CEMC at Home

Grade 9/10 - Monday, May 4, 2020
Contest Day 1

Today's resource features two questions from the 2020 CEMC Mathematics Contests.

2020 Euclid Contest, #2(a)

The three-digit positive integer m is odd and has three different digits. If the hundreds digit of m equals the product of the tens digit and ones (units) digit of m, what is m?

2020 Canadian Team Mathematics Contest, Team Problem #6

On Fridays, the price of a ticket to a museum is \$9. On one particular Saturday, there were 200 visitors to the museum, which was twice as many visitors as there were the day before. The total money collected from ticket sales on that particular Saturday was $\frac{4}{3}$ as much as the day before. The price of tickets on Saturdays is $\$k$. Determine the value of k.

More Info:

Check out the CEMC at Home webpage on Monday, May 11 for solutions to the Contest Day 1 problems.

CEMC at Home

Grade 9/10 - Monday, May 4, 2020
Contest Day 1 - Solution

Solutions to the two contest problems are provided below, including a video for the first problem.

2020 Euclid Contest, #2(a)

The three-digit positive integer m is odd and has three different digits. If the hundreds digit of m equals the product of the tens digit and ones (units) digit of m, what is m?

Solution:

Suppose that m has hundreds digit a, tens digit b, and ones (units) digit c.
From the given information, a, b and c are distinct, each of a, b and c is less than 10, $a = bc$, and c is odd (since m is odd).
The integer $m = 623$ satisfies all of these conditions. Since we are told there is only one such number, then 623 is must be the only answer.
Why is this the only possible value of m?
We note that we cannot have $b = 1$ or $c = 1$, otherwise $a = c$ or $a = b$.
Thus, $b \geq 2$ and $c \geq 2$.
Since $c \geq 2$ and c is odd, then c can equal 3, 5, 7, or 9.
Since $b \geq 2$ and $a = bc$, then if c equals 5, 7 or 9, a would be larger than 10, which is not possible.
Thus, $c = 3$.
Since $b \geq 2$ and $b \neq c$, then $b = 2$ or $b \geq 4$.
If $b \geq 4$ and $c = 3$, then $a > 10$, which is not possible.
Therefore, we must have $c = 3$ and $b = 2$, which gives $a = 6$.

Video

Visit the following link to view a discussion of a solution to the first contest problem:
https://youtu.be/dJ6d0ILAGwE

2020 Canadian Team Mathematics Contest, Team Problem #6

On Fridays, the price of a ticket to a museum is \$9. On one particular Saturday, there were 200 visitors to the museum, which was twice as many visitors as there were the day before. The total money collected from ticket sales on that particular Saturday was $\frac{4}{3}$ as much as the day before. The price of tickets on Saturdays is \$$k$. Determine the value of k.

Solution:

There were 200 visitors on Saturday, so there were 100 visitors the day before. Since tickets cost \$9 on Fridays, the total money collected on Friday was \$900.

Therefore, the amount of money collected from ticket sales on the Saturday was $\frac{4}{3}(\$900) = \$1200$.

Since there were 200 visitors on Saturday, the price of tickets on that particular Saturday was $\frac{\$1200}{200} = \$6$. Therefore, the value of k is 6.

It turns out that you do not need to know the number of visitors to the museum on the Saturday to solve the problem. If this number (200) changes, but all other conditions in the problem are kept the same, then the answer will still be $k = 6$. Can you see why?

CEMC at Home

Grade 9/10 - Tuesday, May 5, 2020
Patterns in Arithmetic

The questions included in this activity can be solved by looking for a pattern and using it to get at the solution.

Problem 1: A Wizard's assistant is paid in an unusual way. The assistant's paycheque for the first week is one dollar. At the end of each week after the first week, the assistant is paid the amount of money earned the previous week plus two dollars for every week worked so far. What is the assistant's paycheque, in dollars, for the fifty-second week?

To get started, calculate the paycheque for a particular earlier week, say the 9^{th} or 10^{th} week, looking for a pattern while you do so.

Problem 2: Suppose that the integer N is the value of the following sum (with 52 terms):

$$1 + 11 + 101 + 1001 + 10001 + \cdots + 1\overbrace{000\ldots000}^{\text{50 zeroes}}1$$

When N is calculated and written as a single integer, what is the sum of its digits?

To get started, consider how each term in the sum is formed.

Problem 3: Using only the digits $1, 2, 3, 4,$ and 5, a sequence is created. The beginning of the sequence in shown below.

$$1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, \ldots$$

The sequence starts with one 1, followed by two 2s, then three 3s, four 4s, five 5s, six 1s, seven 2s, and so on. What is the 1000th term in the sequence?

Some patterns can be tough to explain precisely using few words. The description above is likely sufficient to relay to you how the sequence is defined, but does not precisely define the remaining terms in the sequence. Can you come up with a more formal way to describe how this sequence is defined?

Discovering the correct patterns can lead you to the correct answers for these problems. Think about how you can justify that the pattern you discover in each of the problems is in fact correct.

More Info:

Check out the CEMC at Home webpage on Tuesday, May 12 for a solution to Patterns in Arithmetic.

CEMC at Home

Grade 9/10 - Tuesday, May 5, 2020
Patterns in Arithmetic - Solution

Problem 1: A Wizard's assistant is paid in an unusual way. The assistant's paycheque for the first week is one dollar. At the end of each week after the first week, the assistant is paid the amount of money earned the previous week plus two dollars for every week worked so far. What is the assistant's paycheque, in dollars, for the fifty-second week?

