Part II: For the Teacher

Curriculum Areas

Problem 1 - Problem Solving

Problem 2 - Pattern/Algebra

Problem 3 - Data Management and Problem Solving

Problem 4 - Measurement and Number Sense

Problem 5 - Geometry

Problem 6 - Measurement

Hints and Suggestions:

Problem 1

Hint 1 - Who should go first?

Hint 2 - Who will bring the boat back for the second trip?

Hint 3 - Could either dad cross the river with his son?

Hint 4 - Who must make the last trip?

Suggestion: Have students act out the problem in groups of 4, with one student playing the role of each character.

Problem 2

Hint 1 - b) If you play a Green note after each two-note riff from a), how many three-note riffs will you get? Why?

Hint 2 - c) How could you use your answer from part b) to find how many four-note riffs are possible?

Problem 3

Hint 1 - a) If the numbers are taken in order, which number equals 5?

Hint 2 - a) How many occurrences of the number '1' must there be?

Hint 3 - a) What is the sum of the five numbers?

Hint 3 - b) What is the sum of the six numbers?

Problem 4

Hint 1 - a) If Biff had $2000, how many $10 bills would he have? How did you get your answer?

Hint 2 - b) How much would Biff spend in a year?

Hint 3 - c) How many sheets of photocopy paper are there in a stack of height 1 cm? How many in a stack of height 2 cm?

Extension:

Hint 1 - What are the dimensions of a backpack?

Hint 2 - About how many bills, laid edge to edge, would fit in the bottom of a backpack?

Suggestion: Have students measure the dimensions of several backpacks and come to a consensus on a reasonable size of 'rectangular' box which approximates a backpack.

Problem 5

Hint 1 - a) What is the area of triangle ABC?

Hint 2 - a) What is the height of triangle BDE? Triangle BEF? Triangle BAF?

Hint 3 - a) What is the area of triangle BAF?

Hint 4 - c) Are triangles of equal area necessarily congruent?

Extension:

Hint 1 - What figures appear when you draw the three diagonals through the centre of the hexagon?

Suggestions:

1. You may wish to review the area formula for triangles, and the meaning of 'congruent' beforehand.
2. For many students, the Extension can be solved easily if the 'star' is cut out very carefully, and the outer triangles cut off, or folded along the base of each triangle.
3. This problem could be done in small groups.

Problem 6

Hint 1 - a) How many cubic mm equal one cubic cm?

Hint 2 - b) If the 24 mp3 players were packed in three stacks of 8, what would be the box dimensions? What other sets of stacks might work?

Hint 3 - c) What are the lengths of the sides of each of the boxes you designed in part b)? How many pairs of identical sides are there?

Extension:

Hint 1 - If the 24 mp3 players were packed vertically in three rows of 8, what would be the box dimensions? In what other ways could you choose the rows?

Solutions

Problem 1

The key idea is that the boat has to return to the near shore after each trip except the last one. So after the first trip, there always needs to be at least one person on the far shore, so they can bring the boat back.
Below is a table outlining a sequence of nine trips across the river. The boys and their fathers are represented by their initials, H for Harry, S for Sam, M for Micah, and T for Todd. Their positions on either side of the river are shown AFTER the current trip. (Interchanging H and S, or M and T, or both, gives other solutions.)

Trip	Crossing Over to Far Shore	Crossing Back to Near Shore	On Near Shore	On Far Shore
1	M, T		H, S	M, T
2		M	H, S, M	T
3	S		H, M	T, S
4		T	H, M, T	S
5	M, T		H	S, M, T
6		M	H, M	S, T
7	H		M	S, T, H
8		T	M, T	S, H
9	M, T			S, H, M, T

Suggestion: Ask how we know this is the least possible number of trips...an interesting discussion will ensue.

Problem 2

a) There are three colours, G, R, Y. One way to organize the counting of possible riffs is to list separately those that begin with G, then R, then Y. Thus there are three sets of three riffs, GG, GR, GY, RR, RG, RY, and YY, YG, YR, giving a total of $3 \times 3 = 9$ two-note riffs.

b) The easiest way to think of creating three-note riffs is to add a note to each of the two-note riffs. Since any one of G, R, or Y, can be added to each of the nine two-note riffs, there are three times as many, i.e., $9 \times 3 = 27$ three-note riffs.

c) Similar reasoning show that there are three times as many four-note riffs as three-note riffs, i.e., $3 \times 27 = 81$ four-note riffs, and $3 \times 81 = 243$ five-note riffs.

