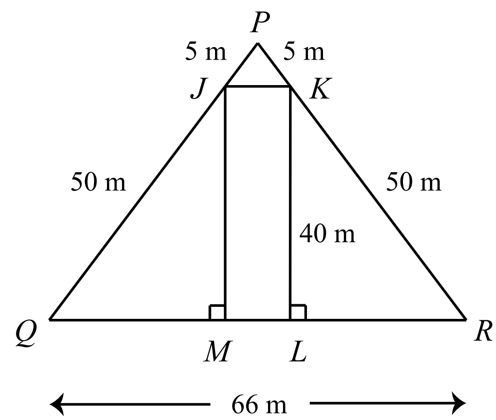
CEMC at Home

Grade 9/10 - Monday, May 25, 2020
Contest Day 4

Today's resource features one question from the recently released 2020 CEMC Mathematics Contests.

2020 Fryer Contest, #2

In the diagram, rectangle $JKLM$ is drawn with its vertices on the sides of $\triangle PQR$ so that $PJ = PK = 5$ m, $JQ = KR = 50$ m, $KL = 40$ m, and $QR = 66$ m, as shown.



(a) What is the length of LR?

(b) What is the length of ML?

(c) Determine the height of $\triangle PJK$ drawn from P to JK.

(d) Determine the fraction of the area of $\triangle PQR$ that is covered by rectangle $JKLM$.

More Info:

Check out the CEMC at Home webpage on Monday, June 1 for solutions to the Contest Day 4 problems.

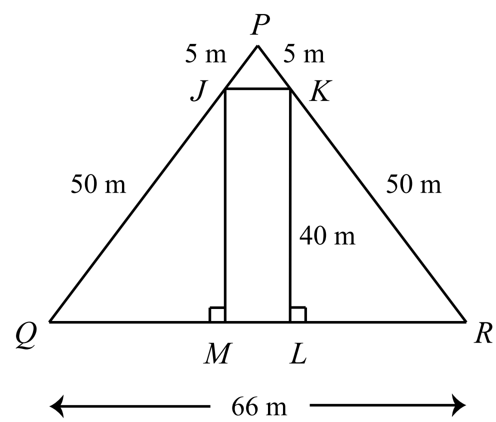
CEMC at Home

Grade 9/10 - Monday, May 25, 2020
Contest Day 4 - Solution

A solution to the contest problem is provided below, along with an accompanying video.

2020 Fryer Contest, #2

In the diagram, rectangle $JKLM$ is drawn with its vertices on the sides of $\triangle PQR$ so that $PJ = PK = 5$ m, $JQ = KR = 50$ m, $KL = 40$ m, and $QR = 66$ m, as shown.



(a) What is the length of LR?

(b) What is the length of ML?

(c) Determine the height of $\triangle PJK$ drawn from P to JK.

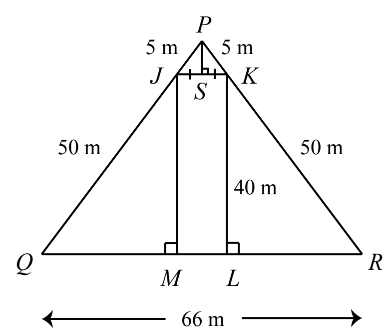
(d) Determine the fraction of the area of $\triangle PQR$ that is covered by rectangle $JKLM$.

Solution:

(a) In $\triangle KLR$, we have $\angle KLR = 90^\circ$ and by using the Pythagorean Theorem, we get $LR^2 = 50^2 - 40^2 = 900$ and so $LR = \sqrt{900} = 30$ m (since $LR > 0$).

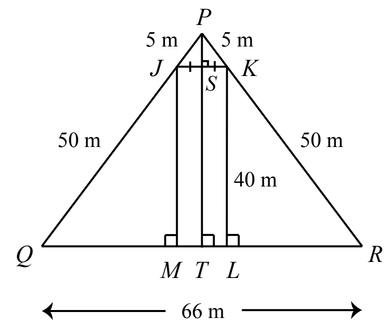
(b) We begin by showing that $\triangle JMQ$ is congruent to $\triangle KLR$.
Since $JKLM$ is a rectangle, then $JM = KL = 40$ m.
In addition, hypotenuse JQ has the same length as hypotenuse KR, and so $\triangle JMQ$ is congruent to $\triangle KLR$ by HS congruence.
Thus, $MQ = LR = 30$ m and so $ML = 66 - 30 - 30 = 6$ m.

(c) Since $PJ = PK = 5$ m, $\triangle PJK$ is isosceles and so the height, PS, drawn from P to JK bisects JK, as shown.
Since $JKLM$ is a rectangle, then $JK = ML = 6$ m and so $SK = \frac{JK}{2} = 3$ m.
Using the Pythagorean Theorem in $\triangle PSK$, we get $PS^2 = 5^2 - 3^2 = 16$ and so $PS = 4$ m (since $PS > 0$).
Thus the height of $\triangle PJK$ drawn from P to JK is 4 m.



See the next page for a solution to part (d) and a link to the video.