Solution:

Looking at the first few weeks' salaries we have:

Week	Pay ($)
1	1
2	$1 + 2(2) = 5$
3	$5 + 2(3) = 11$
4	$11 + 2(4) = 19$
⋮	⋮

The above table may not provide enough information, so let's expand things a bit in another table:

Week	Pay ($)
1	1
2	$1 + 2(2) = 5$
3	$1 + 2(2) + 2(3) = 11$
4	$1 + 2(2) + 2(3) + 2(4) = 19$
⋮	⋮

At this point, it may be easier to see a pattern beginning to emerge. We might guess that the next row in the table will have $1+2(2)+2(3)+2(4)+2(5) = 19+10 = 29$ in the right column. Indeed, the pay in the 5^{th} week is 2×5 dollars more than it was on the 4^{th} week so will be $1+2(2)+2(3)+2(4)+2(5)$. This pattern will continue and so we can deduce a formula for calculating any week's pay directly.

In particular, the pay for the 52^{nd} week can be calculated as follows:

$$1 + 2(2) + 2(3) + \cdots + 2(51) + 2(52) = 1 + 2(2 + 3 + 4 + \cdots + 51 + 52)$$

We are left with the problem of computing the sum

$$S = 2 + 3 + 4 + \cdots + 51 + 52$$

where S represents the sum of the integers from 2 to 52. This can be done in a number of ways.

If you know the formula for the sum of the integers from 1 to n (it is said that Gauss found this formula on his own as a child) then you could use it here. Alternatively, you could notice that the numbers in the sum form an arithmetic sequence and use a formula for summing such a list of numbers.

We will calculate S as follows: First write out the sum S twice, with the second sum written with the terms in the opposite order.

$$\begin{array}{rcl} S &=& 2+\ 3+\ 4+\cdots+50+51+52 \\ +S &=& 52+51+50+\cdots+\ 4+\ 3+\ 2 \\ \hline 2S &=& 54+54+54+\cdots+54+54+54 \end{array}$$

Adding the terms on the left side gives $2S$. Adding the terms on the right side gives

$$54+54+54+\cdots+54+54+54,$$

the sum of 51 copies of 54. Therefore,

$$2S = 54+54+54+\cdots+54+54+54 = 51(54) = 2754$$

This means the assistant's pay for the 52^{nd} week is

$$1+2(2+3+4+\cdots+51+52) = 1+2S = 1+2754 = 2755$$

dollars.

Problem 2: Suppose that the integer N is the value of the following sum (with 52 terms):

$$1+11+101+1001+10001+\cdots+1\overbrace{000\ldots000}^{50\text{ zeroes}}1$$

When N is calculated and written as a single integer, what is the sum of its digits?

Solution:

There are 52 numbers to be added together, and all of them have a 1 in the units position. One way to approach this problem is to first compute $N-52$ by subtracting 1 from each term. That is,

$$\begin{aligned} N-52 &= (1-1)+(11-1)+(101-1)+(1001-1)+\cdots+(1\overbrace{000\ldots000}^{50\text{ zeroes}}1-1) \\ &= 0+10+100+1000+\cdots+1\overbrace{000\ldots000}^{51\text{ zeroes}} \\ &= \overbrace{111\ldots111}^{51\text{ ones}}0 \end{aligned}$$

We can now simply add 52 to both sides to get

$$N = \overbrace{111\ldots111}^{50\text{ ones}}62$$

Therefore, the sum of the digits of N is $50\times 1+6+2 = 58$.

Problem 3: Using only the digits $1, 2, 3, 4,$ and 5, a sequence is created. The beginning of the sequence in shown below.

$$1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, \ldots$$

The sequence starts with one 1, followed by two 2s, then three 3s, four 4s, five 5s, six 1s, seven 2s, and so on. What is the 1000th term in the sequence?

Solution:

Let's think of the sequence as being in "blocks". That is, the first block consists of 1s, the second block consists of 2s, the third block consists of 3s, and so on.
To answer the question, we first need to determine in which block of digits the 1000^{th} term lies.
Note that each block consists of one more term than the previous block and that the first block consists of one term. Convince yourself that after you have written down the first n blocks in the sequence, you will have written down exactly

$$1 + 2 + 3 + \cdots + (n - 1) + n$$

terms in total. How many blocks must be written down before you reach the block containing the 1000^{th} term? To answer this, we need to find when the sum above reaches 1000.

In the solution to Problem 1, we mentioned, but did not use, the formula for the sum of the positive integers from 1 to n. We shall make use of it here. The formula is:

$$1 + 2 + 3 + \cdots + (n - 1) + n = \frac{n(n + 1)}{2}$$

What is the smallest positive integer n satisfying $\frac{n(n + 1)}{2} \geq 1000$? Equivalently, what is the smallest positive integer n satisfying $n(n + 1) \geq 2000$? We could test values of n starting at $n = 1$ until we find such an integer n, but this might take quite a while. We also "know" that n cannot be too small. For example, your intuition probably tells you that n is at least 10, or maybe even at least 20, so you might not start at $n = 1$ and instead start at some larger integer.

Here is a way to choose a "good" starting integer. Note that for any positive integer n, $(n + 1)^2$ is larger than $n(n+1)$, so the n we seek must satisfy $(n+1)^2 \geq 2000$. Using a calculator, you can check that $\sqrt{2000} \approx 44.7$. Since $n + 1$ is an integer, this means $n + 1 \geq 45$, so $n \geq 44$. Thus, we need not check any values of n that are less than 44. With $n = 44$, we have $n(n + 1) = 44(45) = 1980$ which is not ≥ 2000. With $n = 45$, we have $n(n + 1) = 45(46) = 2070$, which is ≥ 2000. This means the smallest positive integer n with $n(n + 1) \geq 2000$ is $n = 45$, and we only had to check two values of n to find it!

We can now see that the first 44 blocks (in total) contain $\frac{44(45)}{2} = 990$ terms, and the first 45 blocks (in total) contain $\frac{45(46)}{2} = 1035$ terms. This means the 1000^{th} term occurs in the 45^{th} block.

