate.
b) Both factors are rounded up, producing an over-estimate.
c), d), and e) One factor is rounded up and one is rounded down.

c)		d)		e)	
895	× 54	102	× 28	58	× 113
900	50	100	30	60	100
round up	round down	round down	round up	round up	round down
gain: 5×50	loss: 4×900	loss: 2×30	gain: 2×100	gain: 2×100	loss: 13×60
= 250	= 3600	= 60	= 200	= 200	= 780
Gain	< Loss	Loss	< Gain	Gain	< Loss
This gives an under-estimate		This gives an over-estimate		This gives an under-estimate	

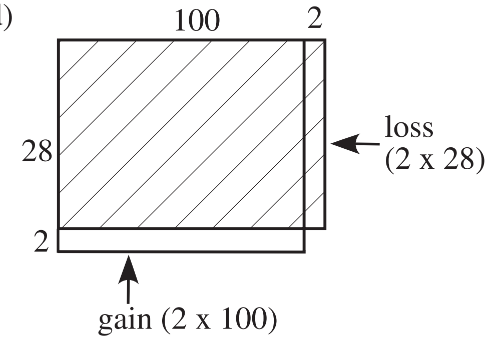
Solutions & Notes

The corresponding diagrams below for c), d), and e) show that the actual loss in each case is slightly less than that used in the boxes above to determine whether the estimate is an over- or under-estimate. This is done in order to use mental arithmetic in this determination.

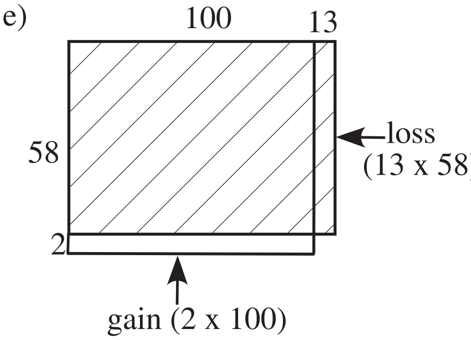
c)



d)



e)



The shaded part represents the initial multiplication - i.e., for c) 895×54 for d) 102×28 for e) 58×113

f) Both factors are rounded up, producing an over-estimate.

g) Both factors are rounded up, producing an over-estimate.

BLM 25: Decoding with Estimates

An extra column has been added to the tables below to show how the estimate was arrived at.

1.

	Question		Estimate	Under	Over
a)	5×89	5×90	450	T	(S)
b)	8×715	8×700	5600	(L)	F
c)	73×21	70×20	1400	(B)	C
d)	8×273	8×300	2400	I	(P)
e)	73×912	70×900	63 000	(E)	O
f)	178×38	200×40	8000	U	(A)
g)	52×595	50×600	30 000	(Y)	D
h)	625×18	600×20	12 000	(Z)	R

Riddle: What catches flies?

B	A	S	E	B	A	L	L
c	f	a	e	c	f	b	b

P	L	A	Y	E	R
d	b	f	g	e	h

Gain: $5 \times 50 = 250$
Loss: $2 \times 600 = 1200$

Gain: $2 \times 600 = 1200$
Loss: $25 \times 20 = 500$

Solutions & Notes

2.

	Question		Estimate	Under	Over	Riddle: What did the mayonnaise say to the salad? “Go away, I’M e a D R E S S I N G h g b c c e d f”
a)	$623 \div 7$	$630 \div 7$	90	B	**M**	
b)	$821 \div 9$	$810 \div 9$	90	**E**	F	
c)	$456 \div 6$	$480 \div 6$	80	P	**S**	
d)	$243 \div 3$	$240 \div 3$	80	**N**	O	
e)	$710 \div 63$	$720 \div 60$	12	L	**I**	Dividend rounded up → gain Divisor rounded down → gain
f)	$397 \div 42$	$400 \div 40$	10	H	**G**	Dividend rounded up → gain Divisor rounded down → gain
g)	$1227 \div 29$	$1200 \div 30$	40	**R**	Q	Dividend rounded down → loss Divisor rounded up → loss
h)	$3716 \div 38$	$3600 \div 40$	90	**D**	A	Dividend rounded down → loss Divisor rounded up → loss

