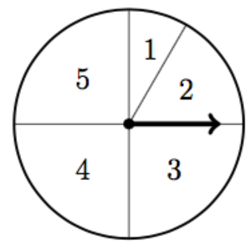
CEMC at Home

Grade 11/12 - Monday, May 4, 2020
Contest Day 1

Today's resource features two questions from the 2020 CEMC Mathematics Contests.

2020 Canadian Team Mathematics Contest, Individual Problem #4

A spinner was created by drawing five radii from the centre of a circle. The first four radii divide the circle into four equal wedges. The fifth radius divides one of the wedges into two parts, one having twice the area of the other. The five wedges are labelled as pictured with the wedge labeled by 2 having twice the area of the wedge labeled by 1. Determine the probability of spinning an odd number.



2020 Euclid Contest, #4(a)

The positive integers a and b have no common divisor larger than 1. If the difference between b and a is 15 and $\frac{5}{9} < \frac{a}{b} < \frac{4}{7}$, what is the value of $\frac{a}{b}$?

More Info:

Check out the CEMC at Home webpage on Monday, May 11 for solutions to the Contest Day 1 problems.

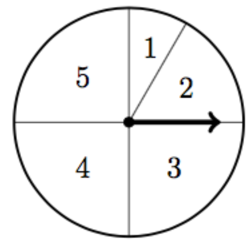
CEMC at Home

Grade 11/12 - Monday, May 4, 2020
Contest Day 1 - Solution

Solutions to the two contest problems are provided below, including a video for the second problem.

2020 Canadian Team Mathematics Contest, Individual Problem #4

A spinner was created by drawing five radii from the centre of a circle. The first four radii divide the circle into four equal wedges. The fifth radius divides one of the wedges into two parts, one having twice the area of the other. The five wedges are labelled as pictured with the wedge labeled by 2 having twice the area of the wedge labeled by 1. Determine the probability of spinning an odd number.



Solution:

The odd numbers on the spinner are 1, 3, and 5. The wedges labelled by 3 and 5 each take up $\frac{1}{4}$ of the spinner and so each will be spun with a probability of $\frac{1}{4}$. If we let the probability of spinning 1 be x, then we have that the probability of spinning 2 is $2x$. The probability of spinning either 1 or 2 is $\frac{1}{4}$, which means $x + 2x = \frac{1}{4}$ or $3x = \frac{1}{4}$ so $x = \frac{1}{12}$.

Therefore, the probability of spinning an odd number is $\frac{1}{4} + \frac{1}{4} + \frac{1}{12} = \frac{7}{12}$.

2020 Euclid Contest, #4(a)

The positive integers a and b have no common divisor larger than 1. If the difference between b and a is 15 and $\dfrac{5}{9} < \dfrac{a}{b} < \dfrac{4}{7}$, what is the value of $\dfrac{a}{b}$?

Solution:

Since $\dfrac{a}{b} < \dfrac{4}{7}$ and $\dfrac{4}{7} < 1$, then $\dfrac{a}{b} < 1$.

Since a and b are positive integers, then $a < b$.

Since the difference between a and b is 15 and $a < b$, then $b = a + 15$.

Therefore, we have $\dfrac{5}{9} < \dfrac{a}{a+15} < \dfrac{4}{7}$.

We multiply both sides of the left inequality by $9(a+15)$ (which is positive) to obtain $5(a+15) < 9a$ from which we get $5a + 75 < 9a$ and so $4a > 75$.

From this, we see that $a > \dfrac{75}{4} = 18.75$.

Since a is an integer, then $a \geq 19$.

We multiply both sides of the right inequality by $7(a+15)$ (which is positive) to obtain $7a < 4(a+15)$ from which we get $7a < 4a + 60$ and so $3a < 60$.

From this, we see that $a < 20$.

Since a is an integer, then $a \leq 19$.

Since $a \geq 19$ and $a \leq 19$, then $a = 19$, which means that $\dfrac{a}{b} = \dfrac{19}{34}$.

Video

Visit the following link for a discussion of two different approaches to solving the second contest problem: https://youtu.be/phNdHo5mE2g.

CEMC at Home

Grade 11/12 - Tuesday, May 5, 2020
Factoring Polynomials without Division

When solving problems we may encounter a polynomial with integer coefficients that needs to be factored. You may have learned some techniques for factoring polynomials that use long division of polynomials. In this activity we will factor some polynomials without using long division.

> **Definition:** Suppose we have a polynomial in the variable x. If the polynomial evaluates to 0 when $x = a$, then we say that a is a *root* of the polynomial.

> **The Factor Theorem:** If a is a root of a polynomial, then $x - a$ is a factor of the polynomial.

Example 1

The number $x = 3$ is a root of the polynomial $x^2 - x - 6$ since $3^2 - 3 - 6 = 0$. The factor theorem tells us that the polynomial $x - 3$ is a factor of the polynomial $x^2 - x - 6$. We can check that indeed $x^2 - x - 6 = (x - 3)(x + 2)$.

Example 2

The number $x = -1$ is a root of the polynomial x^3+5x^2+8x+4 since $(-1)^3+5(-1)^2+8(-1)+4 = 0$. The factor theorem tells us that the polynomial $x - (-1) = x + 1$ is a factor of the polynomial $x^3 + 5x^2 + 8x + 4$. We can check that indeed $x^3 + 5x^2 + 8x + 4 = (x + 1)(x^2 + 4x + 4)$.
How might we find this other quadratic factor?

In this activity, we will focus on factoring polynomials for which all but possibly two of the roots of the polynomial are integers; however, the techniques for factoring that we present below can also be useful in other situations.

Factoring Method

How do we go about factoring a polynomial with integer coefficients?

Let's say we are factoring the cubic polynomial $2x^3 - x^2 - 7x + 6$. If $x = a$ is a root of this polynomial, then $x - a$ is a factor of the polynomial. If a is an integer, then when we factor out $x - a$, we are left with some quadratic polynomial $Ax^2 + Bx + C$ with A, B, and C integers as shown

$$2x^3 - x^2 - 7x + 6 = (x - a)(Ax^2 + Bx + C)$$

If we expand the product on the right side and compare its terms to the like terms on the left side, we observe the following:

- The only term on the right side without an x in it will be the term $-aC$. This means $6 = -aC$. Since a and C are both integers, a must be a factor of 6.
- The only term on the right side with a power of x^3 comes from multiplying the term x by the term Ax^2. This means the term $2x^3$ must be equal to the term Ax^3 and so $A = 2$.
- There are two terms on the right side with a power of x^2, and they come from multiplying the term x by the term Bx and the term $(-a)$ by the term Ax^2. This means the term $-x^2$ on the left must be equal to $(x)(Bx) + (-a)(Ax^2)$ or $Bx^2 - aAx^2$.

Using these three observations, we can factor the polynomial completely! Start by testing all of the factors of 6 to find an integer root $x = a$, and then use this value of a along with the other two observations to solve for the coefficients A, B, and C. The full process is outlined in the examples below.

Example 3: Factor the cubic polynomial $2x^3 - x^2 - 7x + 6$.

$2x^3 - x^2 - 7x + 6$ $= (x-1)(Ax^2 + Bx + C)$	The factors of 6 are $\pm1, \pm2, \pm3$ and ±6. Using these factors, we determine that 1 is a root of the polynomial and so $x - 1$ is a factor. When we factor out $x - 1$ we will be left with a quadratic which we will call $Ax^2 + Bx + C$.
$= (x-1)(2x^2 + Bx - 6)$	The $2x^3$ term from our original polynomial comes from multiplying x by Ax^2. Since $2x^3$ equals Ax^3 we must have $A = 2$. The constant term 6 from our original polynomial comes from multiplying -1 by C and so $C = -6$.
$= (x-1)(2x^2 + x - 6)$	The $-x^2$ from our original polynomial comes from multiplying x by Bx and adding it to -1 times $2x^2$. Since $-x^2$ equals $Bx^2 - 2x^2$, we must have $B = 1$. *Note that we didn't use the $-7x$ term from our original polynomial, but it can be used to check that we didn't make a mistake.*
$= (x-1)(x+2)(2x-3)$	Finally, we factor the resulting quadratic using standard factoring techniques.