(d) We begin by determining the area of $\triangle PQR$.
Construct the height of $\triangle PQR$ drawn from P to T on QR, as shown.
Since PT is perpendicular to QR, then PT is parallel to KL (since KL is also perpendicular to QR).
By symmetry, PT passes through S, and so the height PT is equal to $PS + ST = PS + KL$ or $4 + 40 = 44$ m.
The area of $\triangle PQR$ is $\frac{1}{2} \times QR \times PT = \frac{1}{2} \times 66 \times 44 = 1452$ m^2.
The area of $JKLM$ is $ML \times KL = 6 \times 40 = 240$ m^2.
The fraction of the area of $\triangle PQR$ that is covered by rectangle $JKLM$ is $\frac{240}{1452} = \frac{20}{121}$.



Video
Visit the following link to view a discussion of a solution to this contest problem:
https://youtu.be/gMpResbow9E

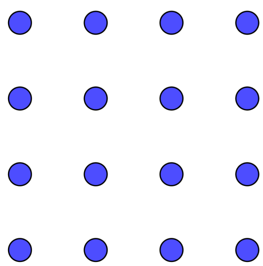
CEMC at Home

Grade 9/10 - Tuesday, May 26, 2020
Perfect Squares

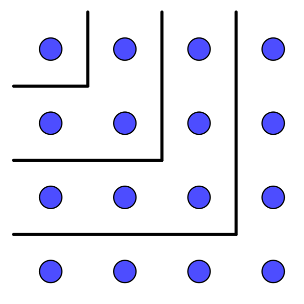
A *perfect square* (or *square number*) is an integer that is the square of an integer. In other words, an integer s is a perfect square if $s = n^2$ for some integer n. There are many different ways to illustrate a perfect square and they often involve the geometric notion of a square.

Consider the perfect square $16 = 4^2$

Since $16 = 4^2$ we can illustrate the perfect square 16 by drawing 16 dots arranged in a 4 by 4 (square) grid as shown.



What do you notice if we group the dots in the grid as shown below? Can you see how this illustrates another way to build the number 16?



You can check directly that $16 = 4^2$ is the sum of the first 4 positive odd integers: $16 = 1+3+5+7$. The illustrations above give an idea of why this is true. If you drew a similar illustration of the perfect square $25 = 5^2$, and grouped the 25 dots as shown above, what would you observe?

Fun Fact: The perfect square $s = n^2$ (where n is a positive integer) is equal to the sum of the first n consecutive positive odd integers. *Take some time to think about why this fact is true.*

Problems: Use the fact above to find an efficient way to answer each of the following questions.

1. What is the sum of the first 99 consecutive positive odd integers?
2. If 1225 is the sum of the first m consecutive positive odd integers, what is the value of m?
3. What is the value of the sum $1 + 3 + 5 + \cdots + 141 + 143 + 145$?
4. What is the value of the sum $17 + 19 + 21 + \cdots + 207 + 209 + 211$?
5. What is the value of the sum $2 + 4 + 6 + \cdots + 296 + 298 + 300$?

More Info:

Check out the CEMC at Home webpage on Tuesday, June 2 for a solution to Perfect Squares.

The sum of the first n consecutive odd numbers, $1 + 3 + 5 + \ldots + (2n - 1)$, is an example of an *arithmetic series* where the first term is 1 and the common difference is 2. Check out this lesson in the CEMC Courseware for more information about arithmetic series.

CEMC at Home

Grade 9/10 - Tuesday, May 26, 2020
Perfect Squares - Solution

1. What is the sum of the first 99 consecutive positive odd integers?

   Solution: The sum of the first 99 consecutive positive odd integers is equal to $99^2 = 9801$.

2. If 1225 is the sum of the first m consecutive positive odd integers, what is the value of m?

   Solution: Since the sum of the first m consecutive positive odd integers is equal to m^2 we must have $m^2 = 1225$. Since m is positive, $m = \sqrt{1225} = 35$.

3. What is the value of the sum $1 + 3 + 5 + \cdots + 141 + 143 + 145$?

   Solution: This is the sum of the first n consecutive positive odd integers for some n. What is the value of n (the number of terms in the sum)? We can write odd numbers in the form $2k-1$ where k is an integer, and so we rewrite this sum as

$$1 + 3 + \cdots + 143 + 145 = \big(2(1) - 1\big) + \big(2(2) - 1\big) + \cdots + \big(2(72) - 1\big) + \big(2(73) - 1\big)$$

   Rewriting the sum in this way allows us to count that there are 73 terms in the sum, and so $n = 73$. Since this sum is the sum of the first 73 consecutive positive odd integers, the sum must be equal to $73^2 = 5329$. (Note that n can be calculated as follows: $n = \frac{145+1}{2} = 73$.)

4. What is the value of the sum $17 + 19 + 21 + \cdots + 207 + 209 + 211$?

   Solution: First, we note that the given sum can be calculated as the following difference of sums:

$$17+19+21+\cdots+207+209+211 = (1+3+5+\ldots+207+209+211)-(1+3+5+\ldots+11+13+15)$$

   You can verify that the sum $1+3+5+\ldots+207+209+211$ has $\frac{211+1}{2} = 106$ terms. Since this is the sum of the first 106 consecutive positive odd integers, the value of the sum is $106^2 = 11\,236$.

   You can verify that the sum $1+3+5+\ldots+11+13+15$ has $\frac{15+1}{2} = 8$ terms. Since this is the sum of the first 8 consecutive positive odd integers, the value of the sum is $8^2 = 64$.

   Therefore, $17 + 19 + 21 + \cdots + 207 + 209 + 211 = 11\,236 - 64 = 11\,172$.