Number of notes	Number of riffs
1	3
2	9
3	27
4	81
5	243

Problem 3

a) There are five numbers in the set. Since the mode is 1, the least two numbers must both be 1. Since the median is 5, the middle number must be 5. Since the mean is 4, the sum of the numbers must be $5 \times 4 = 20$. (By the difinition of mean, 4 = (sum of all five numbers) $\div$ 5, so the sum equals 5×4.) The sum of the first three numbers is $1 + 1 + 5 = 7$, so the last two must sum to 13. Hence the last two numbers must be 6 and 7, because we can't use 5 and 8, or the mode would not be 1, and we can't use 4 and 9 because 5 is the median, so the numbers would be out of order. Thus the set is {1,1,5,6,7}

b) Again, since the mode is 1, at least the first two numbers must be 1 (there could be more 1s in this case). Since the range is 28, the last number must be 29. Since the mean is 14.5, the six numbers must sum to $6 \times 14.5 = 87$, and thus the middle three numbers must sum to $87 - (1 + 1 + 29) = 56$. So Biff's solution can be any set of the form {1,1,M,N,P,29} , where M, N, and P are distinct whole numbers less than or equal to 29, and with a sum of 56. (In the final set of six numbers, 1 must be the number which appears most often.) Some examples are {1,1,1,10,45,29}, {1,1,15,18,23,29}, {1,1,16,19,21,29}. Thus there are many solutions to Biff's second problem.

Problem 4

a) Since one million dollars equals \$1 000 000, Biff will have \$1 000 000 $\div$ \$10 = 100 000 ten dollar bills.

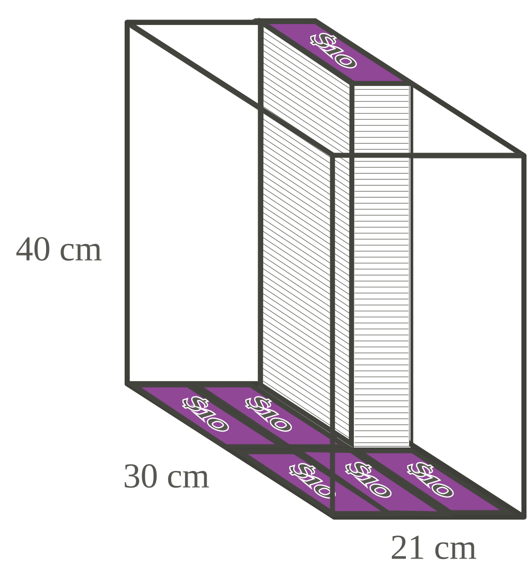
b) First we note that \$500 per week equals \$500 $\times$ 52 = \$26 000 per year. Thus the million dollars would last $\$10^6 \div \$26\,000 = 38.46$ years, or roughly 38.5 years.

c) Since the diagram tells us that 500 sheets of photocopy paper make a stack 5 cm high, and we know a \$10 bill is about the same thickness, every 500 bills would make a stack 5 cm high. Thus 100 bills would make a stack 1 cm high. Since one million dollars equals 100 000 \$10 bills, the stack would be $100\,000 \div 100 = 1\,000$ cm high, or 10 metres high.

Extension:

1. Assuming, as a rough approximation, that an average backpack is a rectangular box about 30 cm wide, 40 cm high, and 21 cm deep, the base will be 21 cm by 30 cm. This would permit about 6 stacks of bills, since the bills are 7 cm wide by 15 cm long, and $21 = 3 \times 7$, while $30 = 2 \times 15$. Each stack 40 cm high would contain $40 \times 100 = 4\,000$ bills, and so the six stacks would contain $6 \times 4\,000 = 24\,000$ ten dollar bills, or \$240 000. Thus an average backpack would only be able to contain about one quarter of the money!

   Note: Answers will vary, depending on the size of backpack; it would have to be a VERY large backpack to hold all the money!



Problem 5

The KEY IDEA here is that each of the six smallest triangles in the diagram has the same height (2) and base length (2). Note that this uses the fact that for an oblique triangle such as ABF (or FBE), the 'height' is the length of the perpendicular BD to the extended base AD (or FD).

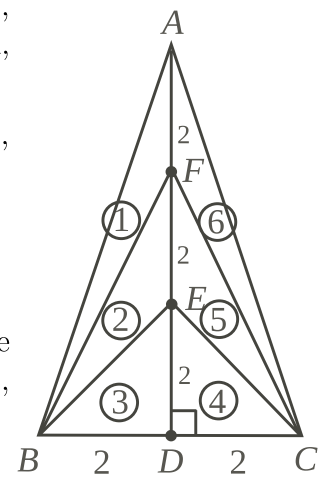
Hence all six of these triangles have the same area, equal to $\frac{1}{6}$ of the area of triangle ABC.

a) To colour a section of the figure with area equal to $\frac{1}{2}$ that of the triangle ABC, all that is needed is to colour any three of the six smallest triangles, since $\frac{1}{6}+\frac{1}{6}+\frac{1}{6}=\frac{3}{6}=\frac{1}{2}$. Using the numbered triangles in the diagram, the 20 different ways to do this are:

(1,2,3), (1,2,4), (1,2,5), (1,2,6), (1,3,4), (1,3,5), (1,3,6), (1,4,5), (1,4,6), (1,5,6);

(2,3,4), (2,3,5), (2,3,6), (2,4,5), (2,4,6), (2,5,6);

(3,4,5), (3,4,6), (3,5,6); (4,5,6).