Example 4: Factor the quartic polynomial $6x^4 - 7x^3 - 13x^2 + 4x + 4$.

$6x^4 - 7x^3 - 13x^2 + 4x + 4$ $= (x-2)(Ax^3 + Bx^2 + Cx + D)$	The factors of 4 are $\pm1, \pm2$ and ±4. Using these factors we determine that 2 is a root of our polynomial and so $x - 2$ is a factor.
$= (x-2)(6x^3 + Bx^2 + Cx - 2)$	We use the $6x^4$ term from our original polynomial to determine that $A = 6$ and the constant term 4 from our original polynomial to determine that $D = -2$. *Notice that $6x^4$ equals $(x)(Ax^3)$ and 4 equals $(-2)(D)$.*
$= (x-2)(6x^3 + 5x^2 + Cx - 2)$	We use the $-7x^3$ term from our original polynomial to determine that $B = 5$. *Notice that $-7x^3$ equals $(-2)(6x^3) + (x)(Bx^2)$.*
$= (x-2)(6x^3 + 5x^2 - 3x - 2)$	We use the $4x$ term from our original polynomial to determine that $C = -3$. *Notice that $4x$ equals $(x)(-2) + (-2)(Cx)$.*
$= (x-2)(x+1)(Ex^2 + Fx + G)$	For the rest of our solution we ignore the $(x - 2)$ factor and focus on factoring the cubic $6x^3 + 5x^2 - 3x - 2$. *Remember that we have already discussed how to factor a cubic.* The factors of -2 are ±1 and ±2. Using these factors we determine that -1 is a root of this cubic and so $x + 1$ is a factor, and then we proceed as in Example 3.

Use these ideas to solve the following problems.

1. Factor $x^3 + 7x^2 + 11x + 5$.
2. Factor $x^4 + 5x^3 - 3x^2 - 17x - 10$. *Hint. Start by verifying that $x = 2$ is a root.*
3. Factor $4x^4 - 16x^3 + x^2 + 39x - 18$.
4. Given that $(Ax^2 + Bx + C)(3x^2 + Dx - 2) = 6x^4 + 3x^3 - 40x^2 + 2x + 4$, determine the values of A, B, C and D.

More Info:

Check out the CEMC at Home webpage on Tuesday, May 12 for a solution to Factoring Polynomials without Division.

When finding the roots of these polynomials we looked at a special case of the Rational Roots Theorem. To learn more about the Rational Roots Theorem check out the lesson Factoring Polynomials Using the Factor Theorem from the CEMC Advanced Functions and Pre-Calculus courseware.

CEMC at Home

Grade 11/12 - Tuesday, May 5, 2020
Factoring Polynomials without Division - Solution

We use the strategy outlined in the activity to factor the first cubic.

Question 1: Factor $x^3 + 7x^2 + 11x + 5$

Solution:

$$x^3 + 7x^2 + 11x + 5$$
$$= (x+1)(Ax^2 + Bx + C)$$

The factors of 5 are ± 1 and ± 5. Using these factors, we determine that -1 is a root and so $x - (-1) = x + 1$ is a factor.
Note that -5 is also a root and so we could instead start with the factor $x + 5$.

$$= (x+1)(x^2 + Bx + 5)$$

The x^3 term from our original polynomial comes from multiplying x by Ax^2 and so $A = 1$. The constant term 5 from our original polynomial comes from multiplying 1 by C and so $C = 5$.

$$= (x-1)(x^2 + 6x + 5)$$

The $7x^2$ term from our original polynomial comes from multiplying x by Bx and adding the result to 1 times x^2. In other words, the term $7x^2$ equals $Bx^2 + x^2$ or $(B+1)x^2$. Comparing coefficients gives $7 = B + 1$ or $B = 6$.

$$= (x+1)(x+1)(x+5)$$

Factor the resulting quadratic.

We use a similar strategy to factor a quartic polynomial with integer coefficients. In this case, once we find an integer root a, and factor out the corresponding linear factor $x - a$, we will be left with a cubic polynomial $Ax^3 + Bx^2 + Cx + D$ with A, B, C, and D integers for which we can solve.

Question 2: Factor $x^4 + 5x^3 - 3x^2 - 17x - 10$

Solution:

$$x^4 + 5x^3 - 3x^2 - 17x - 10$$
$$= (x-2)(Ax^3 + Bx^2 + Cx + D)$$

Remember that we were told in the question that 2 is a root. You should verify this. This means that $x - 2$ is a factor. We factor this term out and are left with a cubic polynomial as shown.

$$= (x-2)(x^3 + Bx^2 + Cx + 5)$$

The x^4 term from our original polynomial comes from multiplying x by Ax^3 and so $A = 1$. The constant term -10 from our original polynomial comes from multiplying -2 by D and so $D = 5$.

$$= (x-2)(x^3 + 7x^2 + Cx + 5)$$

The $5x^3$ term from our original polynomial comes from multiplying x by Bx^2 and adding the result to -2 times x^3. In other words, the term $5x^3$ equals $(x)(Bx^2) + (-2)(x^3)$ or $Bx^3 - 2x^3$. Comparing coefficients, we have $5 = B - 2$ and so $B = 7$.

$$= (x-2)(x^3 + 7x^2 + 11x + 5)$$

The $-17x$ term from our original polynomial comes from multiplying x by 5 and adding the result to -2 times Cx. In other words, the term $-17x$ equals $(x)(5) + (-2)(Cx)$ or $5x - 2Cx$. This means $-17 = 5 - 2C$ and so $C = 11$.

$$= (x-2)(x+1)(x+1)(x+5)$$

Notice that this cubic is identical to the cubic from Question 1 and so we can use the factorization found earlier to finish factoring this quartic.

We use a similar strategy to factor the next quartic polynomial, but this time we need to find a root on our own. Again, it can be shown that if a is an integer root of the polynomial, then a must be a factor of the constant term, -18.

Question 3: Factor $4x^4 - 16x^3 + x^2 + 39x - 18$.

Solution:

$4x^4 - 16x^3 + x^2 + 39x - 18$ $= (x-2)(Ax^3+Bx^2+Cx+D)$	The factors of -18 are $\pm1, \pm2, \pm3, \pm6, \pm9$, and ±18. Using these factors, we determine that 2 is a root and so $x - 2$ is a factor.
$= (x-2)(4x^3+Bx^2+Cx+9)$	The $4x^4$ term from our original polynomial comes from multiplying x by Ax^2 and so $A = 4$. The constant term -18 from our original polynomial comes from multiplying -2 by D and so $D = 9$.
$= (x-2)(4x^3-8x^2-15x+9)$	The $-16x^3$ term from our original polynomial comes from multiplying x by Bx^2 and adding the result to -2 times $4x^3$. Therefore, $B = -8$. The $39x$ term from our original polynomial comes from multiplying x by 9 and adding the result to -2 times Cx. Therefore, $C = -15$.
$= (x-2)(x-3)(Ex^2+Fx+G)$	Now we factor the cubic. The factors of 9 are $\pm1, \pm3$, and ±9. Using these factors, we determine that 3 is a root and so $x - 3$ is a factor of the cubic.
$= (x-2)(x-3)(4x^2+4x-3)$	Using the $4x^3$ term, we determine that $E = 4$ and using the constant term 9, we determine that $G = -3$. Using the $-8x^2$ term, we determine that $F = 4$.
$= (x-2)(x-3)(2x-1)(2x+3)$	Factor the resulting quadratic.

Finally, we use what we have learned to solve the following problem.

Question 4: Given that $(Ax^2 + Bx + C)(3x^2 + Dx - 2) = 6x^4 + 3x^3 - 40x^2 + 2x + 4$, determine the values of A, B, C and D.

Solution:

Consider the equality $(Ax^2 + Bx + C)(3x^2 + Dx - 2) = 6x^4 + 3x^3 - 40x^2 + 2x + 4$.

The $6x^4$ term on the right side must come from multiplying the term Ax^2 by the term $3x^2$ on the left side. Since $6x^4$ is equal to $3Ax^4$ we must have $3A = 6$ and so $A = 2$.

The constant term 4 on the right side must come from multiplying C by -2 on the left side. Since 4 is equal to $-2C$ we must have $C = -2$.