BLM 26: Comparing Estimates

1. Possible estimates are given for each question. Compatible numbers are used in many cases.

a)	29×41 $30 \times 40 = 1200$	**52×48** $50 \times 50 = 2500$	19×89 $20 \times 90 = 1800$
b)	**26×19** $25 \times 20 = 500$	400	21×21 $20 \times 20 = 400$
c)	**$489 + 612$** $500 + 600 = 1100$	31×35 $30 \times 30 = 900$	23×42 $25 \times 40 = 1000$
d)	$900 - 496$ $900 - 500 = 400$	**$615 - 101$** $600 - 100 = 500$	$891 - 525$ $900 - 500 = 400$
e)	$243 \div 8$ $240 \div 8 = 30$	$430 \div 6$ $420 \div 6 = 70$	**$323 \div 4$** $320 \div 4 = 80$
f)	**89×41** $90 \times 40 = 3600$	121×12 $120 \times 10 = 1200$	65×21 $60 \times 20 = 1200$
g)	27×48 $30 \times 50 = 1500$	**672×8** $700 \times 8 = 5600$	136×15 $150 \times 5 \times 3 = 2250$
h)	**$12 \times 8 \times 5$** $12 \times 40 = 480$	$15 \times 4 \times 7$ $60 \times 7 = 420$	$9 \times 5 \times 6$ $9 \times 30 = 270$

Part h) uses grouping of factors, rather than rounding, to make mental arithmetic easy without the need for estimating.

BLM 27: That Can’t Be Right!

Answers will vary. The following are possibilities.

1. Two pair will cost about $60, not $50.

Solutions & Notes

2. A 10% tip for an approximate \$10 bill is \$1, 5% will be 50¢, so 15% is closer to \$1.50 than \$2.00. However, students may feel that an over estimate in this case is appropriate, and accept the \$2.00 estimate as valid.

3. 300 km is a good estimate of the distance, but instead of multiplying this by 10, the driver should divide by 10, giving 30 L necessary.

4. The estimate of area of the bedroom is low. Estimating 3.6 m by $3\frac{1}{2}$ m would give $3\times 3+3\times\frac{1}{2}$ or $10\frac{1}{2}$ m^2, which is still an under-estimate but closer to the acutal value of 10.8 m^2.

5. Buying 2 boxes at about \$8.00 each and getting 1 free means getting the necessary 3 boxes for about \$16.00 if taxes are ignored. Even with taxes, the total cose will be less that \$25.00.

6. A saving of \$9.96 is a saving of 71.4%, not 714%.

BLM 28: When Is An Estimate Not An Estimate?

Answers will vary. Most students should agree that estimates are appropriate for 2, 7, and 10, and that precise values are needed for 1, 3, and 8. The situations in 4, 5, 6, and 9 can be argued either way. If a student can make a good argument for his/her choice, that choice should be accepted.

Suggested Assessment Strategies

Investigations
Investigations involve explorations of mathematical questions that may be related to other subject areas. Investigations deal with problem posing as well as problem solving. Investigations give information about a student's ability to:
- identify and define a problem;
- make a plan;
- create and interpret strategies;
- collect and record needed information;
- organize information and look for patterns;
- persist, looking for more information if needed;
- discuss, review, revise, and explain results.

Journals
A journal is a personal, written expression of thoughts. Students express ideas and feelings, ask questions, draw diagrams and graphs, explain processes used in solving problems, report on investigations, and respond to open-ended questions. When students record their ideas in math journals, they often:
- formulate, organize, internalize, and evaluate concepts about mathematics;
- clarify their thinking about mathematical concepts, processes, or questions;
- identify their own strengths, weaknesses, and interests in mathematics;
- reflect on new learning about mathematics;
- use the language of mathematics to describe their learning.

Observations
Research has consistently shown that the most reliable method of evaluation is the ongoing, in-class observation of students by teachers. Students should be observed as they work individually and in groups. Systematic, ongoing observation gives information about students':
- attitudes towards mathematics;
- feelings about themselves as learners of mathematics;
- specific areas of strength and weakness;
- preferred learning styles;
- areas of interest;
- work habits — individual and collaborative;
- social development;
- development of mathematics language and concepts.

In order to ensure that the observations are focused and systematic, a teacher may use checklists, a set of questions, and/or a journal as a guide. Teachers should develop a realistic plan for observing students. Such a plan might include opportunities to:
- observe a small number of students each day;
- focus on one or two aspects of development at a time.