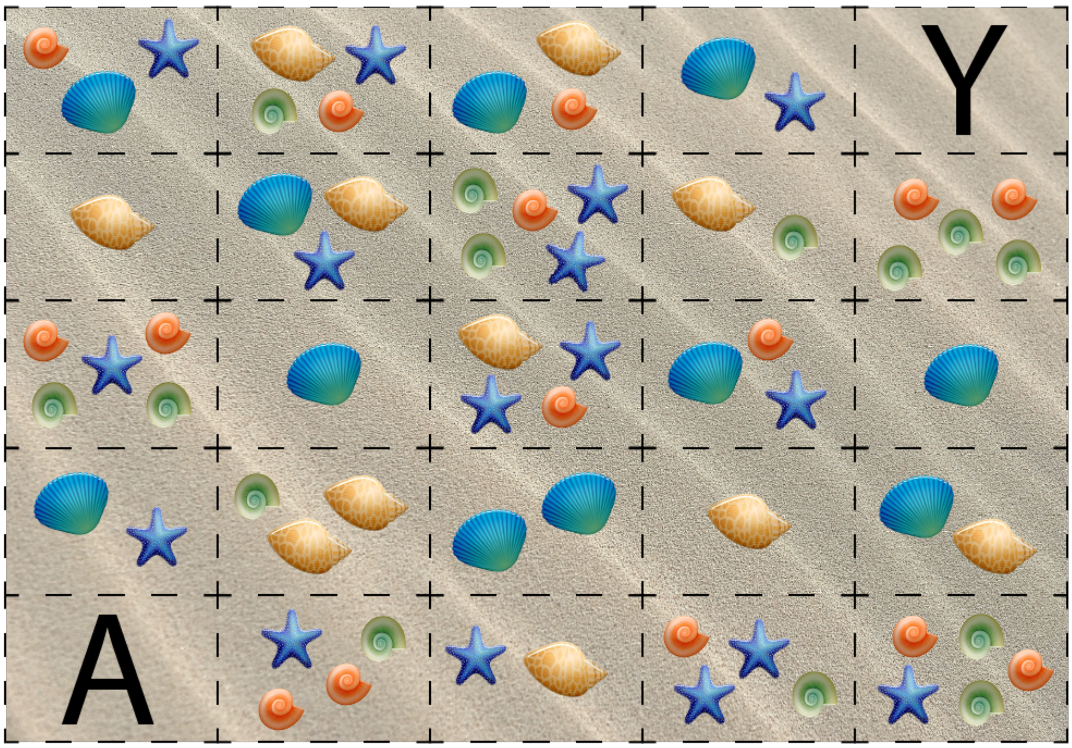
To determine what the 1000^{th} term will be, we need to determine what digit makes up the 45^{th} block.
Note that the 1^{st} block, 6^{th} block, 11^{th} block, and so on, consist of 1s.
The 2^{nd} block, 7^{th} block, 12^{th} block, and so on, consist of 2s.
The 3^{rd} block, 8^{th} block, 13^{th} block, and so on, consist of 3s.
The 4^{th} block, 9^{th} block, 14^{th} block, and so on, consist of 4s.
The 5^{th} block, 10^{th} block, 15^{th} block, and so on, consist of 5s.

In particular, when n is a multiple of 5, the n^{th} block consists of 5s. Since the 1000^{th} term is in the 45^{th} block, the 1000^{th} term will be the digit 5.

CEMC at Home

Grade 9/10 - Wednesday, May 6, 2020
Sheldon's Shells

Sheldon is walking along the beach collecting shells. The shells are scattered across the beach in different areas as shown below.



Sheldon starts in the area marked "A" and ends in the area marked "Y". After collecting all of the shells in an area he either moves up or moves right to a new area. He does not move left or down.

Questions:

1. There are three different paths that Sheldon could take from A to the area located two to the right and one up from A (containing two blue shells). What is the largest number of shells that Sheldon could collect on his way from A to this area (not including these two blue shells)?
2. Sheldon stops part way during a trip from A to Y and notices that he has collected 8 shells so far, including the shells in the area in which he has stopped. What are all of the possible areas in which Sheldon may have stopped?
3. Consider all possible paths that Sheldon could take from A to Y. What is the maximum number of shells that Sheldon could collect on his way from A to Y?

More Info:

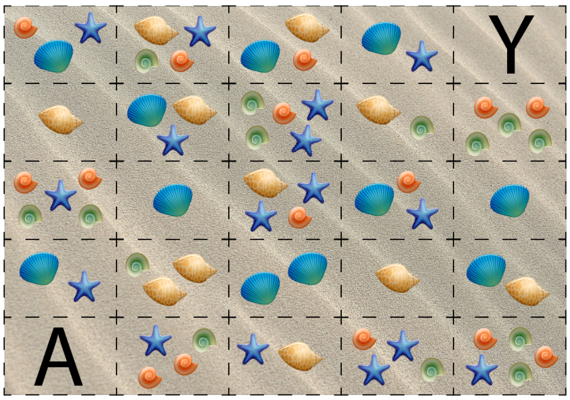
Check out the CEMC at Home webpage on Wednesday, May 13 for a solution to Sheldon's Shells.

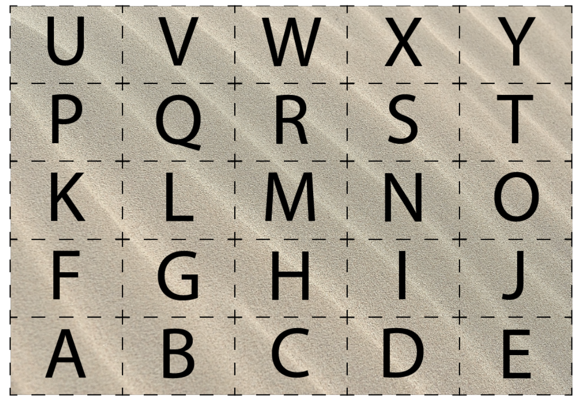
If you enjoyed this problem, check out the problem Coins and Monsters from the 2019 Beaver Computing Challenge, which is a similar problem but with an extra twist!