This means we have $(2x^2 + Bx - 2)(3x^2 + Dx - 2) = 6x^4 + 3x^3 - 40x^2 + 2x + 4$.

The $3x^3$ term must come from multiplying the term $2x^2$ by the term Dx and adding the result to the product of Bx and $3x^2$. Since $2x^2$ is equal to $2Dx^3 + 3Bx^3$, we must have $2D + 3B = 3$.

Similarly, we can show that $2x$ must be equal to $(Bx)(-2) + (-2)(Dx) = -2Bx - 2Dx$. This means $-2B - 2D = 2$.

Since $2D+3B = 3$ and $-2B-2D = 2$, adding the two equations gives $B = 5$. We can then determine that $D = -6$.

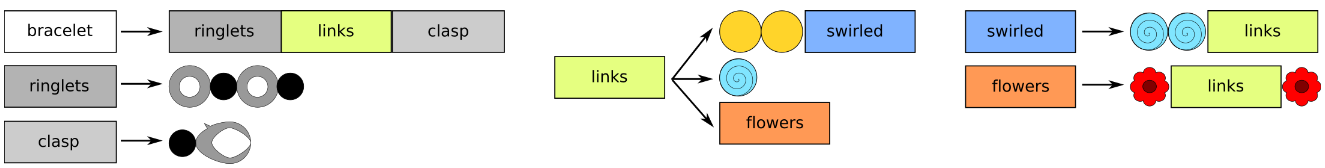
This means we have $A = 2$, $B = 5$, $C = -2$, $D = -6$, and

$$(2x^2 + 5x - 2)(3x^2 - 6x - 2) = 6x^4 + 3x^3 - 40x^2 + 2x + 4$$

CEMC at Home

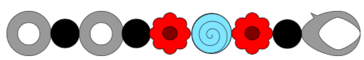
Grade 11/12 - Wednesday, May 6, 2020
Bracelets

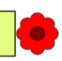
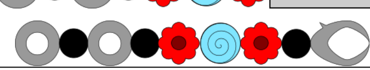
Stephen makes bracelets using the six replacement rules below.



Stephen always starts his pattern with the symbol bracelet. Then, one at a time, he replaces a symbol in the current pattern with a new sequence of symbols based on the rules above. Any symbol that appears on the left side of an arrow can be replaced with a sequence that appears on the right side of a connected arrow. In some, but not all cases, he has a choice about which particular replacement he could make at a particular stage in the process.

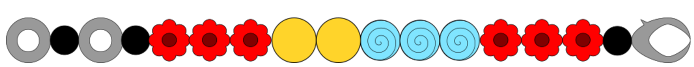
Example

Stephen could make the bracelet  following these steps:

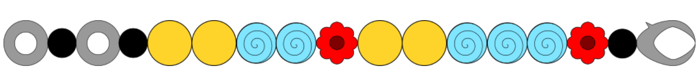
Step	Current Pattern	Explanation
1	bracelet	Stephen always starts with this symbol
2	ringlets links clasp	bracelet is replaced by ringlets links clasp
3	 links clasp	ringlets is replaced by 
4	 flowers clasp	links is replaced by flowers
5	 links  clasp	flowers is replaced by  links 
6	 clasp	links is replaced by 
7		clasp is replaced by 

Problems

1. Give a sequence of steps that Stephen could follow in order to produce the following bracelet:



2. Consider the three bracelets below. Stephen can make exactly two of the three bracelets using the rules. Explain how Stephen can make two of these bracelets, and explain why the remaining bracelet cannot be made using any sequence of steps.

(a) 

(b) 

(c) 

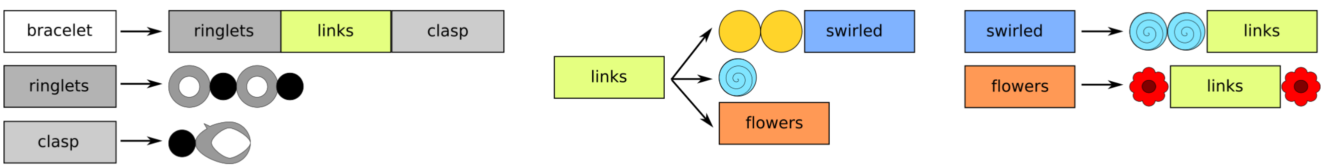
More Info:

Check out the CEMC at Home webpage on Wednesday, May 13 for a solution to Bracelets.

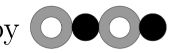
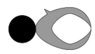
CEMC at Home

Grade 11/12 - Wednesday, May 6, 2020
Bracelets - Solution

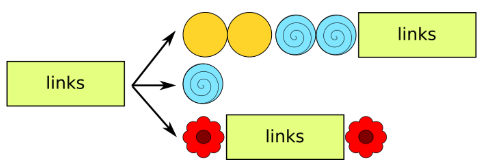
Before we look at particular sequences of symbols, we simplify the original six rules reproduced below.



Observe that five of the six rules do not involve a choice of which pattern to produce.

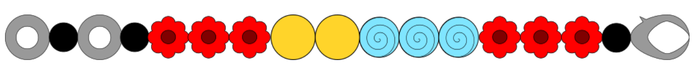
Recall that Stephen always starts with the symbol bracelet and so the next pattern must then be ringlets links clasp. At some point, the leftmost symbol will be replaced by  and the rightmost symbol will be replaced by . This means that to determine if it is possible to make a certain bracelet, we only need to determine if it is possible to create the "middle portion" by starting with links and following the three rules shown above on the right.

Notice that we can substitute the rightmost two rules into the replacement rule for links to give us the following single rule equivalent to the rightmost three original rules.

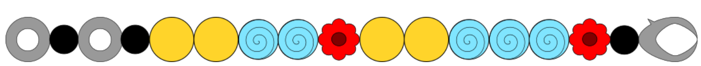


We will call the three choices in the rule above A (top), B (middle), and C (bottom).

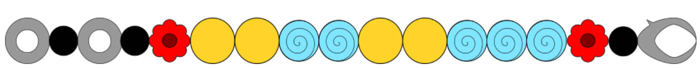
1. Stephen can make the bracelet shown below by making the choices C, C, C, A, B (in that order).



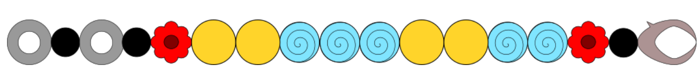
2. Stephen can make bracelets (a) and (c) but not (b).
   Stephen can make the bracelet shown below by making the choices A, C, A, B (in that order).

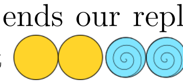
(a) 

Stephen can make the bracelet shown below by making the choices C, A, A, B (in that order).

(c) 

Stephen cannot make the bracelet shown below.

(b) 

To justify this, we read from left to right to see that generating this bracelet would require us to make choice C first and then A. At this point, the next symbol is  which can only be achieved by choosing B next. This ends our replacement choices, but we have the wrong bracelet. (We are missing the rightmost  sequence.)

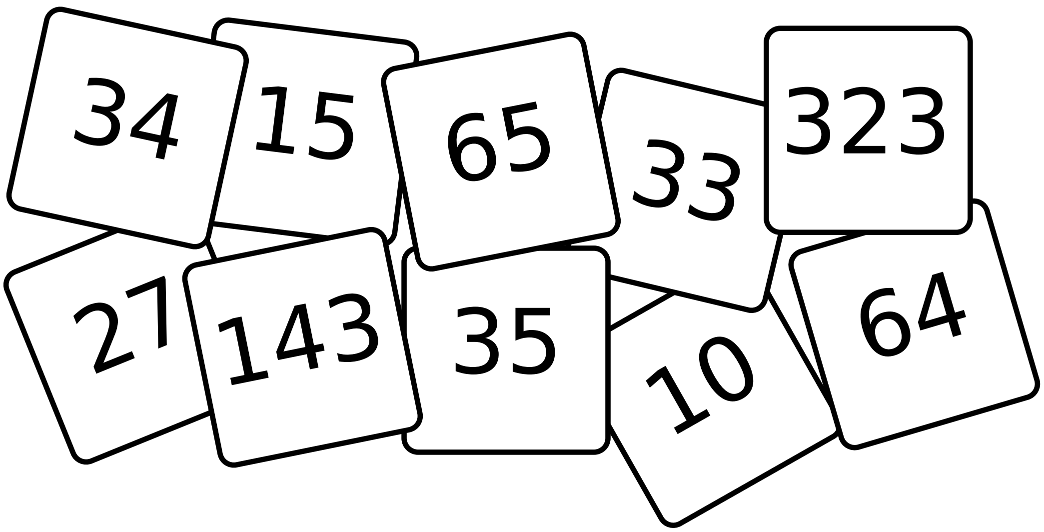
CEMC at Home features Problem of the Week

Grade 11/12 - Thursday, May 7, 2020
The Factor Flip

Dani has 10 cards, each with a number on one side. The numbers on the cards are 10, 15, 27, 33, 34, 35, 64, 65, 143, and 323. The cards are placed on a table with the numbers facing up.