5. What is the value of the sum $2 + 4 + 6 + \cdots + 296 + 298 + 300$?

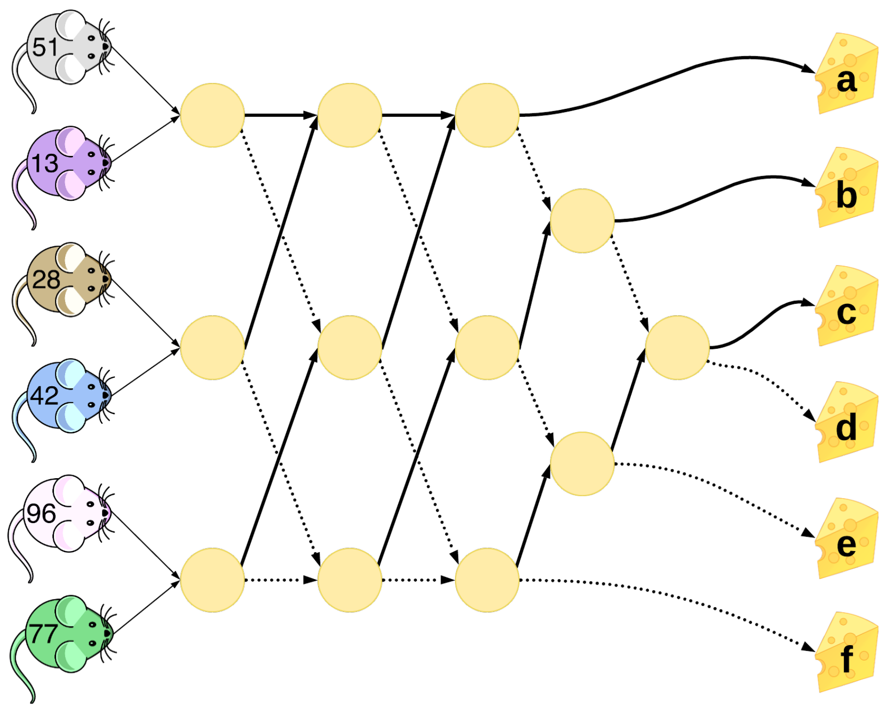
   Solution: The terms in this sum are consecutive positive **even** integers. There are $\frac{300}{2} = 150$ terms in the sum. To create a sum of consecutive positive **odd** integers, we can rewrite each term as an odd number plus 1, and then collect all the extra 1s as follows:

$$\begin{aligned}
&2 + 4 + 6 + \ldots + 296 + 298 + 300 \\
&= (1 + 1) + (3 + 1) + (5 + 1) + \ldots + (295 + 1) + (297 + 1) + (299 + 1) \\
&= 1 + 3 + 5 + \ldots + 295 + 297 + 299 + (1 + 1 + 1 + \ldots + 1) \\
&= 150^2 + 150 \\
&= 22\,650
\end{aligned}$$

CEMC at Home

Grade 9/10 - Wednesday, May 27, 2020
Mixed Up Mice

Six numbered mice are moving through the network of paths shown below in order to reach the cheese. To start, the mice line up randomly on the left side of the network. Then, each mouse moves along a path by following the arrows. When a mouse reaches a yellow circle it waits for another mouse to arrive. When another mouse arrives at the circle, the two mice compare their numbers. The mouse with the smaller number follows the solid arrow out of the circle, while the mouse with the larger number follows the dotted arrow out of the circle.



Questions

1. After all of the mice move through the network of paths shown above, which mouse ends up with which piece of cheese?

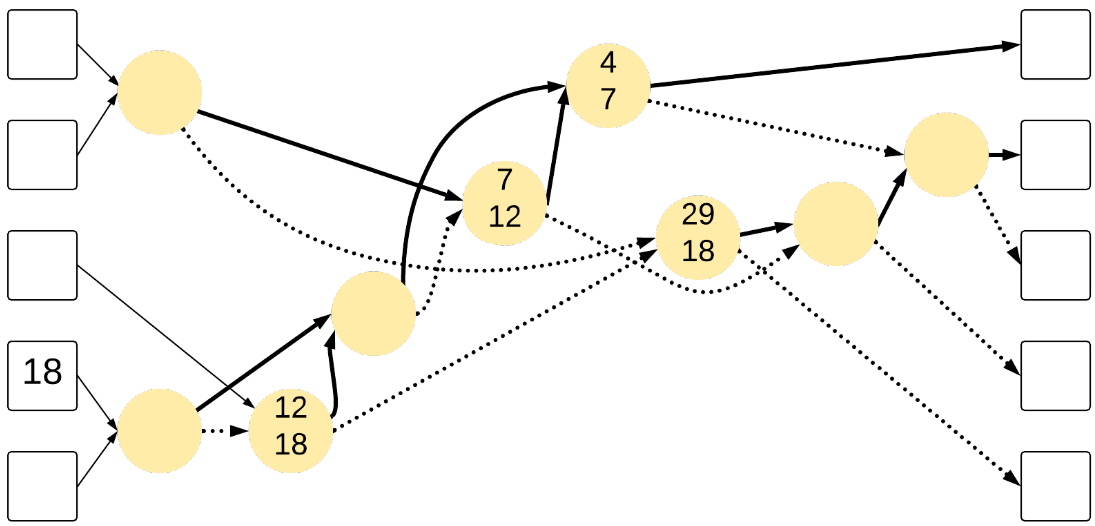
2. Now, line up the mice again at the start of the network, but change their starting order. After all of the mice move through the network of paths again, which mouse ends up with which cheese? Try repeating this a few times, each time with a different starting order for the mice. What do you notice?