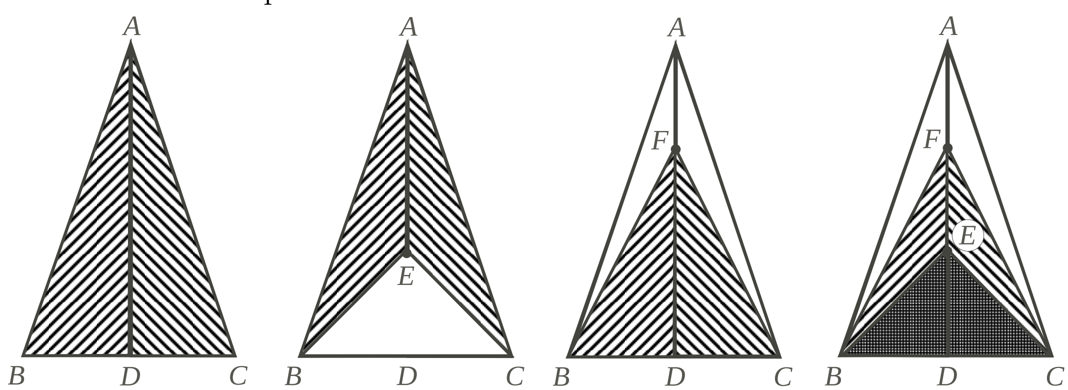


b) To colour a section of the figure with area equal to $\frac{1}{3}$ that of the triangle ABC, all that is needed is to colour any two of the six smallest triangles, since $\frac{1}{6}+\frac{1}{6}=\frac{2}{6}=\frac{1}{3}$. There are 15 different ways to do this:

(1,2), (1,3), (1,4), (1,5), (1,6); (2,3), (2,4), (2,5), (2,6);

(3,4), (3,5), (3,6); (4,5), (4,6); (5,6).

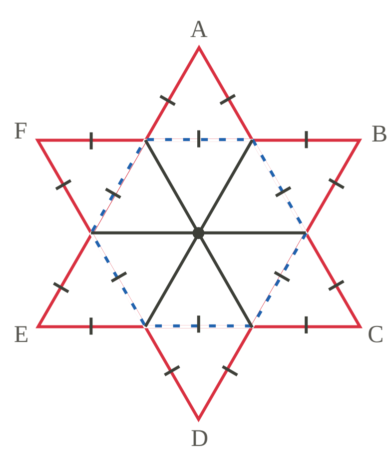
c) Using the symmetry of the figure about AD, the pairs of congruent triangles are: ABD and ADC, ABE and AEC, FBD and FDC, ABF and AFC, FBE and FEC, and EBD and EDC. Thus there are six such pairs.



Extension:

1.a) Since each line segment has the same length, the triangles on the outside of the figure have all three sides of equal length, and hence are equilateral triangles, with each angle being 60°.

b) If the figure is cut out, and the six exterior triangles are folded along the dotted lines as shown in the diagram, the vertices A, B, C, D, E, F all meet at the centre of the hexagon, with the lateral sides matching. Thus the six exterior triangles exactly fill the interior of the hexagon. (See Emmy Noether Circle 1 for 2010-2011, Problem 6 d) for a complementary problem about hexagons.)