CEMC at Home

Grade 9/10 - Wednesday, May 6, 2020
Sheldon's Shells - Solution

Label the areas of the beach as shown:

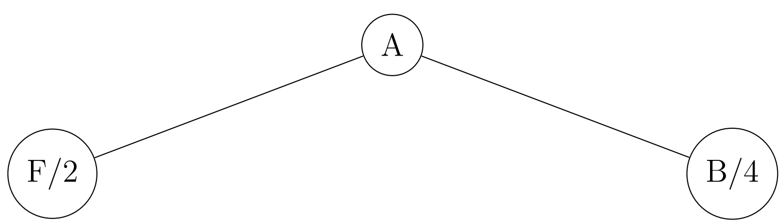




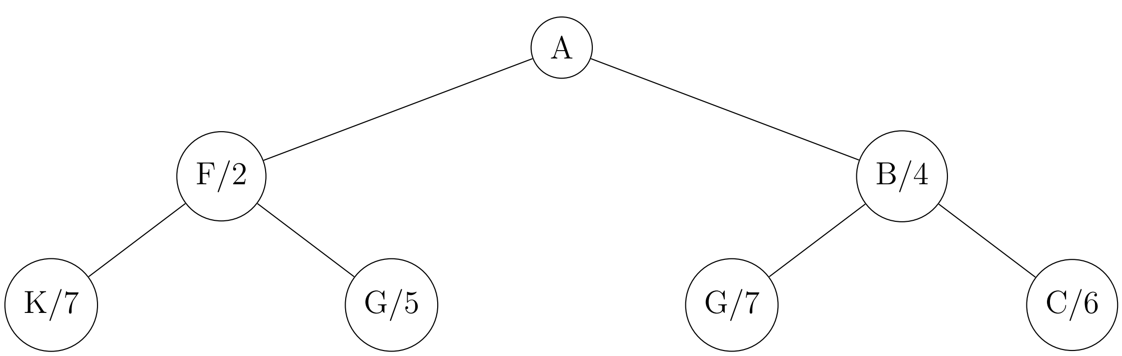
Recall that Sheldon only moves either up or right on his way from area A to area Y.

1. On his way from area A to the area with the two identical blue shells, Sheldon can collect $2 + 3 = 5$ shells (up, right, right), $4 + 3 = 7$ shells (right, up, right) or $4 + 2 = 6$ shells (right, right, up). This means the largest number of shells he can collect on this trip is 7.

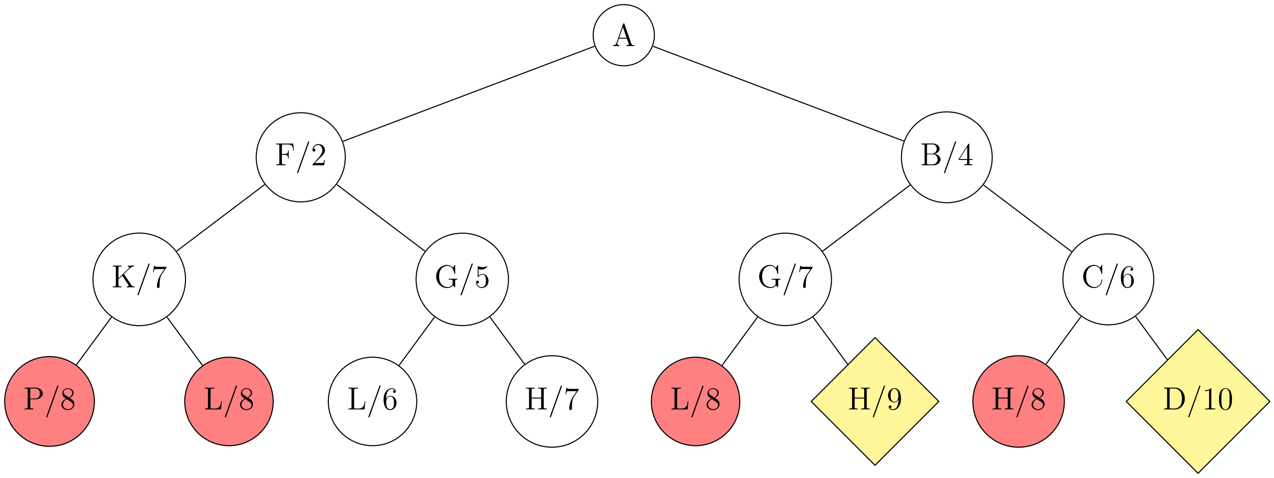
2. Sheldon stops after he has collected 8 shells. One way to figure out where Sheldon might have stopped is to build a tree starting with A and illustrating all the possible paths Sheldon could take. Since from A Sheldon can either move up to F (where he collects 2 shells) or right to B (where he collects 4 shells), the tree begins like this:



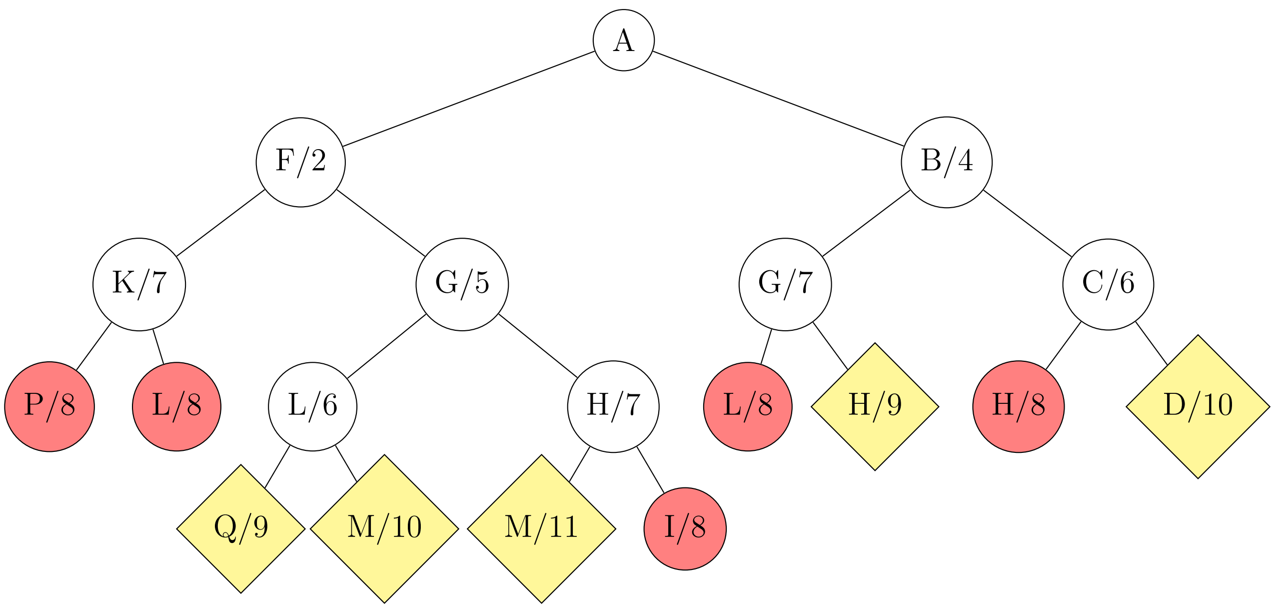
From F Sheldon can either move up to K (where he collects 5 more shells for a total of 7) or right to G (where he collects 3 more shells for a total of 5). From B Sheldon can either move up to G (where he collects 3 more shells for a total of 7) or right to C (where he collects 2 more shells for a total of 6). Those branches are added to the tree like this:



From left to right we can continue to add branches representing Sheldon's possible paths and cumulative amount of shells collected:



The areas shaded in red are areas on the path where Sheldon will have collected exactly 8 shells. We do not need to add any new branches from these areas since any further movement will result in collecting more than 8 shells. We also do not need to add any new branches from the yellow diamond areas since Sheldon has already exceeded 8 shells along these paths.



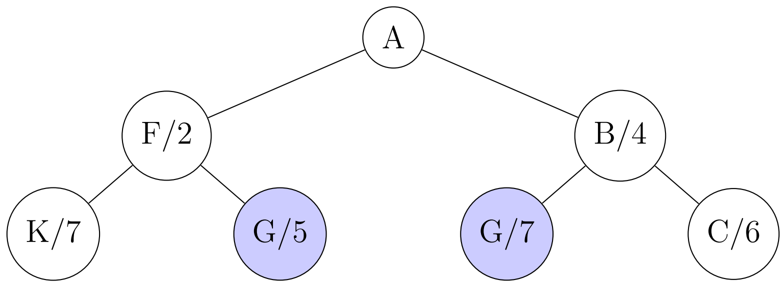
From this tree, we can see that the only areas Sheldon could have stopped in (after collecting exactly 8 shells) are P, L, I, or H.

3. Sheldon can collect 25 shells by following the path "A B G H M R S T Y". It turns out that this is the maximum number of shells that he can collect. How can we be sure of this?

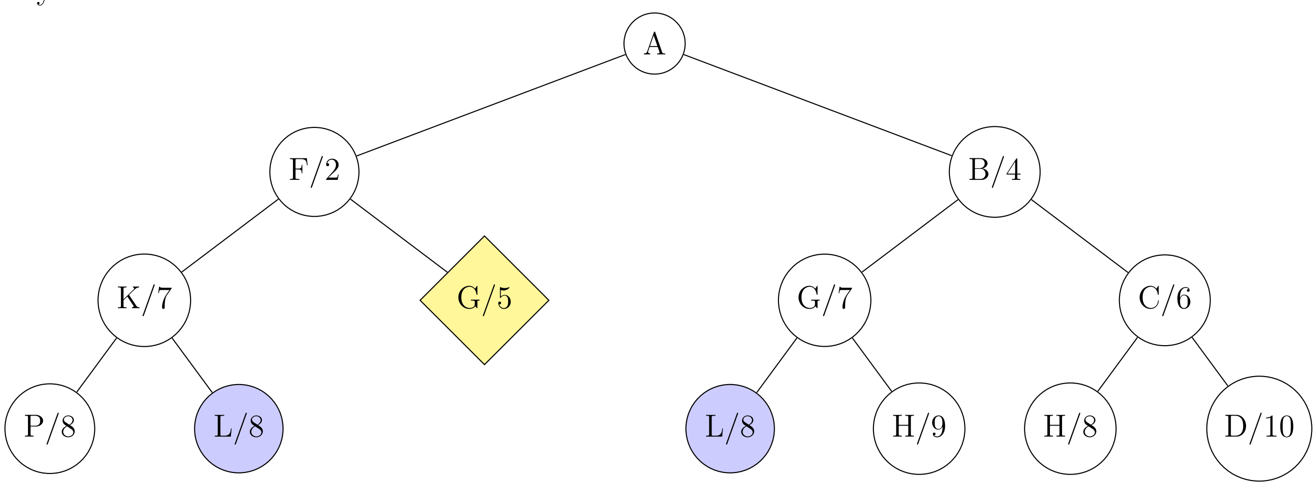
   One way to figure out the maximum number of shells that Sheldon could collect on his way from A to Y is to continue building the tree from the previous question until all paths reach Y. Then the bottom row of the tree can be scanned for the largest number.

   How many possible paths from A to Y are there? To get from A to Y Sheldon must move up four times and right four times, in some sequence. There are 70 different ways to arrange four "up"s and four "right"s, so there are 70 possible paths. That's going to make a big tree!

There are ways to simplify the tree if you would like to carry through with this approach. One way to simplify the tree is to "abandon" a path when a better path is found. For example, consider the beginning of the tree:



There are two ways Sheldon can get to area G. Taking the path "A B G" is better than taking the path "A F G" since it results in more shells. We can abandon path "A F G" by not adding any more branches to it.



In addition, we can simplify the tree by "merging" paths. Since two different paths to L result in the same number of shells (8), we do not lose any information by merging the paths together.

If we continue to expand, abandon, and merge paths we will eventually get a diagram which shows the maximum number of shells Sheldon can collect.