3. What would happen if, upon leaving the circles, the mice with the smaller numbers followed the dotted arrows and the mice with the larger numbers followed the solid arrows? Explain.

Think about questions 1, 2, and 3 before moving on to the next page.

The network shown on the first page is an example of a *sorting network* for six numbers. There are six *inputs* on the left side of the network and six *outputs* on the right side of the network. Each yellow circle represents a *comparison* between two inputs (i.e. which number is larger?) and produces two outputs as demonstrated by the arrows. As the inputs move through the network, they are reordered (or *sorted*) according to their relative sizes.

4. Suppose you were given the following sorting network for five numbers. This network follows the same rules as the network on the first page. The inputs on the left side of the network are the numbers 4, 7, 12, 18, and 29, in some order. Using the information given below about how the numbers moved through the network, is it possible to determine the starting order of the five input numbers?



5. Draw a possible sorting network that sorts exactly four numbers.

   Can you draw two different sorting networks that sort exactly four numbers?

   Activity: Try drawing out your sorting network on a driveway with sidewalk chalk, or find a way to lay it out on a floor (what can you use for the circles and the arrows?). Ask your family members to pick a card from one suit in a deck of cards and randomly line up at the start. Play through a few rounds to convince yourself that you have a proper sorting network.

More Info:

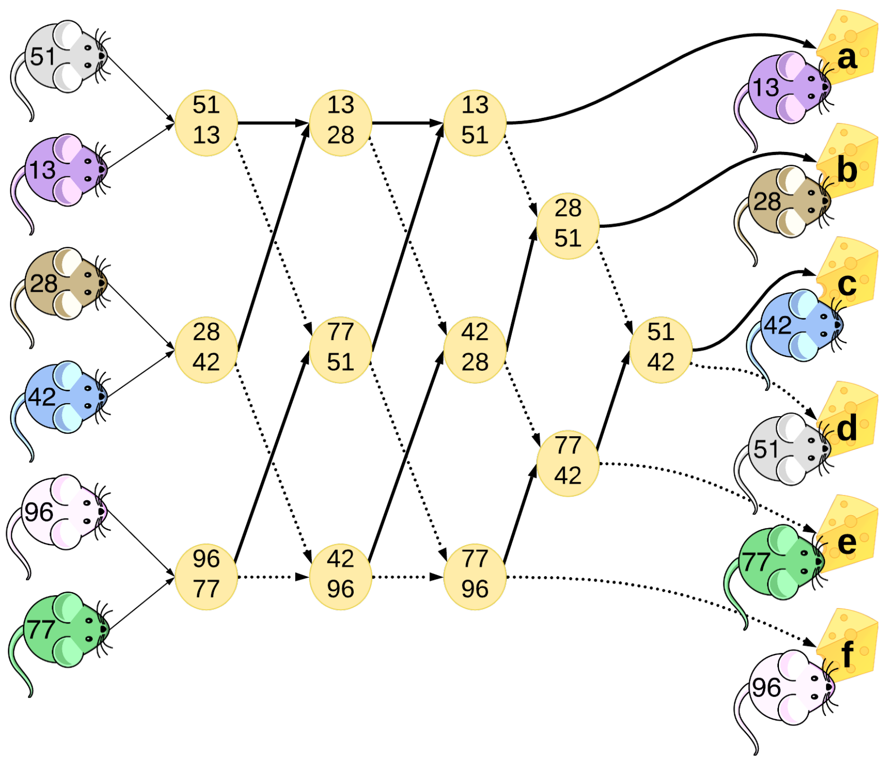
Check out the CEMC at Home webpage on Wednesday, June 3 for a solution to Mixed Up Mice.

CEMC at Home

Grade 9/10 - Wednesday, May 27, 2020
Mixed Up Mice - Solution

In computer science, a sorting network sorts a fixed number of items into a specific order which is predetermined. In this activity, you explored examples of sorting networks that sorted integers.

1. The solution below shows the path that each mouse travelled. Each yellow circle shows numbers that are being compared, and the mouse with the smaller number follows the *solid* line out of the circle, while the mouse with the larger number follows the *dotted* line.