Dani takes a card and flips it over so it is now face down. Of the remaining cards that are still face up, the next card she flips over must have a prime factor in common with the card she last flipped over. She continues in this way until either all the cards have been flipped over, or she is unable to flip any of the cards that remain face up.

List all the possible orders in which Dani can flip the cards so that all cards get flipped over.

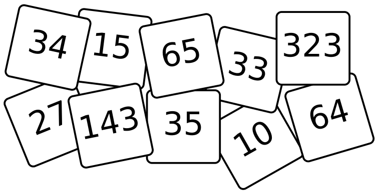


More Info:

Check the CEMC at Home webpage on Thursday, May 14 for the solution to this problem. Alternatively, subscribe to Problem of the Week at the link below and have the solution emailed to you on Thursday, May 14.

This CEMC at Home resource is the current grade 11/12 problem from Problem of the Week (POTW). POTW is a free, weekly resource that the CEMC provides for teachers, parents, and students. Each week, problems from various areas of mathematics are posted on our website and e-mailed to our subscribers. Solutions to the problems are e-mailed one week later, along with a new problem. POTW is available in 5 levels: A (grade 3/4), B (grade 5/6), C (grade 7/8), D (grade 9/10), and E (grade 11/12).

To subscribe to Problem of the Week and to find many more past problems and their solutions visit: https://www.cemc.uwaterloo.ca/resources/potw.php



Problem of the Week

Problem E and Solution

The Factor Flip

Problem

Dani has 10 cards, each with a number on one side. The numbers on the cards are 10, 15, 27, 33, 34, 35, 64, 65, 143, and 323. The cards are placed on a table with the numbers facing up. Dani takes a card and flips it over so it is now face down. Of the remaining cards that are still face up, the next card she flips over must have a prime factor in common with the card she last flipped over. She continues in this way until either all the cards have been flipped over, or she is unable to flip any of the cards that remain face up. List all the possible orders in which Dani can flip the cards so that all cards get flipped over.

Solution

We will start by writing down the prime factors for each of the numbers on the cards.

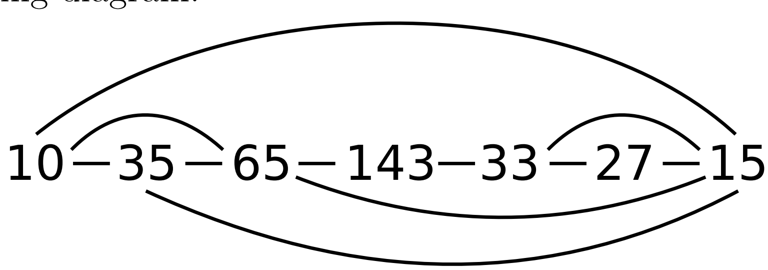
Number	Prime Factors
10	2,5
15	3,5
27	3
33	3,11
34	2,17

Number	Prime Factors
35	5,7
64	2
65	5,13
143	11,13
323	17,19

Notice the number 323 shares a prime factor with only the number 34. That means we must start (or end) with 323.

If we start with 323, then the next number must be 34. From 34, the next number could be 64 or 10. If the next number were 10, then in order to eventually flip over the 64 card, the 64 must follow the 10, and at this point no more cards can be flipped. Therefore, in order to flip all the cards, the first four numbers flipped must be 323, 34, 64, and then 10.

From this point, we can draw lines between numbers that share prime factors to create the following diagram.



We now just need to figure out the number of paths through the diagram, starting at 10 that use each number exactly once.

After 10, the next number can be 35, 65, or 15. In each case, we carefully trace through all possible paths.

Case 1: The number 35 follows the number 10. That gives us the following five possible paths.

35, 15, 65, 143, 33, 27
35, 15, 27, 33, 143, 65
35, 65, 15, 27, 33, 143
35, 65, 143, 33, 27, 15
35, 65, 143, 33, 15, 27

Case 2: The number 65 follows the number 10. That gives us the following two possible paths.

65, 143, 33, 27, 15, 35
65, 35, 15, 27, 33, 143

Case 3: The number 15 follows the number 10. That gives us the following two possible paths.

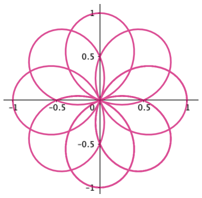
15, 27, 33, 143, 65, 35
15, 35, 65, 143, 33, 27

Starting with the card numbered 323, we have found that there is a total of nine orders for flipping the cards:

323, 34, 64, 10, 35, 15, 65, 143, 33, 27
323, 34, 64, 10, 35, 15, 27, 33, 143, 65
323, 34, 64, 10, 35, 65, 15, 27, 33, 143
323, 34, 64, 10, 35, 65, 143, 33, 27, 15
323, 34, 64, 10, 35, 65, 143, 33, 15, 27
323, 34, 64, 10, 65, 143, 33, 27, 15, 35
323, 34, 64, 10, 65, 35, 15, 27, 33, 143
323, 34, 64, 10, 15, 27, 33, 143, 65, 35
323, 34, 64, 10, 15, 35, 65, 143, 33, 27

Each of these can be reversed, so the total number of possible orders is 18.

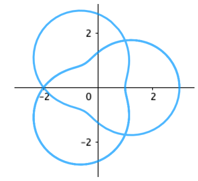
Therefore, there are 18 orders in which Dani can flip the cards so that all cards get flipped over. They are the 9 orders listed above and the reverse of each.



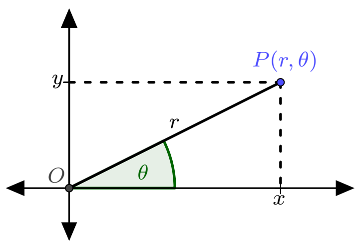
CEMC at Home

Grade 11/12 - Friday, May 8, 2020

Polar Curves



Last week we learned about a different coordinate system for the plane: the *Polar Coordinate System*. Remind yourself about how to work with polar coordinates before you try this activity.



Relationships between Cartesian coordinates and polar coordinates of a point in the plane

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$r = \sqrt{x^2+y^2}$$

Why might we want to view the plane through the lens of polar coordinates? One reason is that simple equations of the form $r = f(\theta)$ involving polar coordinates can lead to interesting graphs!

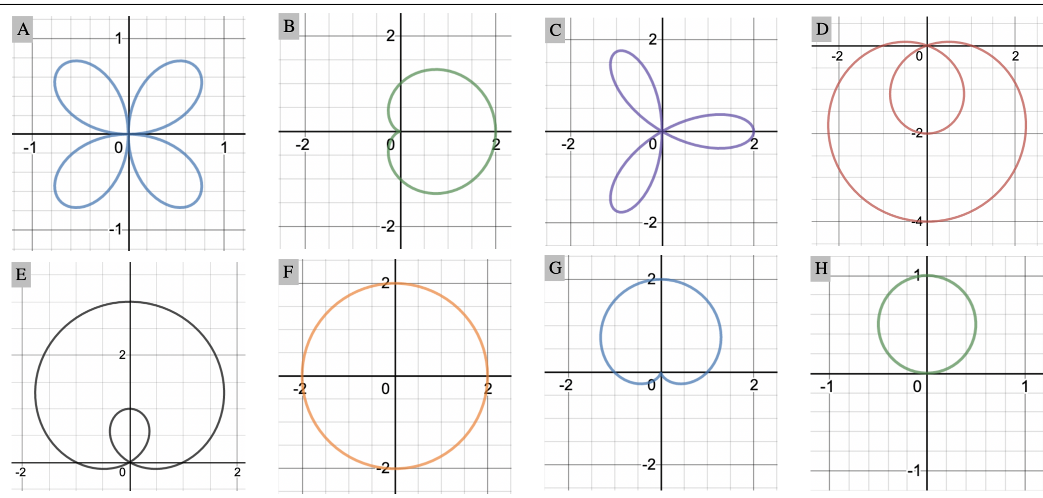
Let f be a function on the real numbers. The graph of the polar equation $r = f(\theta)$ consists of all points in the plane that have polar coordinates, (r, θ), that satisfy the relation $r = f(\theta)$.