See the next page for the completed diagram for the tree approach.

Another way to approach a solution to this problem is to do calculations in a grid using the following observation:

> The maximum number of shells that can be collected by Sheldon upon reaching a particular area of the beach is the number of shells in this area *plus* the maximum of
>
> - the total number of shells that can be collected upon reaching the section to the left, and
> - the total number of shells that can be collected upon reaching the section below.

We can compute each of these maximum numbers of shells starting at the bottom left area and working our way through the grid.

See the next page for the completed grid with these calculations.

Grid approach

In the grid below, the number of shells in each area is displayed in a small box in the bottom left corner of the corresponding square. There is only one way to get to each of areas F and B and the maximum numbers of shells you can collect upon reaching these areas are 2 and 4, respectively. These numbers are entered in the corresponding squares in the grid below. There are two ways to get to area G: through F (from the left) or through B (from below). We see that to collect the maximum number of shells upon reaching G, we want to travel through B, and in doing so we can collect $4+3=7$ shells. See if you can work your way through the rest of the calculations in the table.

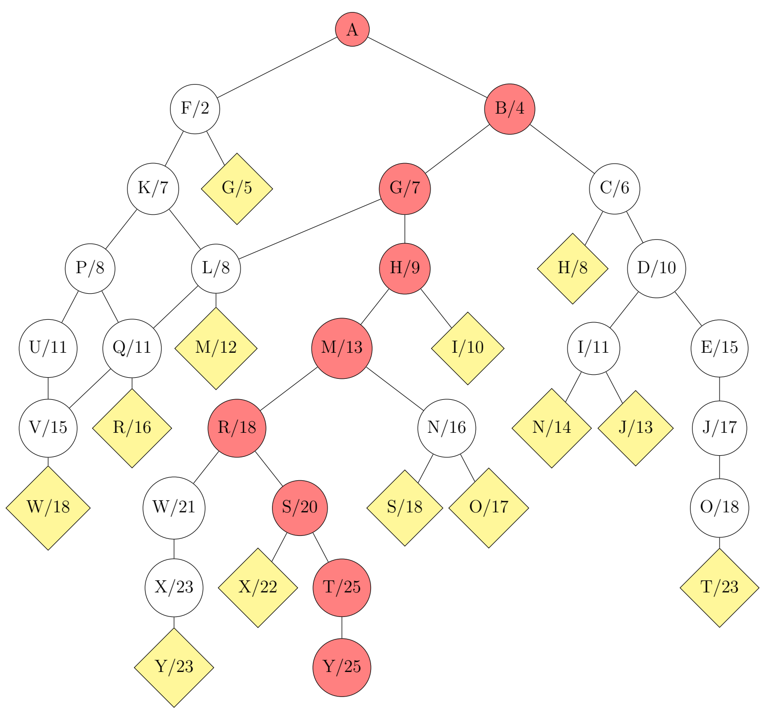
U	V	W	X	Y
P	Q	R	S	T
K	L	M	N	O
F	G	H	I	J
A	B	C	D	E

$8+3=11$ [3]	$11+4=15$ [4]	$18+3=21$ [3]	$21+2=23$ [2]	Y [0]
$7+1=8$ [1]	$8+3=11$ [3]	$13+5=18$ [5]	$18+2=20$ [2]	$20+5=25$ [5]
$2+5=7$ [5]	$7+1=8$ [1]	$9+4=13$ [4]	$13+3=16$ [3]	$17+1=18$ [1]
2 [2]	$4+3=7$ [3]	$7+2=9$ [2]	$10+1=11$ [1]	$15+2=17$ [2]
A [0]	4 [4]	$4+2=6$ [2]	$6+4=10$ [4]	$10+5=15$ [5]

Each arrow pointing to an area indicates whether the maximum comes from the section to the left or comes from the section below (or both). When we reach the top right area, we compute the value 25 which tells us that 25 is the maximum number of shells among all possibilities.

To collect this maximum number of shells, Sheldon should follow the path shown by the red arrows.

Tree approach



Looking at the bottom row, we see that the maximum number of shells is 25.

We can also see that the path Sheldon should take to collect 25 shells is "A B G H M R S T Y".

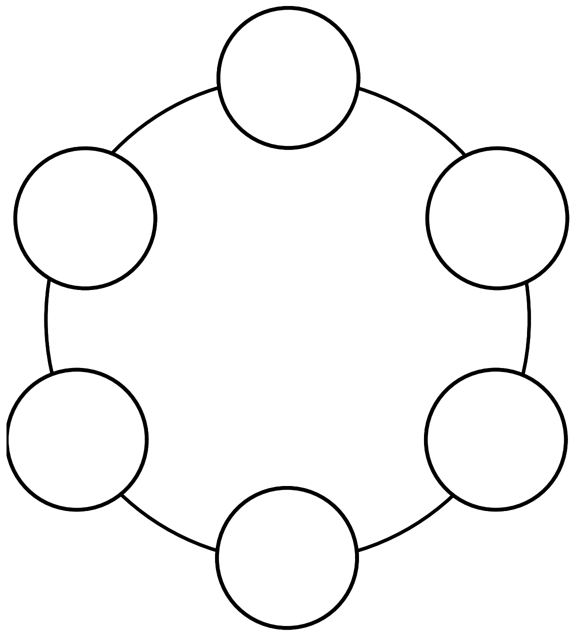
Can you see how the grid and the tree are related?

CEMC at Home features Problem of the Week

Grade 9/10 - Thursday, May 7, 2020
A Circle of Numbers

The numbers 1, 6, 8, 13, 15, and 20 can be placed in the circle below, each exactly once, so that the sum of each pair of numbers adjacent in the circle is a multiple of seven.

In fact, there is more than one way to arrange the numbers in such a way in the circle. Determine all different arrangements. Note that we will consider two arrangements to be the same if one can be obtained from the other by a series of reflections and rotations.