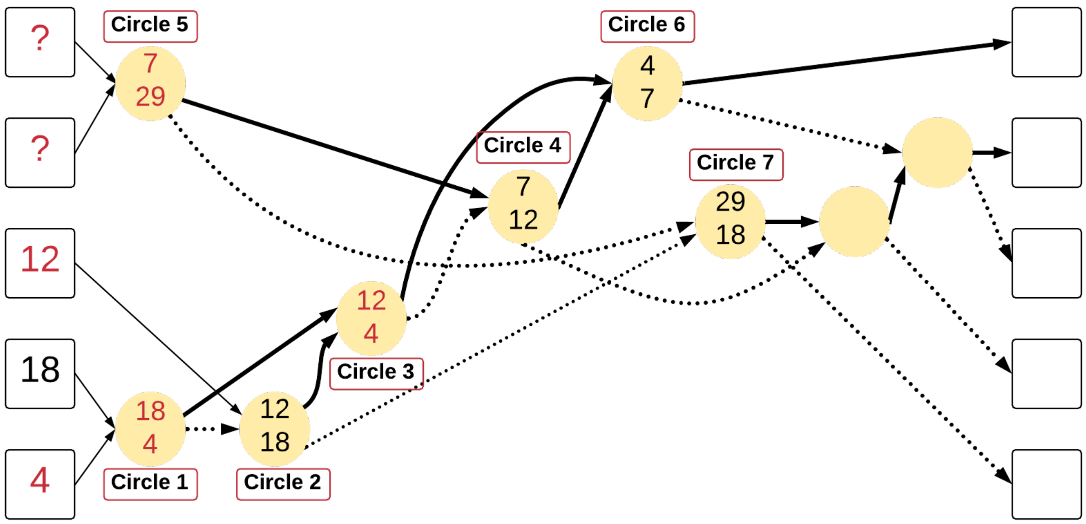
2. Regardless of how the numbers are arranged at the start, each time the mice reach the cheese, their numbers are sorted from smallest to largest, reading from top to bottom. This is not a coincidence as the above network is a *sorting network* for six numbers.

3. If instead the mouse with the smaller number follows the *dotted* line and the mouse with the larger number follows the *solid* line out of the circle, then the mice would end up being sorted from largest to smallest (reading top to bottom).

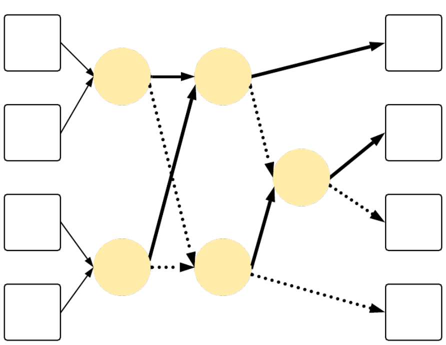
   Note: The numbers we have chosen for the mice are all distinct, and the network rules currently do not specify what to do if two mice have the same number. A computer cannot typically handle this lack of clarity. Instead, it needs to be given clear instructions to cover all possible scenarios. For example, for this sorting network, we could say that if two equal numbers are compared, then the mouse that comes from above follows the solid line out of the circle, while the mouse that comes from below follows the dotted arrow of the circle. If our only goal is to produce a sorted list of numbers, then it does not matter how we choose to handle the comparison of two equal numbers. However, if mice with equal numbers have other information associated with them, then how we make this choice could become more important.

4. It is not possible to completely determine the original order of the five inputs. How much *can* we determine about the order? There are a number of ways to approach this question. One possible approach is described below.

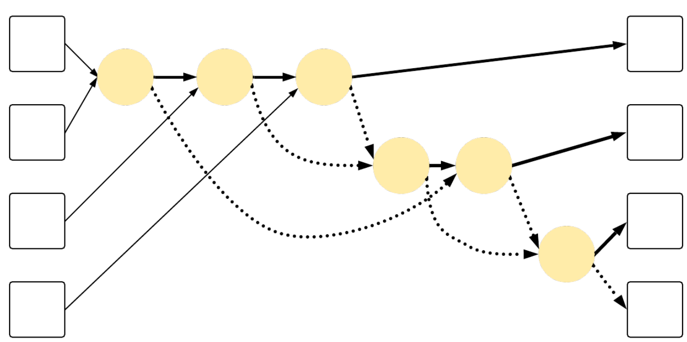
Since we are given the starting position for 18, we know that *Circle 1* has to contain 18 and another number. Next, we see that *Circle 2* contains 12 and 18. From here, we can trace back to the start of the network and place the 12 in the third input from the top. Looking at *Circle 2* again, we know that 12 is the smaller number, so it would proceed to *Circle 3*. From *Circle 4* we can determine that the 7 traces back to *Circle 5* as the 12 traces back to *Circle 3*. Similarly, looking at *Circle 6* we conclude that the 4 traces back to *Circle 3*. Following the arrows even further back to the start of the network, we determine that the 4 was the bottom input. Last, looking at *Circle 7* we determine that 29 comes from *Circle 5* as 18 traces back to *Circle 2*. We cannot determine the original order of inputs 7 and 29.



5. Below are two different sorting networks for four numbers. The first sorting network takes advantage of *parallel processing* while the second does not.



Sorting Network - Parallel Processing
Some comparisons occur simultaneously



Sorting Network - Sequential Processing
Only one comparison occurs at a time

*When a sorting network is used, it can be possible to perform some comparisons simultaneously, also known as *parallel processing*. This can speed up a sorting process overall. For example, in the sorting network from problem 1., the comparison between 51 and 13 occurs at the same time as the comparison between 28 and 42 as well as the comparison between 96 and 77. A common use of parallel processing with sorting networks is in the design of hardware.

CEMC at Home

Grade 9/10 - Thursday, May 28, 2020
Repetition By Product

A positive integer is to be placed in each box below.

Integers may be repeated, but the product of any four adjacent integers is always 120.

Determine all possible values for x.

		2			4			x			3		

More Info:

Check out the CEMC at Home webpage on Friday, May 29 for two different solutions to Repetition By Product.

This CEMC at Home resource is a past problem from Problem of the Week (POTW). POTW is a free, weekly resource that the CEMC provides for teachers, parents, and students during the school year. POTW is wrapped up for the current school year and will resume on September 17, 2020. To subscribe to POTW and to find more past problems and their solutions visit:
https://www.cemc.uwaterloo.ca/resources/potw.php

CEMC at Home

Grade 9/10 - Thursday, May 28, 2020
Repetition By Product - Solution

Problem:

A positive integer is to be placed in each box below.