Student Self-Assessment
Student self-assessment promotes the development of metacognitive ability (the ability to reflect critically on one's own reasoning). It also assists students to take ownership of their learning, and become independent thinkers. Self-assessment can be done following a co-operative activity or project using a questionnaire which asks how well the group worked together. Students can evaluate comments about their work samples or daily journal writing. Teachers can use student self-assessments to determine whether:
- there is change and growth in the student's attitudes, mathematics understanding, and achievement;
- a student's beliefs about his or her performance correspond to his/her actual performance;
- the student and the teacher have similar expectations and criteria for evaluation.

Suggested Assessment Strategies

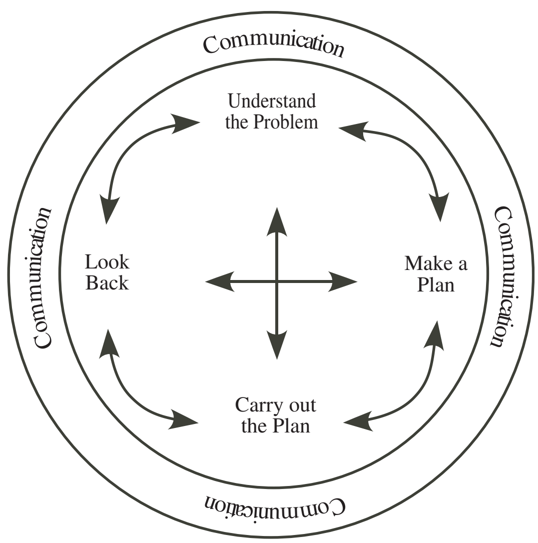
A GENERAL PROBLEM SOLVING RUBRIC

This problem solving rubric uses ideas taken from several sources. The relevant documents are listed at the end of this section.

“US and the 3 R’s”

There are five criteria by which each response is judged:
Understanding of the problem,
Strategies chosen and used,
Reasoning during the process of solving the problem,
Reflection or looking back at both the solution and the solving, and
Relevance whereby the student shows how the problem may be applied to other problems, whether in mathematics, other subjects, or outside school.

Although these criteria can be described as if they were isolated from each other, in fact there are many overlaps. Just as communication skills of one sort or another occur during every step of problem solving, so also reflection does not occur only after the problem is solved, but at several points during the solution. Similarly, reasoning occurs from the selection and application of strategies through to the analysis of the final solution. We have tried to construct the chart to indicate some overlap of the various criteria (shaded areas), but, in fact, a great deal more overlap occurs than can be shown. The circular diagram that follows (from OAJE/OAME/OMCA “Linking Assessment and Instruction in Mathematics”, page 4) should be kept in mind at all times.



There are four levels of response considered:

Level 1: Limited identifies students who are in need of much assistance;
Level 2: Acceptable identifies students who are beginning to understand what is meant by ‘problem solving’, and who are learning to think about their own thinking but frequently need reminders or hints during the process.
Level 3: Capable students may occasionally need assistance, but show more confidence and can work well alone or in a group.
Level 4: Proficient students exhibit or exceed all the positive attributes of the **Capable** student; these are the students who work independently and may pose other problems similar to the one given, and solve or attempt to solve these others.