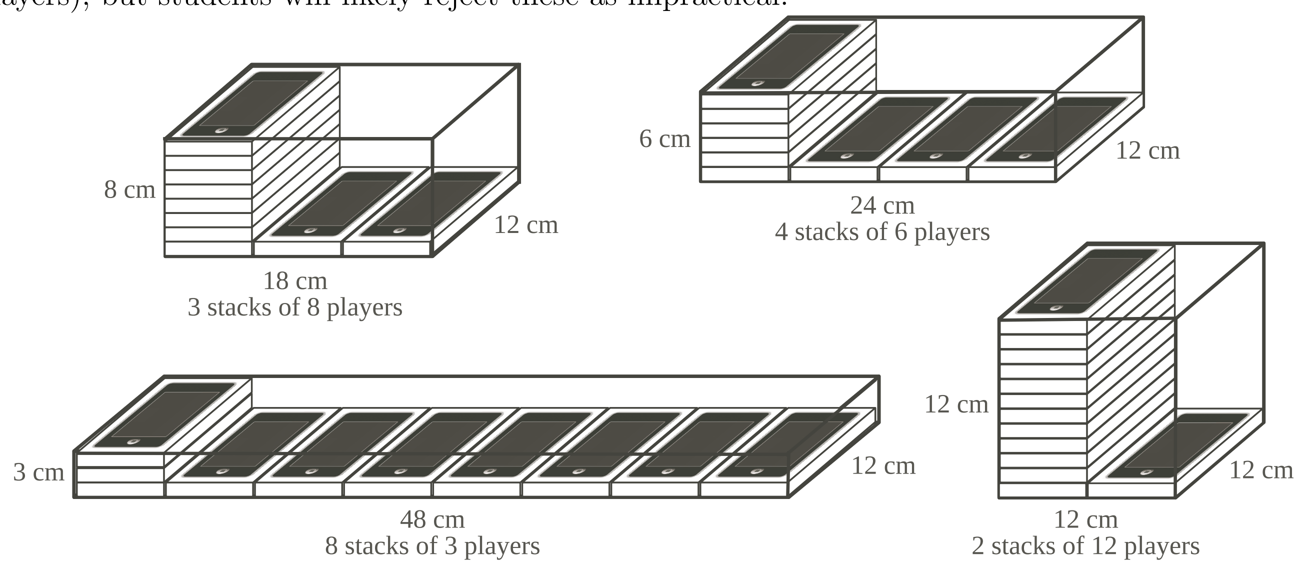
Problem 6

a) The total volume of one mp3 player is 12 cm × 6 cm × 1 cm = 72 cubic centimetres.

b) The KEY IDEA is to make the dimensions of the base of the box some multiple of the 6 cm width and 12 cm height of a single mp3 player (since they are to be packed horizontally), so as not to have any empty space inside the box. Then find how many mp3 players fit into a single layer, and hence how many layers are needed. Here are some sample possibilities (illustrated below):

1. A box with base 18 cm by 12 cm permits 3 mp3 players in each layer, and hence must be 8 cm high (3 stacks of 8 players).
2. A box with base 24 cm by 12 cm permits 4 mp3 players in each layer, and hence must be 6 cm high (4 stacks of 6 players).
3. A box with base 48 cm by 12 cm permits 8 mp3 players in each layer, and hence must be 3 cm high (8 stacks of 3 players).
4. A box with base 12 cm by 12 cm permits 2 mp3 players in each layer, and hence must be 12 cm high (2 stacks of 12 players).

There are other possibilities (e.g., a box with base 12 cm by 6 cm, 24 cm high, for 1 stack of 24 mp3 players, or a box with base 48 cm by 36 cm, 2 cm high, for 12 stacks of 2 mp3 players), but students will likely reject these as impractical.



c) The amount of cardboard required is equal to the surface area of the box. If the box has dimensions length (l), width (w), and height (h) in cm, then the surface area is $S = 2 \times l \times w + 2 \times w \times h + 2 \times l \times h$ square centimetres. For the sample boxes in part b), the surfaces areas are:

1. $S = 2 \times 18 \times 12 + 2 \times 12 \times 8 + 2 \times 18 \times 8 = 912$ square centimetres;
2. $S = 2 \times 24 \times 12 + 2 \times 12 \times 6 + 2 \times 24 \times 6 = 1008$ square centimetres;
3. $S = 2 \times 48 \times 12 + 2 \times 12 \times 3 + 2 \times 48 \times 3 = 1512$ square centimetres;
4. $S = 2 \times 12 \times 12 + 2 \times 12 \times 12 + 2 \times 12 \times 12 = 864$ square centimetres;

For the impractical box, $S = 2 \times 12 \times 6 + 2 \times 6 \times 24 + 2 \times 12 \times 24 = 1008$ square centimetres.

Thus we see that the box which is a cube requires the least amount of cardboard. (This is true for any rectangular box of fixed volume, even when the side lengths are not whole numbers, but the proof requires more advanced mathematics.)

Suggestion: You may wish to discuss with the class whether there are other possible boxes. (There are.) Because the side lengths must be whole numbers, and the base of the box must be some integer multiple of 12 cm by 6 cm (to have no empty space), with a total volume of 24×72 cubic centimetres, there are only a limited number of possibilities.

Extension:

1. Since the 24 mp3 players will exactly fit in 2 vertical rows of 12 in a 12 cm by 12 cm by 12 cm box, the 'minimal' box will be the same as for horizontal packing. Note that this only works because a row of 12 mp3 players packed vertically is 6 cm by 12 cm by 12 cm, which is exactly the same as a column (stack) of 12 mp3 players packed horizontally, due to the thickness being 1 cm.