

### Problem

#### The Palindromic Two-Step (suitable for groups of four students)

A palindrome is an expression which is the same written forwards as backwards. For example, 2, 44, 101, 8118, ‘race car’ and ‘radar’ are palindromes, while 13, 24, 245, and ‘auto’ are not.



The number 13 is not a palindrome, but if you reverse its digits and then sum the two numbers, you get  $13 + 31 = 44$ , which is a palindrome. Numbers like this are called one-step palindromes.

The number 37 is also not a palindrome, but if you reverse its digits and add, you get  $37 + 73 = 110$ , not a palindrome. But if you repeat this process, reversing the digits of 110 and adding, you get  $110 + 011 = 121$  which is a palindrome. Numbers like this are called two-step palindromes.

The number 68 is a three-step palindrome:

$$\begin{array}{r}
 37 \\
 73 \\
 \hline
 110 \\
 011 \\
 \hline
 121
 \end{array}
 \qquad
 \begin{array}{r}
 68 \\
 86 \\
 \hline
 154 \\
 451 \\
 \hline
 605 \\
 506 \\
 \hline
 1111
 \end{array}$$

- a) Working in groups of four, test all the numbers from 10 to 70 and discover how many additional steps are required to achieve a palindrome.

Have one group member do the numbers 10 – 25, one do 26 – 40, one do 41 – 55, and one do 56 – 70.

As you work, fill in the chart below, identifying each number as a one-step, two-step, three-step, or four-step palindrome, and state the palindromic sum.

Number of Steps	Zero-Step	One-Step	Two-Step	Three-Step	Four-Step
Number	1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 22, 33, 44, 55, 66, 77, 88, 99	13,	37,	68,	
Final Palindrome	1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 22, 33, 44, 55, 66, 77, 88, 99	44,	121,	1111,	

- b) Below is a hundred chart. The cells with numbers which are already palindromes are shaded grey. Colour the cells with one-step palindromes red, the two-step green, the three-step blue, and the four-step yellow, for all the other numbers up to 70. Can you see any patterns?

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

***Extension :***

- a) Your answers to part *a)* allow you to predict results for some of the numbers between 71 and 100. What are these predictable numbers? Explain your reasoning. (Note that 100 is a one-step palindrome.)
- b) What are the six remaining numbers that you can't predict? Fill in the remainder of the hundred chart with the appropriate colours, leaving these six unknown numbers white.

**Hints**

**Hint 1** - If 25 is a 1-step palindrome, what can you say about 52?

*Suggestion:*

1. Once the groups have completed their hundred chart, have them check with other groups to verify their results.

**Solution**

a) The results are shown in the chart below.

	Zero-Step	One-Step	Two-Step	Three-Step	Four-Step
N U M B E R		10, 12, 13, 14, 15, 16, 17, 18, 20, 21, 23, 24, 25, 26, 27, 29, 30, 31, 32, 34, 35, 36, 38, 40, 41, 42, 43, 45, 47, 50, 51, 52, 53, 54, 56, 60, 61, 62, 63, 65, 70	19, 28, 37, 39, 46, 48, 49, 57, 58, 64, 67	59, 68	69
		71, 72, 74, 80, 81, 83, 90, 92, 100	73, 75, 76, 82, 84, 85, 91, 93, 94	86, 95	96
P A L I N D R O M E	1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 22,33,44,55, 66	11, 33, 44, 55, 66, 77, 88, 99, 22, 33, 55, 66, 77, 88, 99, 121, 33, 44, 55, 77, 88, 99, 121, 44 55, 66, 77, 99, 121, 55 66, 77 88, 99, 121, 66, 77, 88, 99, 121, 77	121,121,121,363,121,363,484, 363,484,121,484	1111,1111	4884
	77, 88, 99	88,99,121,88,99,121,99,121,101	121,363,484,121,363,484,121, 363,484	1111,1111	4884

b) The completed hundred chart is shown below.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Observed patterns:

- The portion of the chart from 11 to 99, excluding multiples of 10, displays complete symmetry about the diagonal zero-step palindromes 11, 22, ..., 99.
- Most of the numbers are one-step palindromes.
- There are no two-step palindromes above the diagonal formed by 19, 28, 37, ..., 91.
- The three- and four-step palindromes occur for numbers greater than 58.

Some further thought may lead students to see why the above types of palindromes occur where they do in the chart.

- One-step palindromes occur if the sum,  $S$ , of the digits of the number is less than 10, or equal to 11, and the palindrome is  $SS$  if  $S < 10$  (e.g., if the sum is 7 then the palindrome is 77), or 121 if  $S = 11$ .
- Two-step palindromes occur if  $S = 10, 12$ , or 13; three-step palindromes occur if  $S = 14$ , and four-step if  $S = 15$ .

*Extension:*

- a) The numbers 77, 88, 99 are zero-step palindromes. The numbers 8, 9, 17, 18, 27, 29, 38, 47, predict that 80, 90, 71, 81, 72, 92, 83, 74 will be one-step palindromes, as is the number 100. The numbers 19, 28, 37, 39, 48, 49, 57, 58, 67 predict that 91, 82, 73, 93, 84, 94, 75, 85, 76 will be two-step palindromes. The numbers 59, 68 predict 95, 86 will be three step, and 69 predicts 96 will be four-step.
- b) The remaining six numbers are pairs 78 and 87, 79 and 97, and 89 and 98.