Suggested Assessment Strategies

LEVEL OF RESPONSE →

CRITERIA FOR ASSESSMENT ↓	Level 1: Limited	Level 2: Acceptable	Level 3: Capable	Level 4: Proficient
UNDERSTANDING	• requires teacher assistance to interpret the problem • fails to recognize all essential elements of the task	• shows partial understanding of the problem but may need assistance in clarifying	• shows a complete understanding of the problem	• shows a complete understanding of the problem
STRATEGIES	• needs assistance to choose an appropriate strategy	• identifies an appropriate strategy	• identifies an appropriate strategy	• identifies more than one appropriate strategy
STRATEGIES	• applies strategies randomly or incorrectly • does not show clear understanding of a strategy[1] • shows no evidence of attempting other strategies	• attempts an appropriate strategy, but may not complete it correctly[2] • tries alternate strateges with prompting	• uses strategies effectively • may attempt an inappropriate strategy, but eventually discards it and tries another without prompting	• chooses and uses strategies effectively[3] • recognizes an inappropriate strategy quickly and attempts others without prompting
REASONING	• makes major mathematical errors • uses faulty reasoning and draws incorrect conclusions • may not complete a solution	• may present a solution that is partially incorrect	• produces a correct and complete solution, possibly with minor errors	• produces a correct and complete solution, and may offer alternative methods of solution
REASONING	• describes[4] reasoning in a disorganized fashion, even with assistance • has difficulty justifying[5] reasoning even with assisstance	• partially describes[4] a solution and/or reasoning or explains fully with assistance • justification[5] of solution may be inaccurate, incomplete or incorrect	• is able to describe[4] clearly the steps in reasoning; may need assistance with mathematical language • can justify[5] reasoning if asked; may need assistance with language	• explains reasoning in clear and coherent mathematical language • justifies[5] reasoning using appropriate mathematical language
REFLECTION	• shows no evidence of reflection or checking of work • can judge the reasonableness of a solution only with assistance	• shows little evidence of reflection or checking of work • is able to decide whether or not a result is reasonable when prompted to do so	• shows some evidence of reflection and checking of work • indicates whether the result is reasonable, but not necessarily why	• shows ample evidence of reflection and thorough checking of work • tells whether or not a result is reasonable, and why
RELEVANCE	• unable to identify similar[6] problems	• unable to identify similar[6] problems	• identifies similar[6] problems with prompting	• identifies similar[6] problems, and may even do so before solving the problem
RELEVANCE	• unlikely to identify extensions[7] or applications of the mathematical ideas in the given problem, even with assistance	• recognizes extensions[7] or applications with prompting	• can suggest at least one extension[7], variation, or application of the given problem if asked	• suggests extensions[7], variation, or applications of the given problem independently

Suggested Assessment Strategies

Notes on the Rubric

1. For example, diagrams, if used, tend to be inaccurate and/or incorrectly used.

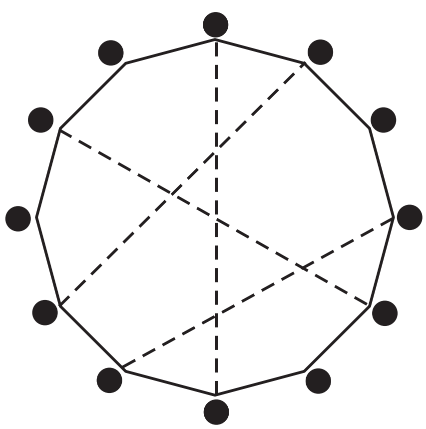
2. For example, diagrams or tables may be produced but not used in the solution.

3. For example, diagrams, if used, will be accurate models of the problem.

4. To *describe* a solution is to tell *what* was done.

5. To *justify* a solution is to tell *why* certain things were done.

6. *Similar* problems are those that have similar structures mathematically, and hence could be solved using the same techniques.
   For example, of the three problems shown below right, the better problem solver will recognize the similarity in structure between Problems 1 and 3. One way to illustrate this is to show how both of these could be modelled with the same diagram:



Problem 1: There were 8 people at a party. If each person shook hands once with each other person, how many handshakes would there be? How many handshakes would there be with 12 people? With 50?

Problem 2: Luis invited 8 people to his party. He wanted to have 3 cookies for each person present. How many cookies did he need?

Problem 3: How many diagonals does a 12-sided polygon have?

   Each dot represents one of 12 people and each dotted line represents either a handshake between two people (Problem 1, second question) or a diagonal (Problem 3).

   The weaker problem solver is likely to suggest that Problems 1 and 2 are similar since both discuss parties and mention 8 people. In fact, these problems are alike only in the most superficial sense.

7. One type of extension or variation is a "what if...?" problem, such as "What if the question were reversed?", "What if we had other data?", "What if we were to show the data on a different type of graph?".

Adapting the Rubric

The problem solving in this unit is spread throughout the activities. That is, not all the components of problem solving as outlined in the rubric are present in each lesson. However, there are examples of each to be found in the series of activities presented.

Examples of these criteria are given below with questions based on a part of one of the activities. This allows you to assess the students' problem-solving abilities in different ways at different times during the unit.

You may wish to share this type of assessment with students. The more aware of the nature of problem solving (as "described" by a rubric) they become, the better problem solvers they will become, and the more willing to try to articulate their solutions and reasons for their choices of various strategies and heuristics.