Activity

Consider the following polar equations and the graphs below. Exactly one of the graphs corresponds to each equation. Can you match each equation with its graph? Think about the following techniques:

- Plot some key points on the curve. For example, when $\theta = \frac{\pi}{2}$, what is the value of r?
- Remember that $-1 \le \sin\theta \le 1$ and $-1 \le \cos\theta \le 1$. What does this mean for the range of r?
- Think about how r changes as θ changes. (See the next pages for help with this.)
- How are points with a negative r-coordinate plotted? (See the next pages for help with this.)

1. $r = 2$
2. $r = \sin\theta$
3. $r = 1 + \cos\theta$
4. $r = 1 + \sin\theta$
5. $r = 1 + 2\sin\theta$
6. $r = 1 - 3\sin\theta$
7. $r = \sin(2\theta)$
8. $r = 2\cos(3\theta)$



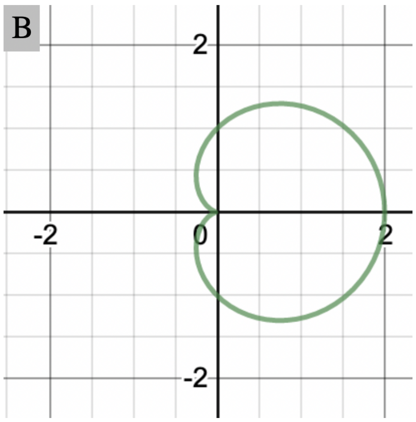
Example 1: Look at graph F. You should recognize this as a circle centred at the origin with radius 2. The points on this curve must be the points having polar coordinates that look like $(2, \theta)$ for some θ (2 units from the origin, at any angle). This means graph F must be matched with equation $r = 2$.

Note that we could also determine what the graph of $r = 2$ must look like by transforming this polar equation into a Cartesian equation. Since $r = \sqrt{x^2+y^2}$, a point's polar coordinates satisfy the equation $r = 2$ exactly when its Cartesian coordinates satisfy the equation $\sqrt{x^2+y^2} = 2$. Squaring both sides reveals the equation $x^2 + y^2 = 4$ which describes the circle shown!

Can you match each of the eight graphs with one of the eight equations without actually trying to sketch the complete graphs of the polar equations? Read the following example to get you started on possible matching strategies that do not involve graphing the polar equations.

Example 2

Consider graph B. Given that this graph is **matched with one of the five equations below**, can you figure out which one by eliminating all but one equation?



1. $r = 2$
2. $r = \sin\theta$
3. $r = 1 + \cos\theta$
4. $r = 1 + \sin\theta$
7. $r = \sin(2\theta)$

Let's see if we can use only the range of r to eliminate several possibilities.

1. Graph B cannot be the graph of $r = 2$: We have already determined that $r = 2$ is matched with another graph.
2. Graph B cannot be the graph of $r = \sin\theta$: Since $\sin\theta$ cannot be larger than 1, no points on the graph of this polar equation can be more than 1 unit from the origin. Graph B has at least one point 2 units from the origin.
3. Graph B might be the graph of $r = 1 + \cos\theta$: Since $-1 \leq \cos\theta \leq 1$, we have $0 \leq 1 + \cos\theta \leq 2$ and so the points on this graph should all be within 2 units of the origin or exactly 2 units from the origin. This is true of the graph B.
4. Graph B might be the graph of $r = 1 + \sin\theta$: Similar reasoning as in 3.
7. Graph B cannot be the graph of $r = \sin(2\theta)$: Similar reasoning as in 2.

By considering the range of r we have narrowed down the choices to two equations: $r = 1 + \cos\theta$ and $r = 1 + \sin\theta$.

Can you see which one must be the correct equation for Graph B? Try plotting a few points.

For equation 3: When $\theta = 0$ we have $r = 1 + \cos 0 = 2$. This matches the graph above.

For equation 4: When $\theta = 0$ we have $r = 1 + \sin 0 = 1$. This does not match the graph above.

This tells us that the equation must be 3: $r = 1 + \cos\theta$.

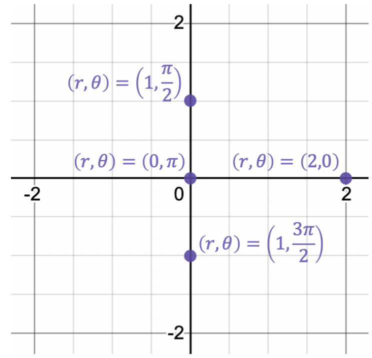
On the next page we will discuss how to sketch the graph of the polar equation $r = 1 + \cos\theta$ to see exactly why Graph B above matches this equation. You do not need to sketch this graph to complete the activity, but you may still want to spend some time thinking about why this is the correct graph.

For many of the eight equations, there are pairs (r, θ) with $r < 0$ that satisfy the equation. We discuss how to interpret negative r-coordinates on the last pages of the resource.

Example 3: Sketch the graph of the polar equation $r = 1 + \cos(\theta)$.

Plot a few key points.

- When $\theta = 0$, $r = 2$.
- When $\theta = \frac{\pi}{2}$, $r = 1$.
- When $\theta = \pi$, $r = 0$.
- When $\theta = \frac{3\pi}{2}$, $r = 1$.
- When $\theta = 2\pi$, $r = 2$.

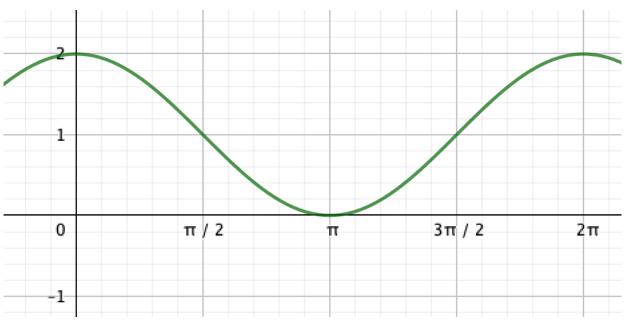


Think about the range of r.

Since $-1 \leq \cos\theta \leq 1$, we must have $0 \leq 1 + \cos\theta \leq 2$. This means all points on the graph must be at most 2 units from the origin.

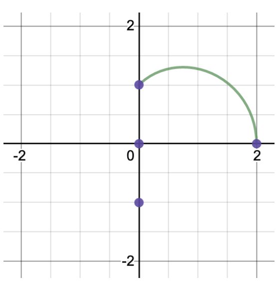
Think about how r changes as θ changes.

Can you describe what happens to r as θ ranges from 0 to 2π? We sketch the graph of $y = 1 + \cos x$ drawn in the usual Cartesian plane. Can you see how to use this information to make the the table?