Integers may be repeated, but the product of any four adjacent integers is always 120.

Determine all possible values for x.

		2			4			x			3		

Solution:

In both Solution 1 and Solution 2, let a_1 be the integer placed in the first box, a_2 the integer placed in the second box, a_4 the integer placed in the fourth box, and so on, as shown below.

a_1	a_2	2	a_4	a_5	4	a_7	a_8	x	a_{10}	a_{11}	3	a_{13}	a_{14}

Solution 1

Consider boxes 3 to 6. Since the product of any four adjacent integers is 120, we have $2 \times a_4 \times a_5 \times 4 = 120$. Therefore, $a_4 \times a_5 = \frac{120}{2\times 4} = 15$. Since a_4 and a_5 are positive integers, there are four possibilities: $a_4 = 1$ and $a_5 = 15$, or $a_4 = 15$ and $a_5 = 1$, or $a_4 = 3$ and $a_5 = 5$, or $a_4 = 5$ and $a_5 = 3$.

In each of the four cases, we will have $a_7 = 2$. We can see why by considering boxes 4 to 7. We have $a_4 \times a_5 \times 4 \times a_7 = 120$, or $15 \times 4 \times a_7 = 120$, since $a_4 \times a_5 = 15$. Therefore, $a_7 = \frac{120}{15\times 4} = 2$.

<u>Case 1:</u> $a_4 = 1$ and $a_5 = 15$

Consider boxes 5 to 8. We have $a_5 \times 4 \times a_7 \times a_8 = 120$, or $15 \times 4 \times 2 \times a_8 = 120$, or $a_8 = \frac{120}{15\times 4\times 2} = 1$.
Next, consider boxes 6 to 9. We have $4 \times a_7 \times a_8 \times x = 120$, or $4 \times 2 \times 1 \times x = 120$, or $x = \frac{120}{4\times 2} = 15$.
Let's check that $x = 15$ satisfies the only other condition in the problem that we have not yet used, that is $a_{12} = 3$.
Consider boxes 9 to 12. If $x = 15$ and $a_{12} = 3$, then $a_{10} \times a_{11} = \frac{120}{15\times 3} = \frac{8}{3}$. But a_{10} and a_{11} must both be integers, so is not possible for $a_{10} \times a_{11} = \frac{8}{3}$. Therefore, it must not be possible for $a_4 = 1$ and $a_5 = 15$, and so we find that there is no solution for x in this case.

<u>Case 2:</u> $a_4 = 15$ and $a_5 = 1$

Consider boxes 5 to 8. We have $a_5 \times 4 \times a_7 \times a_8 = 120$, or $1 \times 4 \times 2 \times a_8 = 120$, or $a_8 = \frac{120}{4\times 2} = 15$.
Next, consider boxes 6 to 9. We have $4 \times a_7 \times a_8 \times x = 120$, or $x = \frac{120}{4\times 2\times 15} = 1$.
Let's check that $x = 1$ satisfies the only other condition in the problem that we have not yet used, that is $a_{12} = 3$.
Consider boxes 7 to 10. Since $a_7 = 2$, $a_8 = 15$ and $x = 1$, then $a_{10} = \frac{120}{2\times 15\times 1} = 4$. Similarly, $a_{11} = \frac{120}{15\times 1\times 4} = 2$. Then we have $x \times a_{10} \times a_{11} \times a_{12} = 1 \times 4 \times 2 \times 3 = 24 \neq 120$. Therefore, it must not be possible for $a_4 = 15$ and $a_5 = 1$. There is no solution for x in this case.

<u>Case 3:</u> $a_4 = 3$ and $a_5 = 5$
Consider boxes 5 to 8. We have $a_5 \times 4 \times a_7 \times a_8 = 120$, or $5 \times 4 \times 2 \times a_8 = 120$, or $a_8 = \frac{120}{5\times4\times2} = 3$.
Next, consider boxes 6 to 9. We have $4 \times a_7 \times a_8 \times x = 120$, or $x = \frac{120}{4\times2\times3} = 5$.
Let's check that $x = 5$ satisfies the only other condition in the problem that we have not yet used, that is $a_{12} = 3$.
Consider boxes 7 to 10. Since $a_7 = 2$, $a_8 = 3$ and $x = 5$, then $a_{10} = \frac{120}{2\times3\times5} = 4$. Similarly, $a_{11} = \frac{120}{3\times5\times4} = 2$. Then we have $x \times a_{10} \times a_{11} \times a_{12} = 5 \times 4 \times 2 \times 3 = 120$. Therefore, the condition that $a_{12} = 3$ is satisfied in the case where $a_4 = 3$ and $a_5 = 5$. If we continue to fill out the entries in the boxes, we obtain the entries shown in the diagram below.

5	4	2	3	5	4	2	3	5	4	2	3	5	4

We see that $x = 5$ is a possible solution. However, is it the only solution? We have one final case to check.