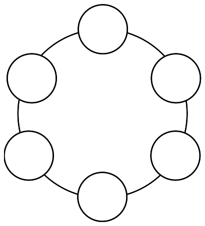


More Info:

Check the CEMC at Home webpage on Thursday, May 14 for the solution to this problem. Alternatively, subscribe to Problem of the Week at the link below and have the solution emailed to you on Thursday, May 14.

This CEMC at Home resource is the current grade 9/10 problem from Problem of the Week (POTW). POTW is a free, weekly resource that the CEMC provides for teachers, parents, and students. Each week, problems from various areas of mathematics are posted on our website and e-mailed to our subscribers. Solutions to the problems are e-mailed one week later, along with a new problem. POTW is available in 5 levels: A (grade 3/4), B (grade 5/6), C (grade 7/8), D (grade 9/10), and E (grade 11/12).

To subscribe to Problem of the Week and to find many more past problems and their solutions visit: https://www.cemc.uwaterloo.ca/resources/potw.php



Problem of the Week
Problem D and Solution
A Circle of Numbers

Problem

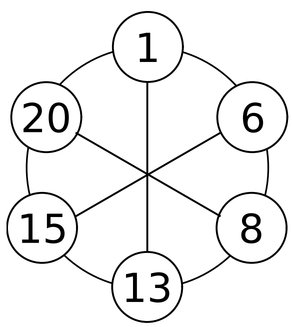
The numbers 1, 6, 8, 13, 15, and 20 can be placed in the circle above, each exactly once, so that the sum of each pair of numbers adjacent in the circle is a multiple of seven. In fact, there is more than one way to arrange the numbers in such a way in the circle. Determine all different arrangements. Note that we will consider two arrangements to be the same if one can be obtained from the other by a series of reflections and rotations.

Solution

We will start by writing down all the pairs of numbers that add to a multiple of 7.

Sum of 7	Sum of 14	Sum of 21	Sum of 28	Sum of 35
1,6	1,13	6,15	13,15	15,20
	6,8	8,13	8,20	
		1,20		

To show these connections visually, we can write the numbers in a circle and draw a line connecting numbers that add to a multiple of 7.

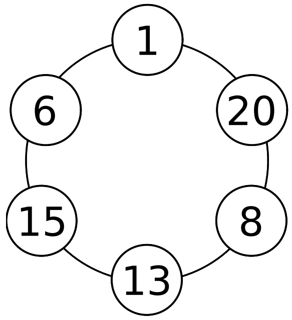


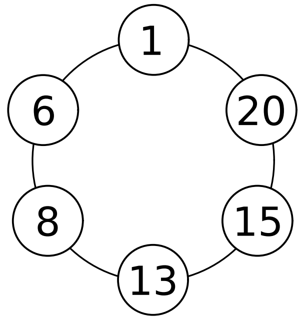
We will now determine all the different arrangements by looking at various cases. Note that in order for two arrangements to be different, at least some of the numbers need to be adjacent to different numbers.

Now, consider the possibilities for the numbers adjacent to 1. Since 6, 13, and 20 are the only numbers in our list that add with 1 to make a multiple of 7, there are three possible cases: 1 adjacent to 6 and 20, 1 adjacent to 6 and 13, and 1 adjacent to 13 and 20. We consider each case separately.

Case 1: 1 is adjacent to 6 and 20

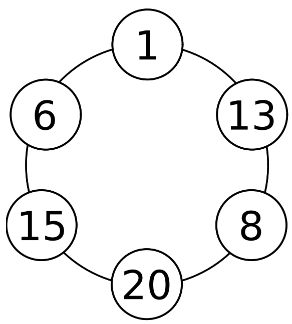
In this case, we can see from our table that 13 must be adjacent to 15 and 8, since 1 is no longer available. The two different ways to write such a circle are shown below.

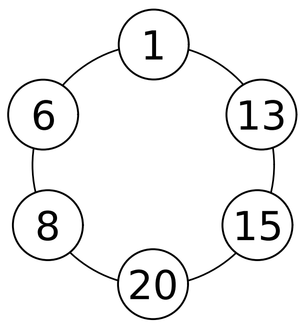




Case 2: 1 is adjacent to 6 and 13

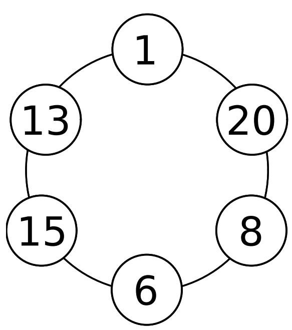
In this case, we can see from our table that 20 must be adjacent to 15 and 8, since 1 is no longer available. The two different ways to write such a circle are shown below.

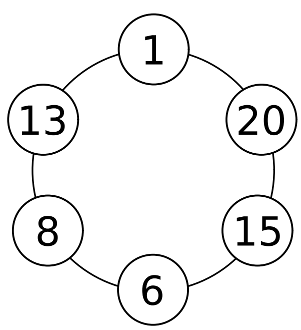




Case 3: 1 is adjacent to 13 and 20

In this case, we can see from our table that 6 must be adjacent to 15 and 8, since 1 is no longer available. The two different ways to write such a circle are shown below.





Therefore, we have found that there are 6 different arrangements. These are the arrangements shown in Cases 1, 2 and 3 above.

CEMC at Home

Grade 9/10 - Friday, May 8, 2020
Unsolved Problems

Odd Perfect Numbers

A *perfect number* is a positive integer that is equal to the sum of its *proper* positive divisors. A divisor is proper if it is smaller than the number itself. For example, the proper positive divisors of 6 are 1, 2, and 3. Since $1 + 2 + 3 = 6$, 6 is a perfect number. The proper divisors of 28 are 1, 2, 4, 7, and 14. Since $1 + 2 + 4 + 7 + 14 = 28$, 28 is also a perfect number.

(a) Is 120 a perfect number?

(b) Is 496 a perfect number?

(c) Are there any odd perfect numbers?