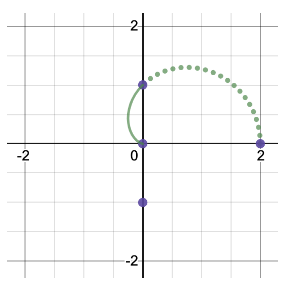


θ	$r = 1 + \cos(\theta)$	Polar Point
0	2	$(2, 0)$
0 to $\frac{\pi}{2}$	r decreases from 2 to 1	
$\frac{\pi}{2}$	1	$(1, \frac{\pi}{2})$
$\frac{\pi}{2}$ to π	r decreases from 1 to 0	
π	0	$(0, \pi)$
π to $\frac{3\pi}{2}$	r increases from 0 to 1	
$\frac{3\pi}{2}$	1	$(1, \frac{3\pi}{2})$
$\frac{3\pi}{2}$ to 2π	r increases from 1 to 2	
2π	2	$(2, 2\pi)$

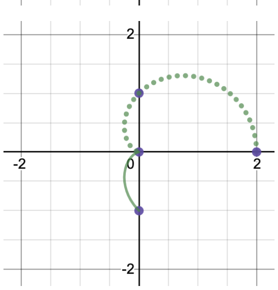
Draw a rough sketch of the curve



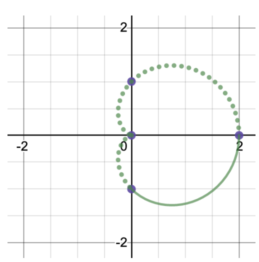
As θ increases from 0 to $\frac{\pi}{2}$, r decreases from 2 to 1. So we connect the polar points $(2, 0)$ and $(1, \frac{\pi}{2})$ through the first quadrant.



As θ increases from $\frac{\pi}{2}$ to π, r decreases from 1 to 0. So we connect the polar points $(1, \frac{\pi}{2})$ and $(0, \pi)$ through the second quadrant.



As θ increases from π to $\frac{3\pi}{2}$, r increases from 0 to 1. So we connect the polar points $(0, \pi)$ and $(1, \frac{3\pi}{2})$ through the third quadrant.



As θ increases from $\frac{3\pi}{2}$ to 2π, r increases from 1 to 2. So we connect the polar points $(1, \frac{3\pi}{2})$ and $(2, 2\pi)$ through the fourth quadrant.

Can you convince yourself that the sketch will take this curved shape? We used technology to plot many points in order to get an accurate curve. Since the function $\cos\theta$ repeats with period 2π, plotting points for more values of θ will just result in drawing this same curve over again!

Example 4: Consider the polar equation $r = 1 + 2\sin\theta$.

Notice that there are values of θ for which the corresponding r is negative. For example, when $\theta = \frac{3\pi}{2}$, we have

$$r = 1 + 2\sin\left(\tfrac{3\pi}{2}\right) = 1 + 2(-1) = -1$$

What does this mean in terms of our graphing activity?

Can we plot points with polar coordinates with negative values of r?

We can extend the definition of polar coordinates to include negative values of r.
How do we interpret the polar coordinates $(1, \frac{\pi}{2})$ versus the polar coordinates $(-1, \frac{\pi}{2})$?

- The fact that they both have the same angle $\frac{\pi}{2}$ tells us that they both describe points that lie on the line passing through the origin and making an angle of $\frac{\pi}{2}$ with the positive x-axis.
- The magnitude of the radii both being 1 tell us that they both describe points that are 1 unit from the origin.
- The different signs tell us that they describe points on opposite sides of the origin. The negative means that we move in the direction *opposite* to the direction defined the ray $\theta = \frac{\pi}{2}$. This means moving in the direction defined by the ray $\theta = \frac{3\pi}{2}$.

So the polar coordinates $(-1, \frac{\pi}{2})$ are equivalent to the polar coordinates $(1, \frac{3\pi}{2})$ and they both represent the Cartesian point $(0, -1)$. Indeed if we use the usual formulas to convert from polar coordinates to Cartesian coordinates, we get the following:

Polar coordinates $(-1, \frac{\pi}{2})$	*Polar coordinates* $(1, \frac{3\pi}{2})$
$x = r\cos\theta = (-1)\cos\left(\frac{\pi}{2}\right) = 0$	$x = r\cos\theta = 1\cos\left(\frac{3\pi}{2}\right) = 0$
$y = r\sin\theta = (-1)\sin\left(\frac{\pi}{2}\right) = -1$	$y = r\sin\theta = 1\sin\left(\frac{3\pi}{2}\right) = -1$

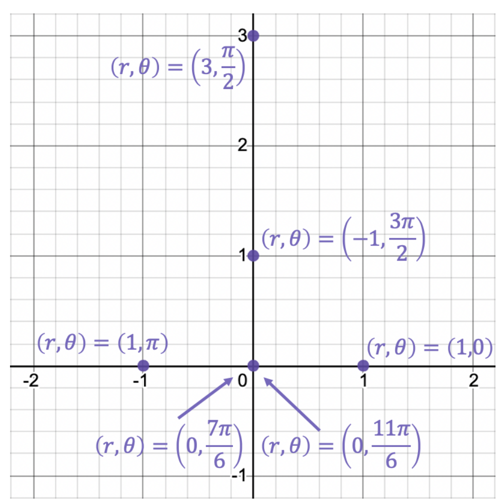
Example 5: Consider the graph of the polar equation $r = 1 + 2\sin\theta$.

Note that it will be important to know where r changes from negative to positive. To find these places, we solve the equation $r = 1 + 2\sin\theta = 0$. *Two solutions are* $\theta = \frac{7\pi}{6}, \frac{11\pi}{6}$.

Plot a few key points.

- When $\theta = 0$ (or $\theta = 2\pi$), $r = 1$.
- When $\theta = \frac{\pi}{2}$, $r = 3$.
- When $\theta = \pi$, $r = 1$.
- When $\theta = \frac{7\pi}{6}$, $r = 0$.
- When $\theta = \frac{3\pi}{2}$, $r = -1$.

  Remember that this pair describes the same point as the pair $\theta = \frac{\pi}{2}$ *and* $r = 1$.
- When $\theta = \frac{11\pi}{6}$, $r = 0$.

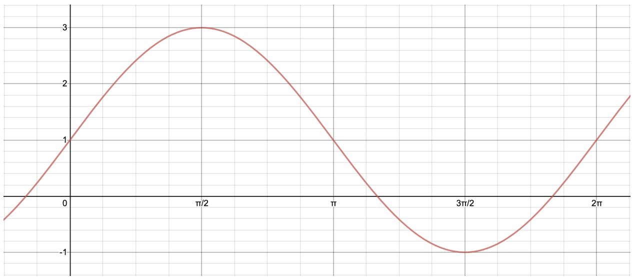


Think about the range of r.

Since $-1 \le \sin\theta \le 1$, we must have $-1 \le 1 + 2\sin\theta \le 3$. Since the magnitude of r must be at most 3, we know that all points on the graph must lie at most 3 units away from the origin.

Think about how r changes as θ changes.

Can you describe what happens to r as θ ranges from 0 to 2π?



θ	$r = 1 + 2\sin(\theta)$	Polar Point
0	1	$(1, 0)$
0 to $\frac{\pi}{2}$	r increases from 1 to 3	
$\frac{\pi}{2}$	3	$(3, \frac{\pi}{2})$
$\frac{\pi}{2}$ to π	r decreases from 3 to 1	
π	1	$(1, \pi)$
π to $\frac{7\pi}{6}$	r decreases from 1 to 0	
$\frac{7\pi}{6}$	0	$(0, \frac{7\pi}{6})$
$\frac{7\pi}{6}$ to $\frac{3\pi}{2}$	r decreases from 0 to -1	
$\frac{3\pi}{2}$	-1	$(-1, \frac{3\pi}{2})$
$\frac{3\pi}{2}$ to $\frac{11\pi}{6}$	r increases from -1 to 0	
$\frac{11\pi}{6}$	0	$(0, \frac{11\pi}{6})$
$\frac{11\pi}{6}$ to 2π	r increases from 0 to 1	
2π	1	$(1, 2\pi)$

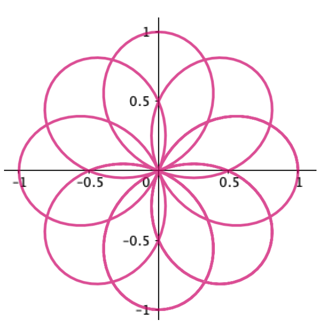
It is not easy to see how to translate the complete information from the table into a sketch of the graph. It takes most people a lot of time to get comfortable sketching these curves when they involve negative values of r. Luckily, you do not need to sketch the *whole* curve in order to figure out which graph matches the equation $r = 1 + 2\sin\theta$. If you can draw a few "pieces" of the graph for $r = 1 + 2\sin\theta$ then you should be able to pick its graph out of the list. In fact, you might be able to pick out the correct graph by using *only* the key points considered in this example!