Activity 2, BLM 9

Understanding: Do students realize that all ten number tiles must be used for each problem? Do they understand that tiles may be moved from initial placements?

Strategies and Reasoning: Do students consider the whole problem and look first for number tile positions for which there may be only one or two choices?

Reflection: Can students explain the steps they followed and why each number tile was placed in each position?

For example,

- The "Limited" student may try to use one or more digits more than once and may resist moving a tile once it is placed.
- The "Acceptable" student may try to complete the boxes in order and may need to be reminded that tiles can be moved, but will eventually place most or all of the tiles completely.
- The "Capable" student feels comfortable moving tiles from one position to another.
- The "Proficient" student will need to move the tiles less ofter.

Activity 3, BLM 13

Understanding: Do students realize that some parts of #1 ask for a specific number, while some ask for a general description (e.g., You multiply your number by a fraction less than one to get a lower value.)

For example,

- The "Limited" student may give a specific number for each part of #1.
- The "Acceptable" student attempts to generalize, but may do so through such statements as "If I multiply my number by $\frac{1}{2}$ or $\frac{1}{4}$ or $\frac{1}{3}$ I get a number of lower value."
- The "Proficient" student will be able to generalize the descriptions of numbers.

Suggested Assessment Strategies

Activity 4, BLM 17

Strategies and Reasoning: To what extent does the student develop a workable strategy?

For example,

- The “Limited” student uses trial and error, and may make arithmetic errors.
- The “Acceptable” student uses trial and adjustment, altering his/her path when estimation shows the path is incorrect.
- The “Capable” may use factoring in specific cases (e.g., if the End number ends in 0, one of the factors must be 5, and another must be even.)
- The “Proficient” student discovers the correct path by factoring the End result.

Resources for Assessment

1. The Ontario Curriculum, Grades 1-8: Mathematics.
2. *Linking Assessment and Instruction in Mathematics: Junior Years*, Ontario Association of Mathematics Educators/OMCA/OAJE, Moore et al., 1996.
   The document provides a selection of open-ended problems tested in grades 4, 5, and 6. Performance Rubrics are used to assess student responses (which are included) at four different levels. Problems could be adapted for use at the Junior Level.
3. *Mathematics Assessment: Myths, Models, Good Questions, and Practical Suggestions*, by Jean Kerr Stenmark (Ed.), NCTM, 1991.
   This book contains a variety of assessment techniques and gives samples of student work at different levels.
4. *How to Evaluate Progress in Problem Solving*, by Randall Charles et al., NCTM, 1987.
   Suggestions for holistic scoring of problem solutions include examples of student work. Also given are ways to vary the wording of problems to increase/decrease the challenge. A section on the use of multiple choice test items shows how these, when carefully worded, can be used to assess student work.

Other Resources

1. *"Number Sense and Operations, Addenda Series," Grades K-6,* ed. Grace Burton, 1992, National Council of Teachers of Mathematics (NCTM), Reston, VA (1-800-235-7566)
   The booklet contains suggested lessons for each grade dealing with numbers. Activities deal with multiplication patterns, exploring large numbers through counting blades of grass, and estimating.

2. "Developing Sense About Numbers", by Carole Greenes, Linda Schulman and Rika Spungin, *Arithmetic Teacher*, pp 279-284, January 1993, NCTM.
   This article lists seven number-sense skills, such as "recognizing the various uses of numbers", "estimating results of computations", and "understanding phrases that establish mathematical relationships". Several "Fit the Facts" activities (similar to BLMs 1, 2, and 3 in Grade 4 "Investigations in Number Sense and Estimation") are included.

3. "The Revolution in Arithmetic", by William A. Brownell, *Mathematics Teaching in the Middle School*, pp 27-30, August 2006, NCTM.
   This is a reprint of an article first printed in 1954. It describes changes in content teaching methods, and text books during the first half of the 20th century. It is work considering whether or not 21st century content, teaching methods, and text books have continued to evolve/improve.

4. "How Much is a Million?", by David M. Schwartz and Steven Kellogg, William Morris and Co., New York, 1985.
   This book helps students come to grips with one million (1 000 000), one billion (1 000 000 000), and one trillion (1 000 000 000 000) through answers to such questions as "If a million kids climbed onto one another's shoulders", how tall would they be? The answers may surprise you.