<u>Case 4:</u> $a_4 = 5$ and $a_5 = 3$
Consider boxes 5 to 8. We have $a_5 \times 4 \times a_7 \times a_8 = 120$, or $3 \times 4 \times 2 \times a_8 = 120$, or $a_8 = \frac{120}{3\times4\times2} = 5$.
Next, consider boxes 6 to 9. We have $4 \times a_7 \times a_8 \times x = 120$, or $x = \frac{120}{4\times2\times5} = 3$.
Let's check that $x = 3$ satisfies the only other condition in the problem that we have not yet used, that is $a_{12} = 3$.
Consider boxes 9 to 12. If $x = 3$ and $a_{12} = 3$, then $a_{10} \times a_{11} = \frac{120}{3\times3} = \frac{40}{3}$. But a_{10} and a_{11} must both be integers, so it is not possible for $a_{10} \times a_{11} = \frac{40}{3}$. Therefore, it must not be possible for $a_4 = 5$ and $a_5 = 3$, and so we find that there is no solution for x in this case.

Therefore, the only possible value for x is $x = 5$.

Solution 2

You may have noticed a pattern for the a_i's in Solution 1. We will explore this pattern.

a_1	a_2	2	a_4	a_5	4	a_7	a_8	x	a_{10}	a_{11}	3	a_{13}	a_{14}

Since the product of any four integers is 120, $a_1a_2a_3a_4 = a_2a_3a_4a_5 = 120$. Since both sides are divisible by $a_2a_3a_4$, and each is a positive integer, then $a_1 = a_5$.
Similarly, $a_2a_3a_4a_5 = a_3a_4a_5a_6 = 120$, and so $a_2 = a_6$.
In general, $a_na_{n+1}a_{n+2}a_{n+3} = a_{n+1}a_{n+2}a_{n+3}a_{n+4}$, and so $a_n = a_{n+4}$.
We can use this along with the given information to fill out the boxes as follows:

x	4	2	3	x	4	2	3	x	4	2	3	x	4

Therefore, $4 \times 2 \times 3 \times x = 120$ and so $x = \frac{120}{4\times2\times3} = 5$.

CEMC at Home

Grade 9/10 - Friday, May 29, 2020
Circle Splash

A group of friends have devised a fun way to stay cool on a hot summer day – bursting water balloons!

Here is how they play:

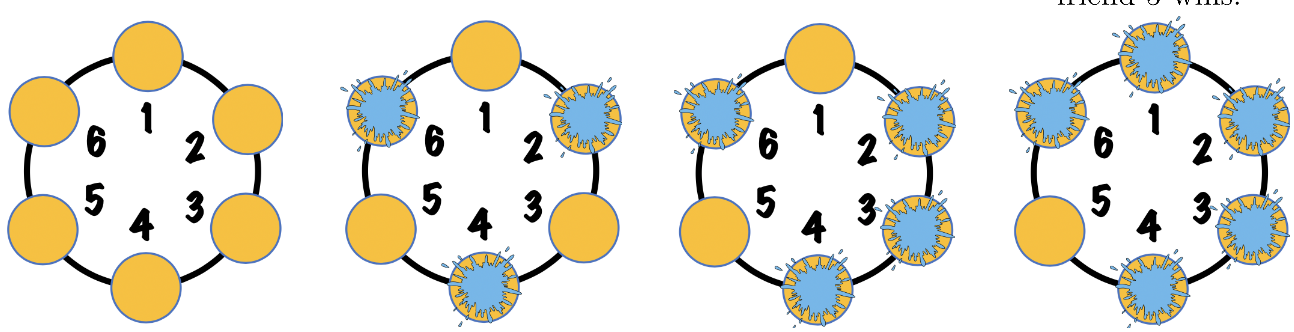
- Everyone stands in a circle. One friend is assigned the number 1 (position 1), and then the other friends in the circle are numbered 2, 3, 4, etc. moving clockwise around the circle until you get back to position 1.
- Starting with the person in position 1 and moving clockwise around the circle, every second person who is still dry bursts a water balloon over their head. (This is explained further below.)
- The last person remaining dry wins the game!

Example: Here is how the game plays out if the circle contains six friends.

The first time around the circle, friends 2, 4, and 6 get water balloons.

Since friend 2 is already out, friend 3 gets the next water balloon.

Since friends 4 and 6 are already out, friend 1 gets the next water balloon, and so friend 5 wins!



Questions:

For each instance of the game below, determine the position of the friend that will win the game.

1. The game is played with 7 friends.
2. The game is played with 16 friends.
3. The game is played with 41 friends.

Challenge: Can you determine the position of the friend that wins the game if the game is played with n friends, where n is a positive integer that is at least 2?

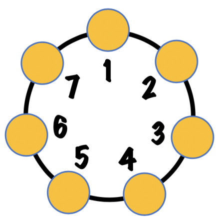
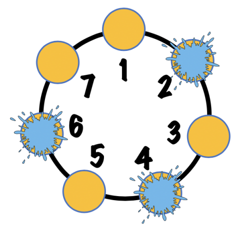
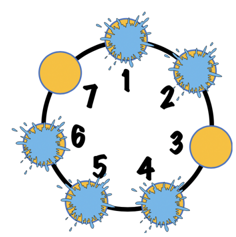
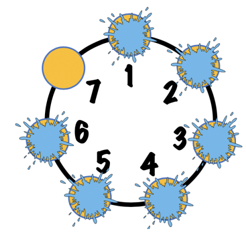
More Info:

Check out the CEMC at Home webpage on Friday, June 5 for a solution to Circle Splash.

