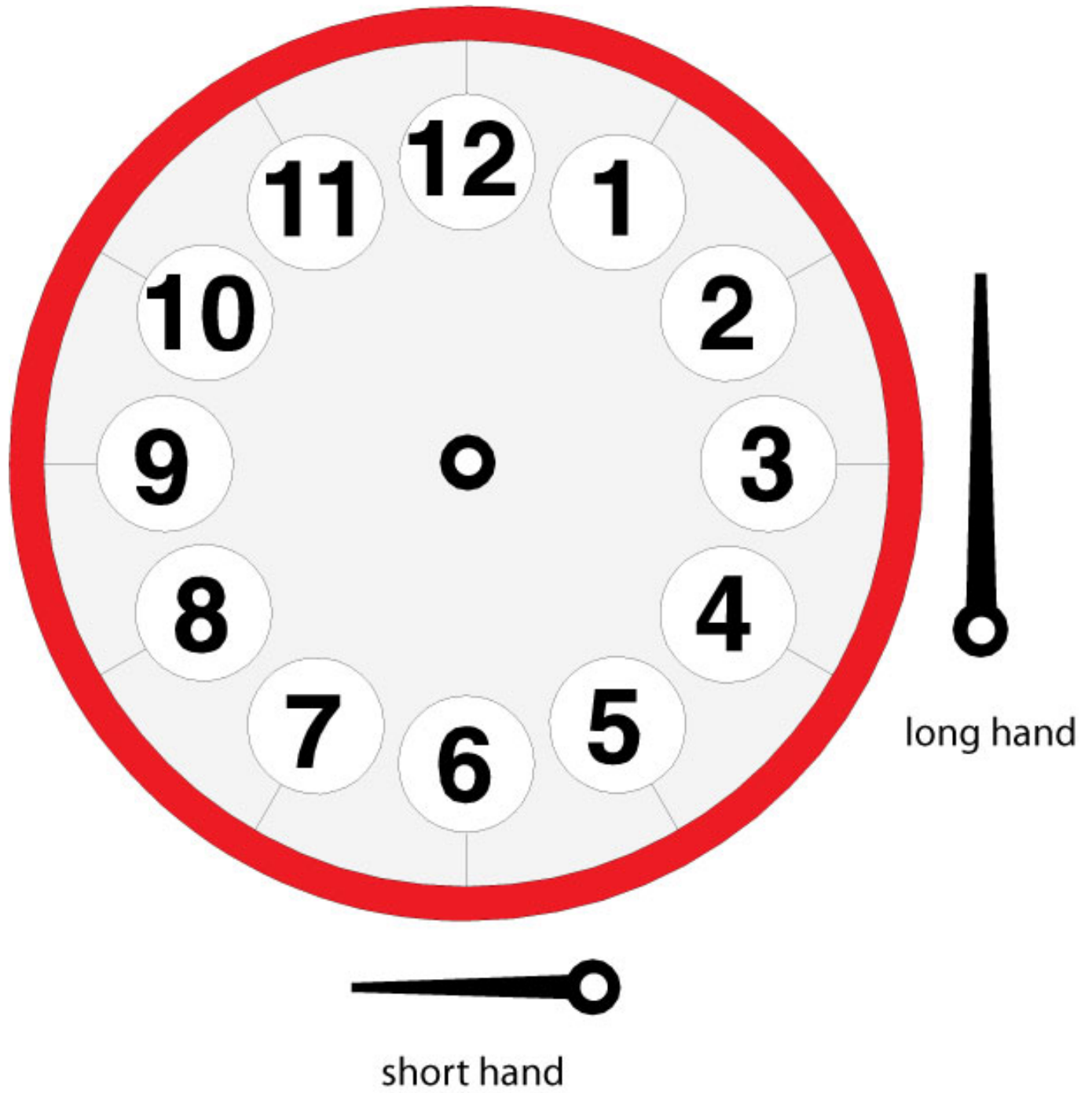# Problem

A 'Timely' Offer! (Suggested for pairs or groups of students)

Sarah's Dad offers her the choice of a weekly allowance of $ 1.00 for every time the hands of her clock make a right angle between the hours of noon and midnight, OR $ 2.00 for every time they make an angle of 180°. Sarah decides on the first choice, believing it will give her a larger allowance. Explain why she is right (or wrong).

To help you decide, here is a useful tool. Place this page on a piece of cardboard. Cut out the hands for the clock and use a stick pin or thumbtack to place them at the center of the clock, pinning the diagram to the cardboard.

long hand

short hand

Hints

Hint 1 - How many 90° angles do the hands make between noon and 1 p.m.? Between 5 and 6 p.m.?

Hint 2 - How many 90° angles do the hands make between 2 p.m. and 4 p.m.?

Suggestions:

1. Have students create a model clock by following the directions on the problem sheet. They will need a piece of cardboard or a cork board in order to pin and manipulate the hands freely.

2. If students have access to the web, you may wish to suggest the following web reference from the National Library of Virtual Manipulatives, which contains a virtual clock manipulative: http://nlvm.usu.edu/en/nav/frames_asid_316_g_2_t_4.html?from=category_g_2_t_4.html

Solution

To see whether Sarah is right, we need to discover how many occurrences there are of 'right-angled hands' (90° angle between the two hands), and of 'straight-line hands' (180° angle between the two hands).

Right-angled hands generally occur twice in each hour; the four clock faces below reveal the two right-angled positions between noon and 1 p.m., and between 5 and 6 p.m.

| Between 2:15 and 12:20 | Between 12:45 and 12:50 | Between 5:10 and 5:15 | Between 5:40 and 5:45 |

This occurs for each one-hour interval between noon and midnight, except for two, namely, between 2 and 3 p.m., and between 8 and 9 p.m. In those two intervals, the second occurence of right-angled hands is actually ON the next hour, and hence coincides with the first occurrence for the next hour, as shown below.

| Between 2:25 and 2:30 | at 3:00 | Between 8:25 and 8:30 | at 9:00 |

Thus, between noon and midnight, there are 10 intervals with 2 right-angled positions, and 2 intervals with only 1, giving a total of $10 \times 2 + 2 = 22$ such positions.

Straight-line hand positions are easier to picture; they occur only once per hour in general. For example, between 12 noon and 1 p.m., the hands make 180° somewhere between 12:30 and 12:35; between 2 and 3 p.m., this occurs somewhere between 2:40 and 2:45; between 8 and 9 p.m., somwhere between 8:10 and 8:15, etc. The 'odd-ball' case here is between 5 and 7 p.m., where there is only one occurrence in 2 hours, namely at 6 p.m. Thus there are only 11 occurrences of 'straight-line' hands between noon and midnight.

We thus conclude that Sarah's allowance will be $ 1 \times 22 = $ 22 for right-angled hands, or $ 2 \times 11 = $ 22 for straight-line hands. So both choices give her the same allowance, i.e., neither gives her a larger allowance.