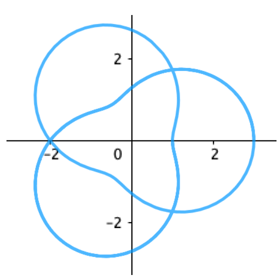
More Info:

Check out the CEMC at Home webpage on Friday, May 15 for a solution to Polar Curves.

You may also want to check out some of the free online graphing calculators for polar curves, like the ones offered by *WolframAlpha* or *Desmos* to verify your answers.

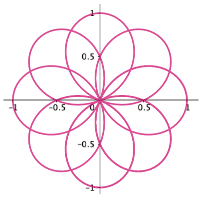
The graphs in the header of the first page of this activity each come from graphing one of the following polar equations. Which equation matches which graph and why?





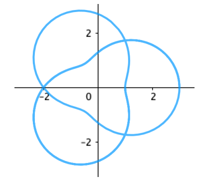
$r = 2 + \cos\left(\frac{3\theta}{2}\right)$

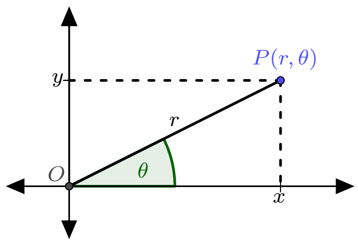
$r = \cos\left(\frac{4\theta}{3}\right)$



CEMC at Home

Grade 11/12 - Friday, May 8, 2020
Polar Curves - Solution





Relationships between Cartesian coordinates and polar coordinates of a point in the plane

$$x = r\cos(\theta)$$
$$y = r\sin(\theta)$$
$$r = \sqrt{x^2 + y^2}$$

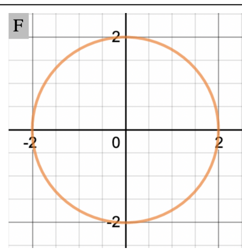
Let f be a function on the real numbers. The graph of the polar equation $r = f(\theta)$ consists of all points in the plane that have polar coordinates, (r, θ), that satisfy the relation $r = f(\theta)$.

Activity

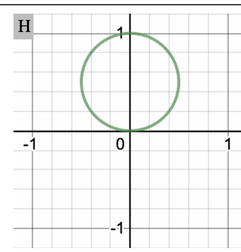
Consider the following polar equations and the graphs below. Exactly one of the graphs corresponds to each equation. Can you match each equation with its graph?

Answers (explanations provided on the pages that follow)

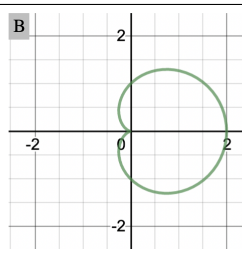
1. $r = 2$



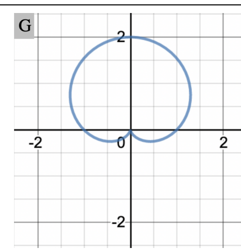
2. $r = \sin\theta$



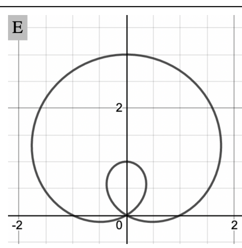
3. $r = 1 + \cos\theta$



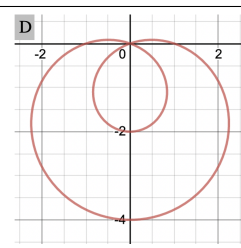
4. $r = 1 + \sin\theta$



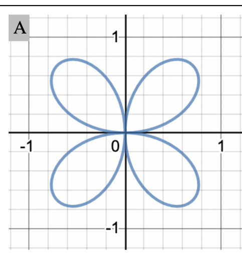
5. $r = 1 + 2\sin\theta$



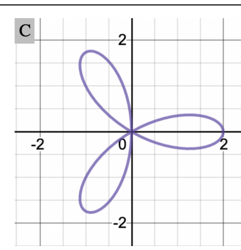
6. $r = 1 - 3\sin\theta$



7. $r = \sin(2\theta)$



8. $r = 2\cos(3\theta)$



Step 1: Think about the range of r. *(This is only one of many possible first steps.)*

For the equations: Using the fact that $-1 \leq \sin\theta \leq 1$ and $-1 \leq \cos\theta \leq 1$, we can determine the range of r values for the polar functions $r = f(\theta)$.

For the graphs: We cannot determine the exact range of r values of the associated equation just by looking at the graph, but we can determine an upper bound on the *magnitude of* r from the graph.

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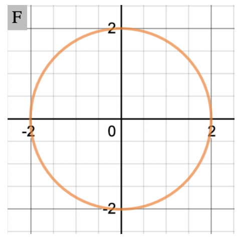
1. $r = 2$

   Range: $2 \leq r \leq 2$



All points appear to be 2 units away from the origin.

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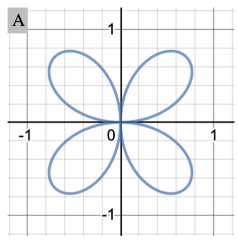
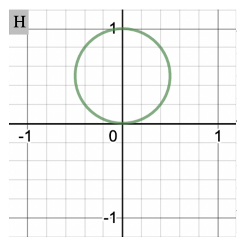
2. $r = \sin\theta$

   Range: $-1 \leq r \leq 1$

7. $r = \sin(2\theta)$

   Range of r: $-1 \leq r \leq 1$





The points that are farthest from the origin appear to be 1 unit away.

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3. $r = 1 + \cos\theta$

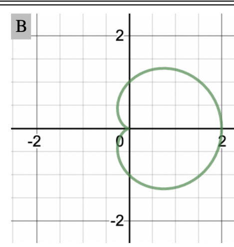
   Range of r: $0 \leq r \leq 2$

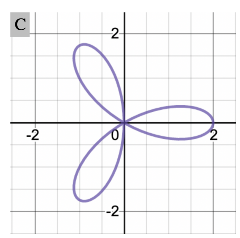
4. $r = 1 + \sin\theta$

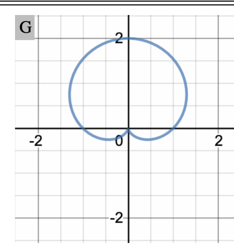
   Range of r: $0 \leq r \leq 2$

8. $r = 2\cos(3\theta)$

   Range of r: $-2 \leq r \leq 2$







The points that are farthest from the origin appear to be 2 units away.

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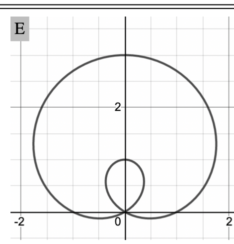
5. $r = 1 + 2\sin\theta$

   Range of r: $-1 \leq r \leq 3$



The points that are farthest from the origin appear to be 3 units away.

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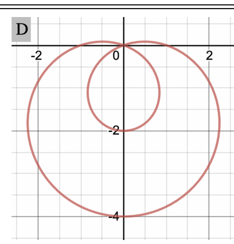
6. $r = 1 - 3\sin\theta$

   Range of r: $-2 \leq r \leq 4$

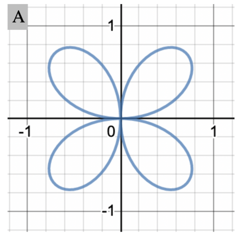


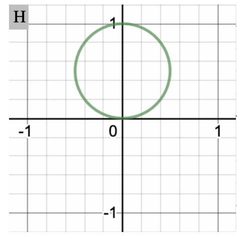
The points that are farthest from the origin appear to be 4 units away.

Step 2: Plot a few key points

Our work on the previous page matches equations 1, 5, and 6 with their graphs, and divides the remaining equations into two different groups as shown below. To determine which equation matches with which graph (within its group) we will think about plotting a few key points.

2. $r = \sin\theta$
7. $r = \sin(2\theta)$

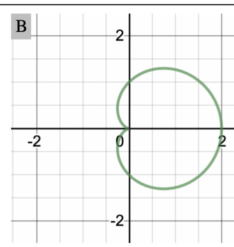


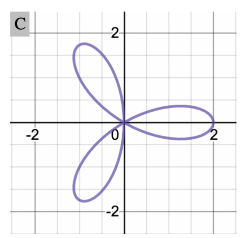


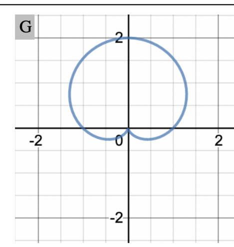
Consider the point in the plane with Cartesian coordinates $(x, y) = (0, 1)$. Notice that this point is on graph H above but not on graph A. One way to describe this point using polar coordinates is $(r, \theta) = (1, \frac{\pi}{2})$.