The answer to part (c) is currently, *we don't know!* The existence of an odd perfect number is an unsolved problem. To date, mathematicians have checked all positive integers up to at least 10^{1500} and have not yet found a single positive integer that is both odd and a perfect number. Does this mean that no such number exists? To conclusively solve this problem of whether or not an odd perfect number exists, we need to either find an example of an odd perfect number, or come up with a justification (a proof) that all perfect numbers are even. To date, we do not have either.

The Goldbach Conjecture

Consider the integer 16. It can be expressed as the sum of two prime numbers: $16 = 5 + 11$.

There is more than one way to express 16 *as the sum of two prime numbers. Can you find them all?*

Remember that a prime number is an integer p that is greater than 1 and whose only positive divisors are 1 and p (itself). The first few prime numbers are $2, 3, 5, 7, 11, \ldots$

What numbers can be expressed as the sum of two prime numbers in at least one way?

(a) Express 34 as the sum of two prime numbers.

(b) Express 52 as the sum of two prime numbers.

(c) Explain why 41 cannot be written as the sum of two prime numbers.

(d) Choose any even integer greater than 2 and express it as the sum of two prime numbers.

The Goldbach conjecture states that every even integer greater than 2 can be expressed as the sum of two prime numbers. It is called a conjecture and not a theorem because all evidence uncovered so far suggests it is true, but it has not been proven. To date, mathematicians have checked all even integers greater than 2 up to around 10^{18}, and have found a way to express each of these integers as the sum of two prime numbers. Does this mean that all even integers greater than 2 must have this property? To conclusively solve this problem, we need to either find an even number greater than 2 that *cannot* be written as the sum of two prime numbers, or come up with a justification (a proof) that all even numbers greater than 2 *can* be written in this way. To date, we do not have either.

Hailstone Sequence

Consider a sequence where the first term is a positive integer, n, and each subsequent term is obtained as follows:

- If the previous term is *even*, then the next term is one half of the previous term.
- If the previous term is *odd*, then the next term is three times the previous term, plus one.

This sequence is known as a *hailstone sequence* and the Collatz conjecture states that no matter what positive integer, n, is chosen for the first term, the sequence will always eventually reach 1. It is called a conjecture and not a theorem because it believed by many but has not been proven.

If you have experience programming, try writing a computer program that takes as input a positive integer, n, and outputs the hailstone sequence, stopping when the sequence reaches 1.

If the program doesn't stop, can you conclude the sequence never reaches 1? Why or why not?

Alternatively, you can experiment with the sequence using a computer program that we have written in Python, by following the instructions below.

Instructions:

1. Open this webpage in one tab of your internet browser. You should see Python code.
2. Open this free online Python interpreter in another tab. You should see a middle panel labelled *main.py*.
3. Copy the code and paste it into the middle panel of the interpreter.
4. Hit *run*. This allows you to interact with the program using the black panel on the right. Enter a positive integer and observe the hailstone sequence.
5. If you encounter an error, or you want to explore a different seqeuence, you can hit *run* to begin again.

More Info:

Check out the CEMC at Home webpage on Friday, May 15 for a discussion of these questions.

CEMC at Home

Grade 9/10 - Friday, May 8, 2020
Unsolved Problems - Solution

Odd Perfect Numbers

(a) Is 120 a perfect number?

The proper divisors of 120 are 1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30, 40, and 60.
Since $1+2+3+4+5+6+8+10+12+15+20+24+30+40+60 = 240$ and not 120, 120 is not a perfect number.

(b) Is 496 a perfect number?

The proper divisors of 496 are 1, 2, 4, 8, 16, 31, 62, 124, and 248.
Since $1+2+4+8+16+31+62+124+248 = 496$, 496 is a perfect number.

(c) Are there any odd perfect numbers?

The answer to this question is currently unknown. It is known that there are no odd perfect numbers between 1 and 10^{1500}.

The Goldbach Conjecture

(a) Express 34 as the sum of two prime numbers.

There are four different ways to express 34 as the sum of two prime numbers:

$$34 = 3 + 31 = 5 + 29 = 11 + 23 = 17 + 17$$

(b) Express 52 as the sum of two prime numbers.

There are three different ways to express 52 as the sum of two prime numbers:

$$52 = 5 + 47 = 11 + 41 = 23 + 29$$

(c) Explain why 41 cannot be written as the sum of two prime numbers.

You could write out a list of all primes less than 41 and check that no pair adds to 41. A faster way to approach this problem is as follows: Since 41 is odd, if 41 can be written as a sum of two primes then it must be an even prime plus an odd prime. The only even prime is 2 and so the only possibility is $41 = 2 + p$ where p is an odd prime. But we can see that p must be $41 - 2 = 39$ which is not prime.

(d) Choose any even integer greater than 2 and express it as the sum of two prime numbers.

It is not known whether there exists a general strategy that can be followed to write any even integer greater than 2 as the sum of two prime numbers. It is known that there is a way to write any even integer from 4 to around 10^{18} as the sum of two primes numbers.

Hailstone Sequence

Consider a computer program that takes as input a positive integer, n, and outputs the hailstone sequence, stopping when the sequence reaches 1. *If the program doesn't stop, can you conclude the sequence never reaches 1? Why or why not?*

If the program doesn't stop you cannot conclude that the sequence never reaches 1. That *might* be the reason why it never stopped, but there are two other possibilities as well.

- There might be an error in the program. For example, the sequence might not be generated correctly.
- You might not have waited long enough. It is possible that the number 1 is a term in the sequence but it does not appear for a very long time. Perhaps the program would stop eventually, given enough time.

You may have noticed that mathematicians have checked many more numbers in an effort to find an odd perfect number (up to around 10^{1500}*) than they have checked to see if the Goldbach conjecture is true (up to around* 10^{18}*). You may want to do some research to get an idea of why this might be the case. Is there some reason to believe that it is much easier to check if a number is perfect than to check if a number is the sum of two prime numbers? How difficult is it to check these conditions for very large numbers?*