CEMC at Home

Grade 9/10 - Friday, May 29, 2020
Circle Splash - Solution

1. When the game is played with 7 friends, the friend in position 7 will win (remain dry). We illustrate this using the following images:

Start	The first time around the circle, friends 2, 4, and 6 get water balloons.	The second time around the circle, friends 1 and 5 get water balloons.	The third time around the circle, friend 3 gets a water balloon and so friend 7 wins!
			

2. When the game is played with 16 friends, the friend in position 1 will win (remain dry). Instead of drawing pictures to illustrate the solution, we start with the sequence of numbers from 1 to 16 and cross off the numbers when this position gets a water balloon.

   Each time we travel through the numbers, we cross out every second number that is not already crossed out. Each time we reach 16, we loop around to 1.

Start:	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16
First pass:	1, ~~2~~, 3, ~~4~~, 5, ~~6~~, 7, ~~8~~, 9, ~~10~~, 11, ~~12~~, 13, ~~14~~, 15, ~~16~~
Second pass:	1, ~~2~~, ~~3~~, ~~4~~, 5, ~~6~~, ~~7~~, ~~8~~, 9, ~~10~~, ~~11~~, ~~12~~, 13, ~~14~~, ~~15~~, ~~16~~
Third pass:	1, ~~2~~, ~~3~~, ~~4~~, ~~5~~, ~~6~~, ~~7~~, ~~8~~, 9, ~~10~~, ~~11~~, ~~12~~, ~~13~~, ~~14~~, ~~15~~, ~~16~~
Fourth pass:	1, ~~2~~, ~~3~~, ~~4~~, ~~5~~, ~~6~~, ~~7~~, ~~8~~, ~~9~~, ~~10~~, ~~11~~, ~~12~~, ~~13~~, ~~14~~, ~~15~~, ~~16~~

   The numbers were crossed out in the following order: 2, 4, 6, 8, 10, 12, 14, 16, 3, 7, 11, 15, 5, 13, 9.

   This leaves 1 at the end.

   Can you see a pattern forming here?

3. When the game is played with 41 friends, the friend in position 19 will remain dry. This can be determined by simulating the game as was done in the solution to either question 1 or question 2, although it will take longer in this case. After discussing the challenge problem, we will return to this answer.

Challenge: Can you determine the position of the friend that wins the game if the game is played with n friends, where n is a positive integer that is at least 2?

Solution:

When the game is played with n friends, a simulation will not work. We need to step back and analyze the game more generally.

Here are some observations:

- The position of the friend who remains dry will never be even since all even numbers are eliminated during the first pass around the circle.
- If a circle has an even number of friends in it, then after one pass, we will be on position 1 again and the number of friends left dry in the circle will be half of what it was at the start.

This second observation above leads to some more observations:

- If, after one pass, the circle still has an even number of friends that are dry, then after pass two, we will be on position 1 again and the number of friends left dry in the circle will be half of what it was.
- If after every pass (except for the final one), the circle continues to have an even number of friends that are dry, eventually we will land on position 1 with only two friends left dry in the circle. On the final pass, the friend in position 1 remains dry and the other friend gets wet.
- In order for the circle to have an even number of friends left dry after every pass (except for the final one), the number of friends at the start, n, needs to be a power of 2.
- We can conclude then that if n is a power of 2, then the friend in position 1 will always remain dry. (Notice that our answer to question 2 agrees with this.)

What if n is not a power of 2? At some point during the game, eliminating people one at a time, the number of friends left dry in the circle *will become* a power of 2. For example, if we started with 12 friends, then after four water balloons, the circle will be left with $8 = 2^3$ people that are still dry. When this happens, whichever position is "next" in the circle acts like position 1 (in the case where n is a power of 2) and will be the position that remains dry for the rest of the game.

Given n, how many people need to be eliminated so that the number of people remaining dry in the circle is a power of 2? Rewrite n as a power of 2 plus m as follows:

$$n = 2^k + m$$

where 2^k is the largest power of 2 less than or equal to n and m is a non-negative integer. Thus, after eliminating m people, there will be 2^k people remaining dry in the circle.

The position that remains dry for the rest of the game is the position in the circle that comes immediately after the mth person that is eliminated. What position is this? Since on the first trip around the circle every second person is eliminated, and it must be the case that $2m < n$, the mth person to be eliminated is in position $2m$. This means the "next" position is position $2m + 1$.

Can you see why it must be the case that $2m < n$ and why this is important for our argument above?

We can conclude that if n is not a power of 2 and we rewrite n as $n = 2^k + m$ (as outlined earlier), then the person that remains dry is the person in position $2m + 1$.

Note 1: If n is a power of 2 then we have $m = 0$ and hence $2m + 1 = 2(0) + 1 = 1$. So this formula tells us the position that remains dry in the case where n is a power of 2 as well.

Note 2: This formula agrees with our earlier answers. In question 1, we have $n = 7$ which can be rewritten as $7 = 4 + 3 = 2^2 + 3$. This means $m = 3$ and so the position that remains dry is $2m + 1 = 2(3) + 1 = 7$. In question 2, we have $n = 16$ which can be rewritten as $16 = 2^4 + 0$. This means $m = 0$ and so $2m + 1 = 2(0) + 1 = 1$. In question 3, we have $n = 41$ which can be rewritten as $n = 32 + 9 = 2^5 + 9$. This means $m = 9$ and so $2m + 1 = 2(9) + 1 = 19$.