Since $1 = \sin\left(\frac{\pi}{2}\right)$, this point must be on the graph of equation 2: $r = \sin\theta$. This means equation 2 must be matched with graph H. It follows that equation 7 must be matched with graph A.

3. $r = 1 + \cos\theta$
4. $r = 1 + \sin\theta$
8. $r = 2\cos(3\theta)$







First, consider the point with Cartesian coordinates $(x, y) = (2, 0)$. Notice that this point is on graphs B and C above but not on graph G. One way to describe this point using polar coordinates is $(r, \theta) = (2, 0)$.

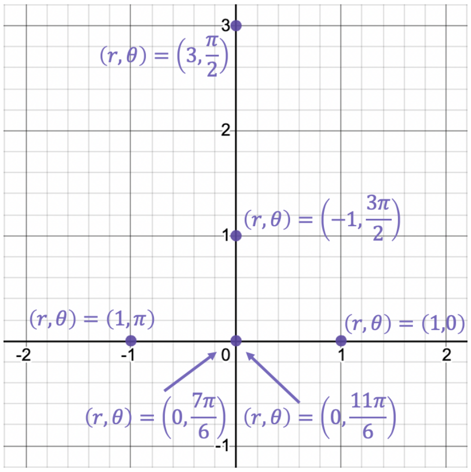
Since $2 = 1+\cos(0)$ and $2 = 2\cos(3\cdot 0)$, this point must be on the graphs of equation 3 ($r = 1+\cos\theta$) and equation 8 ($r = 2\cos(3\theta)$). This means equations 3 and 8 must be matched with graphs B and C, in some order. It follows that equation 4 ($r = 1 + \sin\theta$) must be matched with graph G.

Now, consider the point with Cartesian coordinates $(x, y) = (0, 1)$ and polar coordinates $(r, \theta) = (1, \frac{\pi}{2})$. Notice that this point is on graph B but not on graph C.

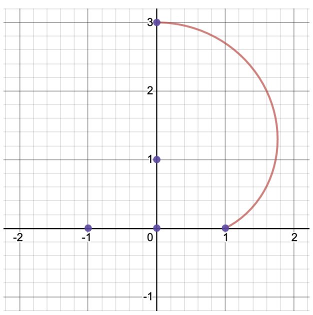
Since $1 = 1 + \cos\left(\frac{\pi}{2}\right)$, this point must be on the graph of equation 3: $r = 1 + \cos\theta$. This means equation 3 must be matched with graph B. It follows that equation 8 must be matched with graph C.

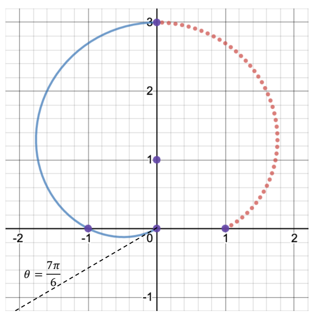
Notice that we have completed the matching activity without actually graphing any of the polar equations completely. We have just picked out certain characteristics of the equations and the graphs in order to find the right matches. We encourage you to follow the strategy outlined in the activity for how to sketch the graphs of the equations from scratch, and confirm the matchings that way as well. On the next page, we revisit the polar equation $r = 1 + 2\sin\theta$ that was discussed in the activity, and outline how to sketch its graph.

Using a few key points on the graph, and the table below, we sketch the graph of $r = 1 + 2\sin\theta$.



θ	$r = 1 + 2\sin(\theta)$	Polar Point
0	1	$(1, 0)$
0 to $\frac{\pi}{2}$	r increases from 1 to 3	
$\frac{\pi}{2}$	3	$(3, \frac{\pi}{2})$
$\frac{\pi}{2}$ to π	r decreases from 3 to 1	
π	1	$(1, \pi)$
π to $\frac{7\pi}{6}$	r decreases from 1 to 0	
$\frac{7\pi}{6}$ to $\frac{3\pi}{2}$	r decreases from 0 to -1	
$\frac{3\pi}{2}$	-1	$(-1, \frac{3\pi}{2})$
$\frac{3\pi}{2}$ to $\frac{11\pi}{6}$	r increases from -1 to 0	
$\frac{11\pi}{6}$ to 2π	r increases from 0 to 1	
2π	1	$(1, 2\pi)$

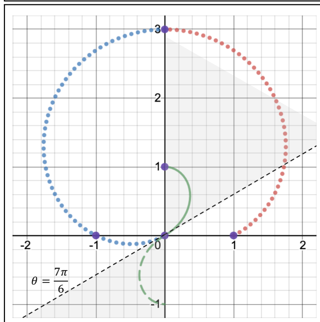




As θ increases from 0 to $\frac{\pi}{2}$, r increases continuously from 1 to 3. So we connect the polar points $(1, 0)$ and $(3, \frac{\pi}{2})$ through the first quadrant as shown in the leftmost image.

As θ increases from $\frac{\pi}{2}$ to π, r decreases from 3 to 1. So we connect polar points $(3, \frac{\pi}{2})$ and $(1, \pi)$ through the second quadrant.

As θ increases from π to $\frac{7\pi}{6}$, r decreases from 1 to 0. So we connect polar points $(1, \pi)$ and $(0, \frac{7\pi}{6})$ through the third quadrant as shown in the rightmost image.

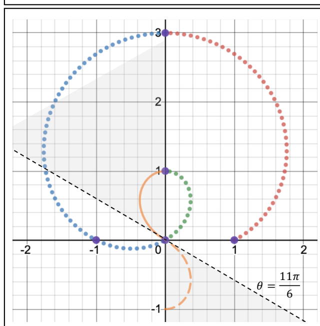


As θ increases from $\frac{7\pi}{6}$ to $\frac{3\pi}{2}$, r decreases from 0 to -1.

Since the point $(-1, \frac{3\pi}{2})$ is equivalent to the point $(1, \frac{\pi}{2})$, we know we are connecting the points $(0, \frac{7\pi}{6})$ and $(1, \frac{\pi}{2})$. But how?

Since r is negative between $\theta = \frac{7\pi}{6}$ and $\theta = \frac{3\pi}{2}$, the points we plot for this interval of θ will not actually lie in the third quadrant; they will lie in the first quadrant as indicated in the image. The magnitude of r tells us how far to plot the points from the origin; the negative sign attached to r tells us to plot the points on the "other side of the origin".

We connect the points $(0, \frac{7\pi}{6})$ and $(1, \frac{\pi}{2})$ through the first quadrant as shown.

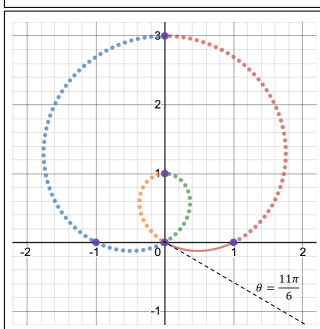


As θ increases from $\frac{3\pi}{2}$ to $\frac{11\pi}{6}$, r increases from -1 to 0.

Since the point $(-1, \frac{3\pi}{2})$ is equivalent to the point $(1, \frac{\pi}{2})$, we know we are connecting the points $(1, \frac{\pi}{2})$ and $(0, \frac{11\pi}{6})$. But how?

Since r is negative between $\theta = \frac{3\pi}{2}$ and $\theta = \frac{11\pi}{6}$, the points we plot for this interval of θ will not actually lie in the fourth quadrant; they will lie in the second quadrant as indicated in the image.

We connect the points $(1, \frac{\pi}{2})$ and $(0, \frac{11\pi}{6})$ through the second quadrant as shown.



Finally, we finish the sketch for θ between $\frac{11\pi}{6}$ and 2π, where r increases from 0 to 1.

Since the function $\cos\theta$ repeats with period 2π, plotting points for more values of θ will just result in drawing this same curve over again.

The completed graph is shown on the right.