5. "How Big is Bill Gates's Fortune?", by Hamp Sherard, *Mathematics Teaching in the Middle School*, pp 250-252, December 2000, NCTM.
   Students explored large numbers by determining the weight or size of Gates' fortune in $100 dollar bills. One student calculated that the fortune was 62.5 miles taller than Mount Everest if the bills were stacked one on top of the other. Recently (summer of 2007) Gates' fortune was estimated at 100 billion dollars. (A $100 bill is approximately 6.6 cm wide, 15.6 cm long, 1 mm thick and 0.9 g mass).

6. "A Game Involving Fraction Squares", by Enrique Ortez, *Teaching Children Mathematics*, pp 218-222, December 2000, NCTM.
   This artical describes a game similar to the idea on BLM 9 (Fraction Tiles) in this book, but with the stress on equivalent fractions.

7. "Multiplication Games: How We Made and Used Them", by Constance Kamii, and Catherine Anderson, *Teaching Children Mathematics*, pp 135-141, November 2003, NCTM.
   Several easily-made multiplication games are described, from those that use one multiplication table at a time (e.g., x 4) to those that use three or more factors. Using these games throughout the year, from the simplest to the more difficult, students showed considerable improvement in their speed and acuracy.

8. "The Influence of Ancient Egypt on Greek and Other Numeration Systems" by Claudia Zaslavsky, *Mathematics Teaching in the Middle School*, pp 174-178, November 2003, NCTM.

Other Resources

The article describes the Egyptian hieroglphic numbers and how they influenced the Greeks who used letters of the alphabet as numberals. Two Black Line Masters for students are given.

9. "Understanding Aztec and Mayan Numeration Systems" by Angela L. E. Walmsley, *Mathematics Teaching in the Middle School*, pp 55-59, August 2006, NCTM.
   A comparison of the Aztec system (lacking a zero) and the Mayan system (including a zero) to the Hindu-Arabic system brings out many properties of numbers and emphasizes place value. Several examples of all systems are given. A source for more information on the Mayan system is given at "www.ancientscripts.com/aztec.html".

10. "Translating Number Words into the Language of Mathematics", by Marlene Kliman, and Susan Janssen, *Mathematics Teaching in the Middle School*, pp 798-800, May 1996, NCTM.
   Grade 6 students examine number names in different languages looking for patterns. For example, in English, "teen" and "ty" represent "$+10$" and "$\times 10$" respectively, as in sixteen $(6+10)$ or sixty (6×10). In French, "ante" represents $\times 10$ as in cinquante (5×10). Students learn that different languages have different patterns in number games. For example, in French, 80 is not huitante (8×10) but quatre-vingt (4×20). Examining these and other number games (Fufulde from Nigeria and Mayan from Mexico) students come to a greater understanding of place value and an appreciation of other cultures.

11. "Building Percent Dolls: Connecting Linear Measurement to Learning Ratio and Proportion", by Joan Moss and Beverly Caswell, *Mathematics Teaching in the Middle School*, pp 68-74, September 2004, NCTM.
   Grade 5 and 6 students determined what percentage of their height various measurements were (e.g., length of hand, shoulder width, arm span). Students then constructed figures using these proportions, and whatever materials they felt were appropriate (e.g., balloons, wooden dowels, modelling clay).

12. "The Thinking of Students: A Fruitful Crop", by Gladis Kersaint, *Mathematics Teaching in the Middle School*, pp 95-99, September 2004, NCTM.
   This column appears in every issue providing a problem for students in grades 5 - 8. This one also includes responses from students describing how they solved the problem. This issue's problem is

   Of 6000 apples harvested, every third apple was too small (S), every fourth apple was too green (G), and every tenth apple was bruised (B). The remaining apples were perfect (P). How many perfect apples were harvested? [Answer:4800]

13. "Mental Computation in the Middle Grades: The Importance of Thinking Strategies", by Alistair McIntosh, Robert E. Reys, an Barbara J. Reys, *Mathematics Teaching in the Middle School*, pp 322-327, March - April 1997, NCTM.
   The article discusses the nature of thinking strategies, number sense, mental computation, and estimation. Students' opinions and thinking strategies are given. Conclusions include the following:
   - Students success with computation is much higher when students see the problem as opposed to when the problem is read to them.
   - The more confident a student is, the more likely he/she is to develop alternate strategies.
   - Students' perceptions of what is meant by mental computation differ greatly.